

*PHENOMENOLOGICAL THEORY OF KINETIC PROCESSES IN FERROMAGNETIC  
DIELECTRICS. II. INTERACTION OF SPIN WAVES WITH PHONONS*

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The interaction between spin waves and phonons is considered. The time for establishment of equilibrium between phonons and spin waves is calculated.

In a previous work,<sup>\*1</sup> a phenomenological method was used to obtain the magnetic part of the energy spectrum of a ferromagnetic dielectric, and the interaction of spin waves with one another was considered. Besides the spin waves there are in a ferromagnetic dielectric, as in any solid body, energy branches connected with vibrations of the crystal lattice (phonons). Three of these, the acoustic, play a basic part both in thermal properties and in kinetic processes. The present work deals with the study of the interaction of spin waves with phonons.

Interaction of elementary excitations (quasi-particles) with each other leads to the establishment of statistical equilibrium in the body. We calculate here the time for establishment of such equilibrium. For this purpose the process of establishment of equilibrium is regarded as a process of equalization of the temperatures of the phonons and of the spin waves. Such an approach is permissible if the time for establishment of equilibrium within each of the subsystems (phonons and spin waves) is considerably shorter than the time of relaxation between the subsystems; as will be shown below, this condition is satisfied over a wide temperature range.

Similar questions have been considered by Akhiezer<sup>2</sup> from a microscopic point of view. Besides the processes studied by him, processes are here considered in which two phonons take part.

### 1. THE INTERACTION HAMILTONIAN

In a phenomenological approach to the study of interaction between spin waves and phonons, it is appropriate to describe the spin waves by the vector density of magnetic moment,  $\mathbf{M} = \mathbf{M}(\mathbf{r}, t)$ , and

<sup>\*</sup>The notation of the present work coincides with the notation of reference 1.

the phonons by the strain tensor,  $u_{ijk} = \frac{1}{2}(\partial u_i / \partial x_k + \partial u_k / \partial x_i)$  [ $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$  = displacement vector]. The interaction Hamiltonian will then be the sum of all those terms, in the expansion of the energy of the ferromagnetic in powers of  $M_i$ ,  $\partial M_i / \partial x_k$ , and  $u_{ijk}$ , that have the form\*

$$\gamma_{iklm} \int M_i M_k u_{lm} dv, \quad \lambda_{iklmrs} \int \frac{\partial M_i}{\partial x_k} \frac{\partial M_l}{\partial x_m} u_{rs} dv.$$

These terms include all the invariant combinations that contain the tensor  $u_{ijk}$  linearly, and the vector  $M_i$  and its spatial derivatives quadratically. The terms quadratic in  $u_{ijk}$  are responsible for the two-phonon processes and will be considered below. Terms linear in  $M_i$  cannot enter, since  $M_i$  is a pseudovector.

The first of the terms written corresponds to magnetostrictive energy, which in the isotropic case (we shall henceforth restrict ourselves to it) can be written in the following form:

$$\gamma_0 \int M^2 \operatorname{div} \mathbf{u} dv + \gamma \int M_i M_k u_{ik} dv.$$

The first of the integrals vanishes, since  $M^2 \equiv M_0^2$ . The second contains an anisotropic combination of the components of the vector  $\mathbf{M}$  and consequently is of relativistic origin. The value of the constant  $\gamma$  may be taken from magnetostriction measurements at low temperatures.

The term quadratic in the derivatives of  $\mathbf{M}$  drops out when the magnetization is uniform and therefore does not enter into the ordinary magnetostrictive effects. The largest among the constants  $\lambda$  are those that are connected with (isotropic) exchange interaction.

On the basis of these remarks, we write the interaction Hamiltonian in the following form:

$$\mathcal{H}_{\text{int}} = \mathcal{H}_{\text{ms}} = \mathcal{H}_{\text{ex}}, \quad (1)$$

<sup>\*</sup>Summation over repeated indices is understood.

where

$$\mathcal{H}_{\text{ms}} = \gamma \int M_l M_{kl} u_{ik} dv; \quad \mathcal{H}_{\text{ex}} = \lambda_1 \int \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_k} u_{ik} dv + \lambda_2 \int \frac{\partial M_l}{\partial x_i} \frac{\partial M_l}{\partial x_i} u_{kk} dv. \quad (2)$$

The energy  $\mathcal{H}_{\text{ex}}$  may be considered the result of expansion of the exchange interaction energy  $\int \alpha_{ijk} (\partial M_l / \partial x_i) (\partial M_l / \partial x_k) dv$  (cf. reference 1) in powers of  $u_{ijk}$ . Therefore the constants  $\lambda_{1,2}$  are conveniently written  $(\beta_{1,2}/2)(\Theta_c / \mu M_0) a^2$ , where  $\beta_{1,2}$  are dimensionless constants of order unity.

In the ground state the density of magnetic moment is uniform and constant and equal to  $M_0$ . We choose the  $z$  axis along the vector  $M_0$ . Then near the ground state, to terms quadratic in the deviation of the magnetic moment from  $M_0$ , we get

$$\begin{aligned} \mathcal{H}_{\text{ms}} &= 2\gamma M_0 \int (M_x u_{xz} + M_y u_{yz}) dv \\ &+ \frac{1}{4} \gamma \int \{ (M^+)^2 (u_{xx} - u_{yy} - 2iu_{xy}) \\ &+ (M^-)^2 (u_{xx} - u_{yy} + 2iu_{xy}) \\ &+ 2M^+ M^- (u_{xx} + u_{yy}) + 2M_0 M'_z u_{zz} \} dv, \end{aligned} \quad (3)$$

$$\begin{aligned} \mathcal{H}_{\text{ex}} &= \frac{\beta_1}{2} \frac{\Theta_c}{\mu M_0} a^2 \int \frac{\partial M^+}{\partial x_i} \frac{\partial M^-}{\partial x_k} u_{ik} dv \\ &+ \frac{\beta_2}{2} \frac{\Theta_c}{\mu M_0} a^2 \int \frac{\partial M^+}{\partial x_i} \frac{\partial M^-}{\partial x_i} u_{kk} dv, \end{aligned} \quad (4)$$

where

$$M^\pm = M_x \pm iM_y; \quad M'_z = M_z - M_0. \quad (5)$$

We remark that  $M'_z$  is quadratic in  $M_x$  and  $M_y$ .

The first integral in the expression (3) describes the connection between elastic and spin waves in a ferromagnetic<sup>3,4</sup> and is not connected with transitions between stationary states of the system of spin waves and phonons. Therefore it is hereafter omitted.

For the study of kinetic processes, the magnetic moment density  $M$  and the strain tensor  $u_{ijk}$  are to be considered operators. In terms of the second-quantization operators  $a_\lambda$  and  $a_\lambda^*$ , the components of the moment are expressed as (cf. reference 1):\*

$$\begin{aligned} M^+ &= (2\mu M_0)^{1/2} \hat{a}^*; \quad M^- = (2\mu M_0)^{1/2} \hat{a}, \quad M'_z = -\mu \hat{a}^* \hat{a}, \\ \hat{a} &= V^{-1/2} \sum_\lambda \hat{a}_\lambda(t) e^{ik_\lambda r}; \quad \hat{a}_\lambda(t) = \hat{a}_\lambda e^{-ie_\lambda t/\hbar}, \end{aligned} \quad (6)$$

$k_\lambda$  and  $e_\lambda$  are the wave vector and energy of the spin wave.

In the temperature range of interest to us ( $T \gg 2\pi\mu M_0$ ), we may, in the spectrum, neglect the magnetic interaction and limit ourselves to the Bloch dispersion law

$$e_\lambda = \Theta_c (ak_\lambda)^2. \quad (7)$$

In the representation based on occupation numbers  $n_\lambda$  of the spin waves, the nonvanishing matrix elements of the operators  $a_\lambda$  and  $a_\lambda^*$  are respectively equal to

$$(n_\lambda | \hat{a}_\lambda | n_\lambda + 1) = \sqrt{n_\lambda + 1}; \quad (n_\lambda | \hat{a}_\lambda^* | n_\lambda - 1) = \sqrt{n_\lambda}. \quad (8)$$

The strain tensor component operators  $u_{ijk}$  are expressed in terms of the creation and annihilation operators  $\hat{b}_{\lambda s}$  and  $\hat{b}_{\lambda s}^*$  for phonons in the following form:

$$\begin{aligned} u_{ik} &= 1/2 (\partial u_i / \partial x_k + \partial u_k / \partial x_i); \\ \mathbf{u} &= V^{-1/2} \sum_{\lambda, s} \sqrt{\hbar / 2\rho\omega_{\lambda s}} \mathbf{p}_{\lambda s} [\hat{b}_{\lambda s}(t) e^{i\mathbf{k}_\lambda \cdot \mathbf{r}} + \hat{b}_{\lambda s}^*(t) e^{-i\mathbf{k}_\lambda \cdot \mathbf{r}}]; \end{aligned} \quad (9)$$

$$\begin{aligned} \hat{b}_{\lambda s}(t) &= e^{-i\omega_{\lambda s} t} \hat{b}_{\lambda s}; \\ (N_{\lambda s} | \hat{b}_{\lambda s} | N_{\lambda s} + 1) &= \sqrt{N_{\lambda s} + 1}, \\ (N_{\lambda s} | \hat{b}_{\lambda s}^* | N_{\lambda s} - 1) &= \sqrt{N_{\lambda s}}. \end{aligned} \quad (10)$$

Here  $\rho$  is the density of matter,  $\mathbf{p}_{\lambda s}$  is a unit vector of polarization of a sound wave, the index  $s$  identifies the polarization, and  $\mathbf{k}_\lambda$  and  $\omega_{\lambda s}$  are the wave vector and frequency of a phonon.

For longitudinal polarization,

$$\omega_{\lambda s} \equiv \omega_{\lambda l} = c_l f_\lambda; \quad (11)$$

for transverse polarization,

$$\omega_{\lambda s} \equiv \omega_{\lambda t} = c_t f_\lambda, \quad (11')$$

$c_l$  and  $c_t$  are the longitudinal and transverse sound velocities ( $c_l > c_t$ ).

On substituting the expansions (6) and (9) in formulas (3) and (4) and integrating over the volume, we get

$$\mathcal{H}_{\text{ms}} = \frac{i\gamma\mu M_0}{2} \sqrt{\frac{\hbar}{2\rho V}} \sum_{\lambda, \mu, \nu, s} \frac{1}{\sqrt{\omega_{\nu s}}} \left\{ p_{\nu s}^- f_\nu a_\lambda^* a_\mu^* b_{\nu s} \right. \quad (12)$$

$$\left. + (p_{\nu s}^+ f_\nu + p_{\nu s}^- f_\nu + p_{\nu s}^z f_\nu) a_\lambda^* a_\mu b_{\nu s} \right\} + \text{complex conjugate.}$$

\*In formulas (6) we have omitted terms of the third order in the operators  $a_\lambda$  and  $a_\lambda^*$ , since they play no part in the processes considered below.

$$\mathcal{H}_{\mathbf{ex}} = -\frac{ia^2\Theta_c}{V^{1/2}} \sqrt{\frac{\hbar}{2\rho}} \sum_{\lambda,\mu,\nu\mathbf{s}} \frac{1}{V\omega_{\nu\mathbf{s}}} \left\{ \beta_1 (p_{\nu\mathbf{s}}^i f_{\nu}^k + p_{\nu\mathbf{s}}^k f_{\nu}^i) k_{\lambda}^i k_{\mu}^k \right. \\ \left. + \beta_2 (k_{\lambda}^i k_{\mu}^i) (f_{\nu}^k p_{\nu\mathbf{s}}^k) \right\} a_{\lambda}^* a_{\mu} b_{\nu\mathbf{s}} + \text{complex conjugate.} \quad (13)$$

Here  $\mathbf{p}^{\pm} = \mathbf{p}^{\mathbf{x}} \pm i\mathbf{p}^{\mathbf{y}}$ ,  $\mathbf{f}^{\pm} = \mathbf{f}^{\mathbf{x}} \pm i\mathbf{f}^{\mathbf{y}}$ . The operator  $a_{\lambda}^* a_{\mu} b_{\nu\mathbf{s}}$  describes the process of creation of a phonon by a spin wave. To this process corresponds the following law of conservation of momentum:

$$\mathbf{k}_{\lambda} = \mathbf{k}_{\mu} + \mathbf{f}_{\nu}. \quad (14)$$

The operator  $a_{\lambda}^* a_{\mu}^* b_{\nu\mathbf{s}}$  describes the annihilation of two spin waves with formation of a phonon. To it corresponds the conservation law

$$\mathbf{k}_{\lambda} + \mathbf{k}_{\mu} = \mathbf{f}_{\nu}. \quad (15)$$

Both conservation laws (14) and (15) are obtained naturally upon integration over the volume.

## 2. GENERATION AND ABSORPTION OF PHONONS BY SPIN WAVES

Since the dispersion law (7) for spin waves coincides with the dispersion law for ordinary free particles, the generation of phonons by spin waves is analogous to the Cerenkov radiation of sound waves by particles in motion at supersonic speed. From this it follows that the condition for radiation of a phonon has the form: for longitudinal phonons  $v_{\mathbf{s}} > c_l$ , for transverse phonons  $v_{\mathbf{s}} > c_t$ , where  $v_{\mathbf{s}} = 2\Theta_c a^2 k_{\lambda} / \hbar$  is the speed of the spin wave. By expressing  $k_{\lambda}$  in terms of  $\epsilon_{\lambda}$ , we may write the radiation condition

$$\epsilon_{\lambda} > \Theta_c^2 / 4\Theta_c, \quad (16)$$

where  $\Theta_0$  is the Debye temperature;  $\Theta_0 = \hbar c_l / a$  for longitudinal phonons, and  $\Theta_0 = \hbar c_t / a$  for transverse phonons. From the condition (16) it is clear that the generation of a phonon by a spin wave is an activation process.

As follows from formulas (12) and (13), the Hamiltonian that describes this process has the form

$$\mathcal{H}_e = \frac{i}{2} \sqrt{\frac{\hbar}{2\rho V}} \sum_{\lambda,\mu,\nu\mathbf{s}} \frac{1}{V\omega_{\nu\mathbf{s}}} \{ \gamma(\mu) M_0 (p_{\nu\mathbf{s}}^+ f_{\nu}^- + p_{\nu\mathbf{s}}^- f_{\nu}^+ - p_{\nu\mathbf{s}}^z f_{\nu}^z) \\ + \Theta_c a^2 [\beta_1 (p_{\nu\mathbf{s}}^i f_{\nu}^k + p_{\nu\mathbf{s}}^k f_{\nu}^i) k_{\lambda}^i k_{\mu}^k + 2\beta_2 k_{\lambda}^i k_{\mu}^i f_{\nu}^k p_{\nu\mathbf{s}}^k] \} a_{\lambda}^* a_{\mu} b_{\nu\mathbf{s}} + \text{c. c.}$$

On comparing the terms inside the curly brackets we find that for  $T \gg \mu M_0$  the second term is considerably larger than the first, since  $\Theta_c a^2 k_{\lambda}^i k_{\mu}^k \sim \epsilon \sim T$ . Therefore

$$\mathcal{H}_e = -\frac{i}{2} a^2 \Theta_c \sqrt{\frac{\hbar}{2\rho V}} \sum_{\lambda,\mu,\nu\mathbf{s}} \frac{1}{V\omega_{\nu\mathbf{s}}} [\beta_1 (p_{\nu\mathbf{s}}^i f_{\nu}^k + p_{\nu\mathbf{s}}^k f_{\nu}^i) k_{\lambda}^i k_{\mu}^k \\ + 2\beta_2 k_{\lambda}^i k_{\mu}^i f_{\nu}^k p_{\nu\mathbf{s}}^k] a_{\lambda}^* a_{\mu} b_{\nu\mathbf{s}} + \text{c. c.} \quad (17)$$

The nonvanishing matrix elements of the Hamiltonian  $\mathcal{H}_e$  correspond to a transition  $n_{\lambda}, n_{\mu}, N_{\nu\mathbf{s}} \rightarrow n_{\lambda} - 1, n_{\mu} + 1, N_{\nu\mathbf{s}} + 1$  and to the reverse transition. According to (8) and (10) we have for the indicated transitions

$$(n_{\lambda}, n_{\mu}, N_{\nu\mathbf{s}} | \mathcal{H}_e | n_{\lambda} - 1, n_{\mu} + 1, N_{\nu\mathbf{s}} + 1) \\ = \frac{i}{2} \sqrt{\frac{\hbar}{2\rho V\omega_{\nu\mathbf{s}}}} \Theta_c a^2 \{ \beta_1 (p_{\nu\mathbf{s}}^i f_{\nu}^k + p_{\nu\mathbf{s}}^k f_{\nu}^i) k_{\lambda}^i k_{\mu}^k + 2\beta_2 k_{\lambda}^i k_{\mu}^i f_{\nu}^k p_{\nu\mathbf{s}}^k \} \\ \times \sqrt{n_{\lambda} (n_{\mu} + 1) (N_{\nu\mathbf{s}} + 1)} \exp \left\{ \frac{i}{\hbar} (\epsilon_{\lambda} - \epsilon_{\mu} - \hbar\omega_{\nu\mathbf{s}}) t \right\}.$$

Hence the transition probability is

$$(n_{\lambda}, n_{\mu}, N_{\nu\mathbf{s}} | W | n_{\lambda} - 1, n_{\mu} + 1, N_{\nu\mathbf{s}} + 1) \\ = \frac{\pi \Theta_c^2 a^4}{4\rho V\omega_{\nu\mathbf{s}}} \{ \beta_1 (p_{\nu\mathbf{s}}^i f_{\nu}^k + p_{\nu\mathbf{s}}^k f_{\nu}^i) k_{\lambda}^i k_{\mu}^k + 2\beta_2 k_{\lambda}^i k_{\mu}^i f_{\nu}^k p_{\nu\mathbf{s}}^k \}^2 \\ \times n_{\lambda} (n_{\mu} + 1) (N_{\nu\mathbf{s}} + 1) \delta(\epsilon_{\lambda} - \epsilon_{\mu} - \hbar\omega_{\nu\mathbf{s}}). \quad (18)$$

The  $\delta$ -function guarantees the energy conservation law

$$\epsilon_{\lambda} = \epsilon_{\mu} + \hbar\omega_{\nu\mathbf{s}}. \quad (19)$$

We recall that the momentum conservation law for the process under study has the form (14).

We now consider the transfer of energy from spin waves to phonons. The energy received by the phonon system in unit time is

$$\dot{U}_e = \sum_{\nu\mathbf{s}} \hbar\omega_{\nu\mathbf{s}} \dot{N}_{\nu\mathbf{s}}, \quad (20)$$

where  $\dot{N}_{\nu\mathbf{s}}$  is the change in the number of phonons in unit time resulting from radiation and absorption of sound waves by spin waves. According to (18),

$$\dot{N}_{\nu\mathbf{s}} = \frac{\pi \Theta_c^2 a^4}{4\rho V\omega_{\nu\mathbf{s}}} \sum_{\lambda,\mu} A_s^2 (n_{\lambda} (n_{\mu} + 1) (N_{\nu\mathbf{s}} + 1) \\ - (n_{\lambda} + 1) n_{\mu} N_{\nu\mathbf{s}}) \delta(\epsilon_{\lambda} - \epsilon_{\mu} - \hbar\omega_{\nu\mathbf{s}}), \quad (21)$$

$$A_s = \beta_1 [(\mathbf{p}_{\nu\mathbf{s}} \cdot \mathbf{k}_{\lambda}) (\mathbf{f}_{\nu} \cdot \mathbf{k}_{\mu}) + (\mathbf{p}_{\nu\mathbf{s}} \cdot \mathbf{k}_{\mu}) (\mathbf{f}_{\nu} \cdot \mathbf{k}_{\lambda})] + 2\beta_2 (\mathbf{k}_{\lambda} \cdot \mathbf{k}_{\mu}) (\mathbf{f}_{\nu} \cdot \mathbf{p}_{\nu\mathbf{s}}).$$

Here and hereafter, the sum extends over those values of the wave vectors that satisfy the momentum conservation law.

On inserting the value of  $\dot{N}_{\nu\mathbf{s}}$  into (20), we get

$$\dot{U}_e = \frac{\pi \Theta_c^2 a^4 \hbar}{4\rho V} \sum_{\lambda,\mu,\nu\mathbf{s}} A_s^2 \{ n_{\lambda} (n_{\mu} + 1) (N_{\nu\mathbf{s}} + 1) \\ - (n_{\lambda} + 1) n_{\mu} N_{\nu\mathbf{s}} \} \delta(\epsilon_{\lambda} - \epsilon_{\mu} - \hbar\omega_{\nu\mathbf{s}}). \quad (22)$$

Upon replacement of  $n_\lambda$  and  $N_{\nu S}$  by the Bose equilibrium distribution,  $\dot{U}_e$  naturally vanishes. On deviation of the system from the equilibrium state, the return to the equilibrium state will proceed thus: equilibrium will be established first within each subsystem (spin waves and phonons), and then — considerably more slowly — between the subsystems. The second stage can be described as a process of equalization of the temperatures of the spin waves and of the phonons. Clearly such a view is correct if the time for temperature equalization is much longer than the relaxation time of spin waves.<sup>1</sup> As will be clear below, this condition is satisfied over a wide temperature range.

Thus we shall consider that the spin waves and the phonons are described by Bose equilibrium distribution functions with different temperatures, respectively equal to  $T_S = T$  and to  $T_{ph} = T - \Delta T$ . Hereafter we shall always consider that  $\Delta T \ll T$ .

On the basis of these remarks, we get from formula (22)

$$\dot{U}_e = \frac{\pi \theta_c^2 a^4 \hbar^2}{4 \rho V T^2} \Delta T \sum_{\lambda, \nu, S} A_s^2 \omega_{\nu S} \frac{e^{\epsilon_\lambda / T} \delta(\epsilon_\lambda - \epsilon_\nu - \hbar \omega_{\nu S})}{(e^{\epsilon_\lambda / T} - 1)(e^{\epsilon_\nu / T} - 1)(e^{\hbar \omega_{\nu S} / T} - 1)} \quad (23)$$

or

$$\dot{U}_e = \dot{U}_e^l + \dot{U}_e^t, \quad (24)$$

where  $\dot{U}_e^l$  is the amount of energy transferred to a longitudinal (transverse) phonon in one second.  $\dot{U}_e^l$  is determined by formula (23) with  $A_s^2$  replaced by  $A_{l,t}^2$ . From expression (21), with account taken of the momentum conservation law (14), we have:

$$A_l = 2 \{ \beta_1 f_\nu k_\lambda \cos \theta_{\lambda\nu} (k_\lambda \cos \theta_{\lambda\nu} - f_\nu) + \beta_2 f_\nu k_\lambda (k_\lambda - f_\nu \cos \theta_{\lambda\nu}) \}, \quad (25)$$

$$A_t = \beta_1 f_\nu k_\lambda \sin \theta_{\lambda\nu} (2k_\lambda \cos \theta_{\lambda\nu} - f_\nu) \quad (26)$$

( $\theta_{\lambda\nu}$  is the angle between the vectors  $\mathbf{k}_\lambda$  and  $\mathbf{f}_\nu$ ).

On going over from summation to integration, we get after integration over angles

$$\begin{aligned} \dot{U}_e^l &= \frac{V \hbar}{64 (2\pi)^3 \rho a^3} \left( \frac{T}{\theta_l} \right)^8 \Delta T J_{el}(\alpha_l), \\ \dot{U}_e^t &= \frac{V \hbar}{16 (2\pi)^3 \rho a^3} \frac{T^5}{\theta_c^3 \theta_t^2} \Delta T J_{et}(\alpha_t), \end{aligned} \quad (27)$$

where

$$J_{el}(\alpha) = \int_0^\infty \frac{y^4 dy}{e^y - 1} \times \int_{(1/4)\alpha}^\infty \frac{e^x \{ \beta_1 (\alpha^2 - y^2) + 2\beta_2 [2\alpha x - y(y + \alpha)] \}^2 dx}{(e^x - 1)(e^{x-y} - 1)}; \quad (28)$$

$$J_{et}(\alpha) = \beta_1^2 \int_0^\infty \frac{y^4 dy}{e^y - 1} \int_{(1/4)\alpha}^\infty \frac{e^x [x - (y + \alpha)^2 / 4\alpha] dx}{(e^x - 1)(e^{x-y} - 1)}, \quad (29)$$

$$\alpha_l = \theta_l^2 / \theta_c T, \quad \alpha_t = \theta_t^2 / \theta_c T.$$

We consider the limiting cases of high and low temperatures.

High temperatures ( $\alpha_l, \alpha_t \ll 1$ ). Here

$$J_{el}(\alpha) \approx \frac{27\pi^4 \alpha^4}{15} \frac{\beta_1^2 + 2(\beta_1 + \beta_2)^2}{3}; \quad J_{et}(\alpha) \approx 8\zeta(3) \alpha^2 \beta_1^2. \quad (30)$$

On substituting in (22) the asymptotic value of the integrals, we find

$$\dot{U}_e^l \approx \frac{\pi V \hbar}{60 \rho a^3} \frac{\beta_1^2 + 2(\beta_1 + \beta_2)^2}{3} \left( \frac{T}{\theta_c} \right)^4 \Delta T \quad (T \gg \theta_l^2 / \theta_c), \quad (31)$$

$$\dot{U}_e^t \approx \frac{\zeta(3) \beta_1^2 V \hbar}{16 \pi^3 \rho a^3} \frac{\theta_l^2}{\theta_c^2} \left( \frac{T}{\theta_c} \right)^3 \Delta T \quad (T \gg \theta_l^2 / \theta_c). \quad (32)$$

On comparison of expressions (31) and (32), we find that for  $T \gg \theta_l / \theta_c$ , always  $\dot{U}_e^l \gg \dot{U}_e^t$  ( $\beta_1 \sim \beta_2$ ). Therefore

$$\dot{U}_e \approx \frac{\pi V \hbar}{60 \rho a^3} \frac{\beta_1^2 + 2(\beta_1 + \beta_2)^2}{3} \left( \frac{T}{\theta_c} \right)^4 \Delta T, \quad (T \gg \theta_l^2 / \theta_c). \quad (33)$$

Low temperatures ( $\alpha_l, \alpha_t \gg 1$ ). In this case

$$J_{el}(\alpha) \approx 24 \cdot 31 \zeta(5) \alpha^4 e^{-\alpha/4} (\beta_1 + \beta_2)^2,$$

$$J_{et}(\alpha) \approx 24 \cdot 31 \zeta(5) \beta_1^2 e^{-\alpha/4}.$$

Hence

$$\begin{aligned} \dot{U}_e^l &\approx \frac{3V \hbar}{2\pi^3 \rho a^3} (\beta_1 + \beta_2)^2 \left( \frac{T}{\theta_c} \right)^4 \exp \left\{ -\frac{\theta_l^2}{4\theta_c T} \right\} \Delta T, \\ &\quad (T \ll \theta_l^2 / \theta_c), \end{aligned} \quad (34)$$

$$\dot{U}_e^t \approx \frac{6\beta_1^2 V \hbar}{\pi^3 \rho a^3} \frac{T^5}{\theta_c^3 \theta_t^2} \exp \left\{ -\frac{\theta_t^2}{4\theta_c T} \right\} \Delta T, \quad (T \ll \theta_t^2 / \theta_c). \quad (35)$$

Since  $\theta_l > \theta_t$  (cf. reference 5), the fundamental role in energy transfer at low temperatures is played by the process of generation of transverse phonons, i.e.,

$$\dot{U}_e \approx \frac{6\beta_1^2 V \hbar}{\pi^3 \rho a^3} \frac{T^5}{\theta_c^3 \theta_t^2} \exp \left\{ -\frac{\theta_t^2}{4\theta_c T} \right\} \Delta T, \quad (T \ll \theta_t^2 / \theta_c). \quad (36)$$

As was to be expected, the coefficient of heat emission for the activation process of radiation and absorption of phonons decreases exponentially with temperature.

### 3. ANNIHILATION OF TWO SPIN WAVES WITH FORMATION OF A PHONON, AND DISSOCIATION OF A PHONON INTO TWO SPIN WAVES

The processes indicated in the section title occur by virtue of the magnetostrictive part (12) of the Hamiltonian. Because of the relativistic origin of the magnetostrictive energy, the probability of these processes is small. However, as will be evident below, the corresponding coefficient

cient of heat transfer does not contain an exponential dependence on temperature (at low temperatures). Therefore it becomes necessary to make a comparison of the coefficient of heat emission resulting from generation and absorption of phonons with the corresponding coefficient resulting from the processes to be investigated.

By use of the Hamiltonian (12) and of a formula analogous to formula (20), we find [with the aid of the expressions (8) and (10) for the matrix elements of the operators  $a_\lambda$  and  $b_\lambda$ ] that the energy given up by the spin waves to the lattice in unit time by virtue of the processes under investigation is equal to

$$\dot{U}_a = \frac{\pi \hbar \gamma^2 (\mu M_0)^2}{\rho V} \sum_{\lambda, \mu, \nu, s} B_s^2 \{ (N_{\nu s} + 1) n_\lambda n_\mu - N_{\nu s} (n_\lambda + 1) (n_\mu + 1) \} \delta(\varepsilon_\lambda + \varepsilon_\mu - \hbar \omega_{\nu s}), \quad (37)$$

where

$$B_s = |p_{\nu s}^\dagger f_\nu^\dagger|. \quad (38)$$

As before, we get from (37) and (38)

$$\dot{U}_a = \dot{U}_a^l + \dot{U}_a^t,$$

where

$$\dot{U}_a^l = \frac{\gamma^2 V \hbar}{80 \pi^3 \rho a^3} \left( \frac{\mu M_0}{\Theta_c} \right)^2 \left( \frac{T}{\Theta_l} \right)^4 \Delta T J_a(\alpha_l), \quad (39)$$

$$\dot{U}_a^t = \frac{\gamma^2 V \hbar}{40 \pi^3 \rho a^3} \left( \frac{\mu M_0}{\Theta_c} \right)^2 \left( \frac{T}{\Theta_t} \right)^4 \Delta T J_a(\alpha_t),$$

here

$$J_a = \int_0^{2\alpha} \frac{y^4 e^y dy}{e^y - 1} \int_{x_1}^{x_2} \frac{dx}{(e^x - 1)(e^{y-x} - 1)}, \quad (40)$$

$$x_{1,2} = \frac{y}{2} \left( 1 \mp \sqrt{\frac{2y}{\alpha} - \frac{y^2}{\alpha^2}} \right),$$

and the quantities  $\alpha_l$  and  $\alpha_t$  are determined by equations (29).

On finding the asymptotic expressions for the integral (40) for the cases of high and of low temperatures, we find:

$$\dot{U}_a \approx \frac{7}{90 \pi^2} \frac{\gamma^2 V \hbar}{\rho a^3} \left( \frac{\mu M_0}{\Theta_c} \right)^2 \frac{\Theta_c^2 + 3/2 \Theta_l^2}{\Theta_c^2} \frac{T}{\Theta_c} \Delta T \quad (T \gg \Theta_l^2 / \Theta_c). \quad (41)$$

$$\dot{U}_a \approx 0.25 \frac{\gamma^2 V \hbar}{\rho a^3} \left( \frac{\mu M_0}{\Theta_c} \right)^2 \Theta_c^5 \left( \frac{1}{\Theta_l^5} + \frac{3}{2 \Theta_t^5} \right) \left( \frac{T}{\Theta_c} \right)^{3/2} \Delta T, \quad (42)$$

$$(T \ll \Theta_l^2 / \Theta_c).$$

By comparison of formulas (33) and (41), it can easily be shown that for  $T \gg \Theta_l^2 / \Theta_c$  the fundamental role is played by the processes of generation and absorption of phonons (i.e.,  $\dot{U}_a \ll \dot{U}_e$ ).

For  $T \ll \Theta_l^2 / \Theta_c$  two cases are possible, as is clear from comparison of formulas (36) and (42).

For

$$\Theta_l^2 / 8 \Theta_c \ln(\Theta_l^2 / \mu M_0 \Theta_c) \ll T \ll \Theta_l^2 / \Theta_c,$$

as for high temperatures, spin-wave annihilation processes play no part in the transfer of energy to phonons. For

$$T \ll \Theta_l^2 / 8 \Theta_c \ln(\Theta_l^2 / \mu M_0 \Theta_c)$$

these processes should be the ones of greatest importance. However, since the whole treatment is valid only down to temperatures of order  $\mu M_0$ , the last statement may prove to be incorrect [in case  $\mu M_0 \gtrsim \Theta_l^2 / 8 \Theta_c \ln(\Theta_l^2 / \mu M_0 \Theta_c)$ , which usually occurs].

Practically, therefore, for all temperatures above  $\mu M_0$  the fundamental role, among single-phonon processes, is played by the processes of generation and absorption of phonons by spin waves.

#### 4. THE ROLE OF TWO-PHONON PROCESSES

Two-phonon processes are described by the quadratic terms in the expansion of the energy of the ferromagnetic in powers of the strain tensor. Here, as in the single-phonon processes, there can be terms of exchange origin and of relativistic origin. It can be shown that the relativistic terms are small in comparison with the analogous terms in the single-phonon processes by virtue of the smallness of the parameter  $M^2 / \rho c^2$ .

Therefore it is necessary to compare with the single-phonon processes only the exchange terms, which have the form

$$\mathcal{H}_2 = L_{lmrs}^{ik} \int \frac{\partial M_q}{\partial x_i} \frac{\partial M_q}{\partial x_k} u_{lm} u_{rs} dv, \quad (43)$$

where the coefficients  $L_{lmrs}^{ik}$  are in order of magnitude equal to  $\Theta_c a^2 / \mu M_0$ .

On going over to a second-quantization representation, we get from (38)

$$\mathcal{H}_2 = - \frac{\Theta_c a^2 \hbar}{4V\rho} \sum_{\lambda, \mu, \nu, s, s'} F_{\lambda, \mu, \nu}^{s, s'} \frac{k_\lambda k_\mu k_\nu}{\sqrt{\omega_{\mu s} \omega_{\nu s'}}} \times (a_{\lambda\lambda} a_{\mu s} b_{\nu s'} - a_{\lambda\lambda} a_{\mu s} b_{\nu s'}^* + \text{c.c.}). \quad (44)$$

The coefficients  $F$  are of order unity. They are complicated functions of the angles; their precise form is not important for the estimates carried out below.

As is clear from equation (44), there are two types of process: scattering of a phonon by a spin wave (or vice versa) and radiation (or absorption) of two phonons by a spin wave. The radiation of two phonons, like the radiation of a single phonon,

is an activation process. It is less probable (by comparison with the radiation of a single phonon), as being a higher-order process. On the other hand the scattering of a phonon by a spin wave is a non-activation process. Therefore at low temperatures ( $T \ll \Theta_0^2/\Theta_c$ , where  $\Theta_0$  is the Debye temperature) the energy transferred to phonons by spin waves by virtue of this process varies with temperature in a moderate fashion, and so, it might seem, can play a larger part at low temperatures than does the activation process of phonon radiation. A calculation analogous to the preceding one gives, apart from a numerical factor,

$$\dot{U}_2 \sim \frac{\beta^2 V \hbar}{\rho a^3} \frac{T^5}{\Theta_c^3 \Theta_0^2} \left(\frac{T}{\Theta_1}\right)^{1/2} \left(\frac{T}{\Theta_2}\right)^4 \Delta T \quad (T \ll \Theta_1), \quad (45)$$

where

$$\Theta_1 = \Theta_0^2/\Theta_c; \quad \Theta_2 = mc^2 \Theta_0^3 \quad (m = \rho a^3).$$

On comparison of equations (45) and (36), we find that two-phonon processes can be important at temperatures below  $\Theta_1/[52 \ln 2 + 16 \ln(\Theta_2/\Theta_1)] \sim 0.1^\circ\text{K}$ , where our treatment, as we have pointed out, does not apply.

Thus, finally, according to formulas (33) and (36),

$$\dot{U} \approx \frac{6\beta_1^2 V \hbar}{\pi^3 \rho a^3} \frac{T^5}{\Theta_c^2 \Theta_0^2} \exp\left\{-\frac{\Theta_0^2}{4\Theta_c T}\right\} \Delta T, \quad (T \ll \Theta_0^2/\Theta_c), \quad (46)$$

$$\dot{U} \approx \frac{\pi V \hbar}{60 \rho a^3} \left(\frac{T}{\Theta_c}\right)^4 \frac{\beta_1^2 + 2(\beta_1 + \beta_2)^2}{3} \Delta T, \quad (T \gg \Theta_0^2/\Theta_c).$$

## 5. THE TIME FOR ESTABLISHMENT OF EQUILIBRIUM

By use of the results obtained, we shall calculate the time for establishment of thermal equilibrium (the relaxation time) between spin waves and phonons.

The heat balance equation clearly has the form

$$C_s \partial T_s / \partial t = \dot{U}; \quad C_{\text{ph}} \partial T_{\text{ph}} / \partial t = -\dot{U}, \quad (47)$$

where  $C_s$  and  $C_{\text{ph}}$  are the specific heats of spin waves and of phonons:

$$C_s \approx \frac{15\zeta(5/2)}{32\pi^{5/2}} \frac{V}{a^3} \left(\frac{T}{\Theta_c}\right)^{5/2}; \quad C_{\text{ph}} \approx \frac{2\pi^2}{5} \frac{V}{a^3} \left(\frac{T}{\Theta_0}\right)^3, \quad (48)$$

( $3/\Theta_0^3 = 1/\Theta_1^3 + 2/\Theta_2^3$ ), and  $\dot{U}$  is determined by formula (46).

At low temperatures ( $T \ll \Theta_0^2/\Theta_c$ ), the phonon specific heat is much smaller than the spin ( $C_{\text{ph}} \ll C_s$ ). Therefore the temperature of the spin-wave gas may be considered constant.

Then for the temperature difference  $\Delta T$  we get

the equation

$$C_{\text{ph}} \frac{\partial \Delta T}{\partial t} = -\alpha \Delta T, \quad \alpha = \frac{6\beta_1^2 V \hbar}{\pi^3 \rho a^3} \frac{T^5}{\Theta_c^2 \Theta_0^2} \exp\left\{-\frac{\Theta_0^2}{4\Theta_c T}\right\}.$$

Hence

$$\frac{1}{\tau_{\text{sph}}} = \frac{15\beta_1^2}{\pi^5} \frac{\Theta_0^3}{\Theta_c m c^2 \hbar} \left(\frac{T}{\Theta_c}\right)^2 \exp\left\{-\frac{\Theta_0^2}{4\Theta_c T}\right\} \quad (T \ll \Theta_0^2/\Theta_c). \quad (49)$$

At high temperatures ( $T \gg \Theta_0^2/\Theta_c$ ), the temperature of the phonons must be considered constant. As before we find

$$\frac{1}{\tau_{\text{sph}}} = \frac{8\pi^{5/2}}{225\zeta(5/2)} \frac{\Theta_0^2}{m c^2 \hbar} \left(\frac{T}{\Theta_c}\right)^{1/2} \frac{\beta_1^2 + 2(\beta_1 + \beta_2)^2}{3} \quad (T \gg \Theta_0^2/\Theta_c). \quad (50)$$

As was indicated in the introduction, the expressions obtained here for the relaxation time must be compared with the relaxation time within the spin system. For the relaxation time  $\tau_{\text{SS}}$  within the spin system, there were obtained in reference 1 the formulas

$$\frac{1}{\tau_{\text{SS}}} \approx \begin{cases} (\Theta_c/\hbar)(T/\Theta_c)^4 & \text{for } T \gg \mu M_0 (\Theta_c/\mu M_0)^{3/2}, \\ g M_0 (\mu^2/a^3 \Theta_c)(T/\Theta_c)^{1/2} & \text{for } T \ll \mu M_0 (\Theta_c/\mu M_0)^{3/2}. \end{cases} \quad (51)$$

We have omitted in the second formula a logarithmic factor unimportant for our estimate. The characteristic temperature for spin-spin relaxation  $\mu M_0 (\Theta_c/\mu M_0)^{3/2}$ , agrees in order of magnitude with the temperature  $\Theta_0^2/\Theta_c$ . Therefore the expression (49) must be compared with the second formula (51), and the expression (48) with the first. The comparison shows that both for  $\mu M_0 \ll T \ll \Theta_0^2/\Theta_c$  and for  $\Theta_0^2/\Theta_c \ll T \ll \Theta_0, \Theta_c$ , the relaxation time  $\tau_{\text{SS}}$  is much smaller than the time  $\tau_{\text{sph}}$ . This justifies the introduction of two temperatures in the treatment of spin-phonon relaxation.

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<sup>2</sup> A. I. Akhiezer, J. Phys. (U.S.S.R.) **10**, 217 (1946).

<sup>3</sup> Akhiezer, Bar'yakhtar, and Peletminskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 228 (1958), Soviet Phys. JETP **8**, 157 (1959).

<sup>4</sup> Yu. P. Irkhin and E. A. Turov, *Физика металлов и металловедение* (Physics of Metals and Metal Research) **3**, 15 (1956).

<sup>5</sup> L. D. Landau and E. M. Lifshitz, *Механика сплошных сред* (*Mechanics of Continuous Media*) M.-L. Gostekhizdat, 1953.

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