CONTRIBUTION TO THE THEORY OF MULTIPLE SCATTERING

M. L. TER-MIKAELYAN

Physics Institute, Academy of Sciences, Armenian S.S.R.

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The article treats a method of calculating multiple-scattering curves with allowance for the finite dimensions of the nucleus. In the calculation use is made of the experimental results on the scattering of fast electrons by nuclei.

LET us denote by $f(\theta_x, \theta_y, t)$ the angular distribution function of charged particles at a depth t (g/cm²). The scattering angles of the particle are assumed small and be presented in the form of a vector $\boldsymbol{\theta}$ in the plane perpendicular to the direction of motion; θ_x and θ_y are the projections of the scattering angle on two mutually perpendicular planes through the initial direction of motion. The distribution function, for the case of small scattering angles, obeys the following kinetic equation

$$\frac{df(\theta_x\theta_y t)}{dt} = -f(\theta_x\theta_y t)\iint\limits_{-\infty}^{\infty}\frac{N}{A}\sigma(\theta_x'\theta_y')\,d\theta_x'\,d\theta_y' + \iint\limits_{-\infty}^{\infty}\frac{N}{A}\sigma(\theta_x'\theta_y')\,f(\theta_x-\theta_x';\ \theta_y-\theta_y')\,d\theta_x'\,d\theta_y'. \quad (1)$$

Here N is Avogadro's number, A the atomic weight, $\sigma(\theta_x, \theta_y)$ is the transverse scattering cross section. The boundary condition is chosen in the following form

$$f(\theta_x; \theta_y; 0) = 2\delta(\theta_x)\,\delta(\theta_y). \quad (2)$$

We shall be interested henceforth in the distribution function for one angle only, either θ_x, or θ_y. This is caused by the fact that it is much easier to measure in a cloud chamber the projection of the scattering angle, than the three-dimensional scattering angle. We must therefore integrate the solution of Eq. (1) over one of the projection angles. The final result can be represented in the following form

$$f(\theta_x, t)\,d\theta_x = \frac{d\theta_x}{\pi}\int\limits_{-\infty}^{\infty} e^{ik_x\theta_x} \times\exp\left\{\frac{Nt}{A}\int\limits_{-\infty}^{\infty} e^{ik_x\theta_x'}\sigma(\theta_x', \theta_y')\,d\theta_x'\,d\theta_y' - \frac{Nt}{A}\sigma(0, 0)\right\}dk_x, \quad (3)$$

where

$$f(\theta_x, t) = \int\limits_{-\infty}^{\infty} f(\theta_x, \theta_y, t)\,d\theta_y \quad (4)$$

with the normalization condition $\int\limits_0^{\infty} f(\theta_x, t)\,d\theta_x = 1$. The solution (3) is valid for any scattering law, provided the scattering angles are small.

We are interested so far only in pure Coulomb scattering from a stationary nucleus. This means that $\sigma(\theta_x, \theta_y)$ is the Rutherford scattering cross section with account for both the atomic and nuclear form factors:

$$(Nt/A)\,\sigma(\theta_x, \theta_y) = 2\chi_c^2 q^2 F^2/(\theta_x^2+\theta_y^2)^2; \quad (5)$$

we employ the quantity universally used in the theory of multiple scattering

$$\chi_c^2 = (4\pi Nt/A)\,Z(Z+1)\,z^2e^4/p^2v^2, \quad (6)$$

where Z is the charge of the scattering nucleus, z the charge of the scattered quantity (henceforth taken as unity); p and v are the momentum and velocity of the scattered particles, q the atomic form factor, and F the nuclear form factor.

Moliere[1,2] has shown how to obtain an exact formula that takes into account the screening of the Coulomb center. This reduces to replacing $q^2/(\theta_x^2+\theta_y^2)^2$ with $(\theta_x^2+\theta_y^2+\varphi_m^2)^{-2}$ where

$$\varphi_m = (\lambda\!\!\!^{-}/0.885R_0Z^{-1/3})\sqrt{1.13+3.76\alpha^2}; \quad (7)$$

Here $\lambda\!\!\!^{-}$ is the wavelength of the incident particle, R_0 is the radius of the hydrogen atom, while $\alpha = zZe^2/\hbar v$.

The nuclear form factor was measured by Hofstadter[3] for a number of elements. For electrons with energies on the order of several hundred Mev, the quantity F^2 in (5) can be approximated by the following expression:[4]

$$F^2 = \{1 + k^2a^2(\theta_x^2+\theta_y^2)\}^{-4} \quad (8)$$

The values of a characterize the scattering nucleus. For example, $a = 2.36\times10^{-13}$ cm for lead. It must be noted that the choice of the analytic form (8) for the form factor is dictated exclusively by its suitability for further calculations. From the physical point of view, this analytical form is not acceptable since it leads to an exponential distribution of the nuclear density. We should consider expression (8) only as an interpolation of the experimental results in the region of electron momenta on the order of 100 Mev/c. It is natural to assume that expression (8) remains valid also for scattering of particles of a different kind in the same range of momentum variation.

Using (5), (7), and (8) we can integrate the exponential function (3) with respect to θ'_y. We obtain

$$f(\theta_x)\,d\theta_x = \frac{2}{\pi}\,d\theta_x \int_0^\infty \cos k_x\theta_x$$

$$\times \exp\left\{-\frac{2}{3}\chi_c^2 \int_0^\infty (1-\cos k_x\theta'_x)\,B(\theta'_x)\,d\theta'_x\right\}, \qquad (9)$$

where

$$B(\theta_x) = 3/2\,(\varphi_m^2+\theta_x^2)^{5/2} - 12/\varphi_{\rm nuc}^2(\theta_x^2+\varphi_m^2)^{1/2}$$

$$+ 15\varphi_{\rm nuc}^2/16\,(\theta_x^2+\varphi_{\rm nuc}^2)^{7/2} + 9\varphi_{\rm nuc}^2/4\,(\theta_x^2+\varphi_{\rm nuc}^2)^{5/2}$$

$$+ 9/2\,(\theta_x^2+\varphi_{\rm nuc}^2)^{3/2} + 12/\varphi_{\rm nuc}^2(\theta_x^2+\varphi_{\rm nuc}^2)^{1/2}. \qquad (10)$$

We denote here $\varphi_{\rm nuc} = 1/ka$. It is easy to see that the probability of a single scattering in the interval of the planar angle $\theta_x - \theta_x + \Delta\theta_x$ while traversing the thickness t is $1/3\,\chi_c^2 B(\theta_x)\,d\theta_x$.

The integral in the exponential of expression (9) is also found analytically; we obtain finally

$$f(\theta_x)\,d\theta_x = \frac{2}{\pi}\,d\theta_x \int_0^\infty d\xi \cos\theta_x\xi$$

$$\times \exp\left\{-\frac{\chi_c^2\xi^2}{2}\left(\frac{1}{2} - 0.577 - \ln\frac{\xi\varphi_m}{2}\right)\right.$$

$$-\frac{8\chi_c^2}{\varphi_{\rm nuc}^2}\left[-0.577 + \ln\frac{2}{\xi\varphi_{\rm nuc}} - K_0(\xi\varphi_{\rm nuc})\right]$$

$$-\frac{13}{3}\frac{\chi_c^2}{\varphi_{\rm nuc}^2}\left[1 - \xi\varphi_{\rm nuc}K_1(\xi\varphi_{\rm nuc})\right]$$

$$\left. + \frac{2}{3}\chi_c^2\xi^2 K_0(\xi\varphi_{\rm nuc}) + \frac{\xi^3\varphi_{\rm nuc}\chi_c^2}{24}K_1(\xi\varphi_{\rm nuc})\right\}. \qquad (11)$$

We have incidentally expanded $K_1(\xi\varphi_m)$ and $K_0(\xi\varphi_m)$ in a series, since the important region $\xi\varphi_m \ll 1$. When $\varphi_{\rm nuc} \gg \chi_c$ we arrive at the expression obtained by Moliere[1] for the distribution function on a screened point Coulomb center.

Expression (11) can be reduced to a form amenable to numerical computations. We introduce the variables.

$$\theta_x/\varphi = \theta'_x, \quad \xi\varphi_{\rm nuc} = y, \qquad (12)$$

i.e., we measure the scattering angles in units of the nuclear angle.

$$f(\theta_x)\,d\theta_x = \frac{2d\theta'_x}{\pi}\int_0^\infty \cos\theta'_x y$$

$$\times \exp\left\{-\frac{\chi_c^2}{\varphi_{\rm nuc}^2}\left[\frac{y^2}{2}\left(\frac{1}{2} - 0.577 - \ln\frac{y\varphi_m}{2\varphi_{\rm nuc}}\right)\right.\right.$$

$$+ 8\left(-0.577 + \ln\frac{2}{y} - K_0(y)\right)$$

$$\left.\left. + \frac{13}{3}(1 - yK_1(y)) - \frac{2}{3}\,y^2K_0(y) - \frac{y^3}{24}K_1(y)\right]\right\}dy. \qquad (13)$$

The results of the integration will depend on the two parameters

$$\varphi_{\rm nuc}/\chi_c = (vh/aZe^2)(4\pi Nt/A)^{-1/2}, \qquad (14)$$

$$\varphi_m/\varphi_{\rm nuc} = (1.14m_ecZ^{1/3}a/137h)\,[1.13 + 3.76\,(Ze^2/hv)^2]^{1/2}. \qquad (15)$$

It is seen from these expressions that both parameters depend only on the velocity of the particle v. This makes it possible to use identical curves for different particles of the same velocity for specified values of t, A, and Z.

Formula (13) was used to calculate the curves for scattering in lead plates 4.5 and 8.5 g/cm^2 thick for velocities of 0.61 c (muon momentum 80 Mev/c), 0.73 c (p = 110 Mev/c), 0.78 c (p = 130 Mev/c), and 0.85 c (p = 170 Mev/c), used in references 5 and 6 to process the experimental results on the scattering of pions and muons, and also protons. It must be noted that it is customary to express the angles in units of the characteristic multiple-scattering angle,[1] $\varphi = \theta_x/\chi_c\sqrt{B}$. We, however, have obtained curves [Eq. (13)] in which the angles are expressed in units of the "nuclear" angle, $\theta'_x = \theta_x/\varphi_{\rm nuc}$. If we rewrite (13) in the following form

$$f(\theta_x)\,d\theta_x = f_1(\theta'_x)\,d\theta'_x = f_2(\varphi)\,d\varphi,$$

it becomes clear that it is easy to change to the variables φ. It is important to know that the use of the scale $(f_1 \cdot \chi_c\sqrt{B}/\varphi_{\rm nuc})$, which must be performed, depends only on the velocity of the particle (but not on its mass), since D is a function of the velocity only. This is why even the recalculated curves can be applied to different particles of the same velocity. The graphs thus obtained were numerically integrated to find the mean values of the scattering angles and their mean squares. For the sake of illustration we list in the table several numbers (lead plate 8.5 g/cm^2 thick). Some boxes contain two values. The upper values are computed for a distribution curve that is cut off at $\varphi = 2.4$, and the lower ones for $\varphi = 2$. The values of $\overline{\varphi}$ and $\overline{\varphi^2}$ for a point nucleus have been calculated from the Moliere distribution curve while $\overline{\varphi}$ was calculated from the following Moliere formula

$$\overline{\varphi}=\frac{2}{\pi}\overline{\vartheta}=\frac{1}{\pi}\left\{1+\frac{0.982}{B}-\frac{0.117}{B^2}\right\}$$

(here ϑ is the mean value of the absolute magnitude of the three-dimensional scattering angle). In the calculation of $\overline{\varphi^2}$ for a point nucleus, the same procedure was used as in our case, but the cut-off was at $\varphi = 4$. The fourth line yields $\chi_c\sqrt{B}$ for muons. This quantity must be used to multiply the numbers in the table to obtain the results in degrees.

v/c	0.61	0.78	0.85	Point nucleus
$\overline{\varphi}$	0.53 0.52	0.54 0.537	0.56 0.54	0.60
$(\overline{\varphi^2})^{1/2}$	0.66 0.65	0.69 0.67	0.71 0.66	0.77
$\chi_c\sqrt{B}$	34°.6	20°.2	15°.5	
A	532	550	561	611
B	672	694	711	776

The theory developed above can be used to determine the mass from the scattering angle and range, measured in the cloud chamber. We introduce, after Olbert,[7] the quantity

$$\eta=\theta R^{\alpha}, \tag{16}$$

where R is the range and θ is the scattering angle in radians, while α is determined from the following empirical relation between the range and the quantity $\Pi = pv$:

$$R/mc^2=A_z(\Pi/mc^2)^{1/\alpha}; \tag{17}$$

m is the mass of the particle, and A_z is a constant which equals in our case 0.32 g/cm² Mev. This relation is valid for the interval $0.05 \le \Pi/mc^2 \le 2$. It is easy to see that the distribution function for the quantity η differs from the distribution function for the scattering angles φ only in scale. The values of $\overline{\eta}$ and $\overline{\eta^2}$ are connected with those of $\overline{\varphi}$ and $\overline{\varphi^2}$ as follows

$$\overline{\eta}=\overline{\varphi}\left(\chi_c\sqrt{B}\,\frac{pv}{m_ec^2}\right)(A_zm_ec^2)^{\alpha}\left(\frac{m_e}{m}\right)^{1-\alpha}=A\left(\frac{m_e}{m}\right)^{1-\alpha}, \tag{18}$$

$$(\overline{\eta^2})^{1/2}=(\overline{\varphi^2})^{1/2}\left(\chi_c\sqrt{B}\,\frac{pv}{m_ec^2}\right)(A_zm_ec^2)^{\alpha}\left(\frac{m_e}{m}\right)^{1-\alpha}=B\left(\frac{m_e}{m}\right)^{1-\alpha}; \tag{19}$$

Here m_e is the electron mass. These formulas can be used to determine the mass of the unknown particle m, since the dependence on pv drops out.

The table lists the values of the coefficients [factor in front of $(m_e/m)^{1-\alpha}$], calculated for our case. The coefficients are given for a cut-off at $\varphi = 2.4$. The values of A and B for a different cut-off parameter can be found if the corresponding values of $\overline{\varphi}$ and $\overline{\varphi^2}$ are known for a suitable cut-off (the values given in the table correspond to a cut-off at $\varphi_{max} = 2.4$ and $\varphi_{max} = 2$). It is seen from the above that the results do not depend greatly on the cut-off parameter. Details on the determination of the mass can be found in reference 7. As to the theory of multiple scattering with allowance for the finite dimensions of the nucleus, developed in the same reference, it is a poor approximation to the actual state of affairs.

In conclusion we make several remarks regarding the limits of applicability of the formulas obtained.

We consider first the scattering of particles with nuclear interaction in addition to Coulomb interaction (pions, protons, etc.). In this case one must replace $\sigma(\theta'_x, \theta'_y)$ and $\sigma(0, 0)$ of Eq. (3) by a particle-scattering cross section that accounts for the nuclear interaction, too. It should be noted that the nuclear scattering has a diffraction character and the cross section for nuclear scattering may amount to a considerable fraction of the total differential cross section (we note that at small angles the cross sections for diffraction scattering may be considerably greater than the geometric cross section).

The plotting of the multiple-scattering curve for the exact differential scattering cross section is made difficult by the lack of sufficient experimental and theoretical data on the single differential cross section. However, certain conclusions with respect to the multiple scattering of nuclear-active charged particles can be made.[5]

Many estimates have been on the scattering of muons. Consideration was given to inelastic interactions that lead to the excitation of the nucleus and that imitate elastic scattering; the possibility of the muon having an anomalous magnetic moment was investigated; estimates were also made of the approximations made in the kinetic equation (for example, the small-angle approximation, of the momentum losses in the plate, and of other corrections. We shall not cite these estimates, in view of the many recent special papers devoted to these problems (for example, references 8 and 9).

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