THE SHAPIRO INTEGRAL TRANSFORMATION

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We give an integral transformation which is related to the decomposition, into irreducible representations of the proper Lorentz group, of the representation according to which the wave function of a particle with mass M and spin s transforms.

THE problem mentioned in the abstract was stated and solved in 1955 by I. S. Shapiro.[1] Unfortunately, he used a wave-function transformation law which is incorrect for $s > 0$ (the transformation law for the spin under a pure Lorentz transformation[2]), and thus the equations he obtained can be used only for spin zero.

In this paper we solve Shapiro's problem for arbitrary spin using a wave-function transformation law previously obtained;[2] the principles by which the integral transformation is obtained are the same as those used by Shapiro.

In the first two sections we state some known facts which we shall need later.[2,3] In the third and fourth sections we derive the integral transformation.

1. IRREDUCIBLE REPRESENTATIONS OF THE PROPER LORENTZ GROUP

Let

$$k = (k_1 k_2 k_3 k) \tag{1.0}$$

be the 4-momentum of a particle with mass 0, and let

$$k_0 = (00\,kk). \tag{1.0'}$$

Obviously $k = R(\mathbf{n})k_0$, where $R(\mathbf{n})$ is the rotation* which carries the third axis into the direction given by $\mathbf{n} = \mathbf{k}/k$; if θ and φ are the spherical angle coordinates of $\mathbf{n}$, we may write

$$R(\mathbf{n}) = R_3(\varphi + \pi/2)\,R_1(\theta), \tag{1.1}$$

where $R_3(\alpha)$ is a counterclockwise rotation about the 3-axis through an angle α. Let $k' = Sk$, where S is a transformation of the Lorentz group. We shall denote the relation between $\mathbf{n} = \mathbf{k}/k$ and $\mathbf{n}' = \mathbf{k}'/k'$ by

$$\mathbf{n}' = S\mathbf{n}; \tag{1.2}$$

then $\mathbf{n}' = S_1\mathbf{n}$ and $\mathbf{n}'' = S_2\mathbf{n}'$ implies that $\mathbf{n}'' = S_2S_1\mathbf{n}$. Arbitrary transformation t of the Lorentz group can be written in the form

$$t = rK, \tag{1.3}$$

where $K\mathbf{n}_0 = \mathbf{n}_0$ in the sense of (1.2), and r is a pure rotation such that $r\mathbf{n}_0 = t\mathbf{n}_0$ [again in the sense of (1.2)]. Here the choice of $\mathbf{n}_0$ is arbitrary; (1.3) obviously determines r up to an arbitrary rotation about the $\mathbf{n}_0$ direction. We shall choose $\mathbf{n}_0$ as the unit vector along the third axis; we define r by the condition $r = R_3(\varphi_1)\,R_1(\theta)$, which means that $\varphi_2 = 0$, and then the factorization implied by (1.3) is clearly unique.

Now let $x' = Sx$. It is well known that if $H(x) = x_0 + (\sigma\cdot\mathbf{x})$ and $b(S)$ is the matrix corresponding to S in the 2×2 representation of the Lorentz group, then $H(x') = b(x)H(x)b^*(S)$. If we now choose $x = k_0$ [see Eq. (1.0')] we see that in the 2×2 representation a K transformation of the type in (1.3) is represented by a matrix of the form

$$K = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix}. \tag{1.4}$$

We now define the transformation $K(S, \mathbf{n})$ by applying (1.3) to the matrix $t = S^{-1}R(\mathbf{n})$ (the definition of $R(\mathbf{n})$ is given in (1.1)):

$$S^{-1}R(\mathbf{n}) = R(S^{-1}\mathbf{n})\,K(S, \mathbf{n}) \tag{1.5}$$

[the vector $S^{-1}\mathbf{n}$ is defined in (1.2)]. Applying both sides of (1.5) to $\mathbf{n}_0$, we see that in the present case $r = R(S^{-1}\mathbf{n})$.

Further, making use of the uniqueness of the factorization given by (1.3), we find that $K(S, \mathbf{n})$ has the property that

$$K(S_2, \mathbf{n})\,K(S_1, S_2^{-1}\mathbf{n}) = K(S_2S_1, \mathbf{n}), \tag{1.6}$$

which we shall need later.

*In what follows we shall always denote by R a pure rotation (or the matrix in the appropriate function space corresponding to such a rotation); L will denote a pure Lorentz transformation.

Consider the association

$$f(\mathbf{n}) \xrightarrow{S} f'(\mathbf{n}) = \alpha(S, \mathbf{n}) f(S^{-1}\mathbf{n}). \quad (1.7)$$

Let us find the properties that $\alpha(S, \mathbf{n})$ must satisfy in order that this association form a representation of the Lorentz group. We have

$$f'(\mathbf{n}) \xrightarrow{S_1} f''(\mathbf{n}) = \alpha(S_1, \mathbf{n}) f'(S_1^{-1}\mathbf{n})$$
$$= \alpha(S_1, \mathbf{n}) \alpha(S, S_1^{-1}\mathbf{n}) f(S^{-1}S_1^{-1}\mathbf{n}).$$

It is thus necessary that

$$\alpha(S_1, \mathbf{n}) \alpha(S, S_1^{-1}\mathbf{n}) = \alpha(S_1 S, \mathbf{n}). \quad (1.8)$$

It is seen from this that we may choose as $\alpha(S, \mathbf{n})$ any matrix element $[K(S, \mathbf{n})]_{22}$ to any power [see (1.4) to (1.6)]. Thus

$$\alpha = |k_{22}|^a \exp\{i \arg k_{22} \cdot b\}.$$

Now (1.4) implies that if $k' = S^{-1}k$, then $|k_{22}|^2 = |\mathbf{k}|/|\mathbf{k}'|$. Denoting the association $\mathbf{k} \to \mathbf{n} = \mathbf{k}/|\mathbf{k}|$ by

$$|\mathbf{k}| = k(\mathbf{n}), \quad (1.9)$$

we may write $|k(S, \mathbf{n})_{22}|^2 = k(\mathbf{n})/k(S^{-1}\mathbf{n})$.

Further, let us write*

$$\arg k_{22} = {}^1/_2\, \varphi(S, \mathbf{n}), \quad (1.9a)$$

so that $\alpha(S, \mathbf{n})$ becomes

*Let us study the angle $\varphi(S, \mathbf{n})$ in more detail.

(a) $S = R$. In this case the definition of $\varphi(s, \mathbf{n})$ in (1.9a) is equivalent to

$$R^{-1}R(\mathbf{n}) = R(R^{-1}\mathbf{n}) R_3(\varphi(R, \mathbf{n})). \quad (1.9b)$$

(b) $S = L_p$ [the definition of L_p is given in (2.1)]. Let the rotation $R(L, \mathbf{n})$ be defined (clearly not uniquely) by

$$L^{-1}\mathbf{n} = R(L, \mathbf{n})\,\mathbf{n}. \quad (1.9c)$$

Then

$$L^{-1}R(\mathbf{n}) = R(L, \mathbf{n}) R^{-1}(L, \mathbf{n}) L^{-1}R(\mathbf{n})$$
$$= R(L, \mathbf{n}) R(\mathbf{n}) \{R^{-1}(\mathbf{n}) R^{-1}(L, \mathbf{n}) L^{-1}R(\mathbf{n})\}.$$

The transformation in curly brackets does not alter the direction of the 3-axis. If we take advantage of its arbitrariness and choose the rotation $R(L_p, \mathbf{n})$ to be about the axis parallel to $\mathbf{p} \times \mathbf{n}$, the k_{22} matrix element of the transformation in the brackets will be real. (To see this we note that the expression in curly brackets is equal to $R^{-1}(L_{p_0}, \mathbf{n}_0) L_{p_0}^{-1}$. Here p_0 is obtained from $\mathbf{p}$ by the same rotation as leads to $\mathbf{n}_0$, the unit vector in the direction of the 3-axis from $\mathbf{n}$. The proof is completed by a simple calculation involving the matrices of the 2×2 representation.) The angle $\varphi(L, \mathbf{n})$ therefore satisfies the relation

$$R(L_p, \mathbf{n}) R(\mathbf{n}) = R(R(L_p, \mathbf{n})\,\mathbf{n}) R_3(\varphi(L_p, \mathbf{n})). \quad (1.9d)$$

The angle involved in the rotation $R(L_p, \mathbf{n})$ is given in the Appendix.

$$\alpha_{m\rho}(S, \mathbf{n}) = [k(\mathbf{n})/k(S^{-1}\mathbf{n})]^{1-i\rho/2} e^{im\varphi(S,\mathbf{n})}. \quad (1.10)$$

Equations (1.7) and (1.10) define the complete set of representations of the proper Lorentz group, with ρ taking on all complex values, while m takes on only integral and half-integral values.[3] It can be shown that all these representations are nondecomposable and that all except those for which $i\rho$ is an integer are irreducible; they are all inequivalent, except for pairs one of which has parameters ρ and m, and the other of which has parameters $-\rho$ and $-m$. It is easily shown that when ρ is real, (1.7) gives a unitary representation of the Lorentz group in the sense that for such representations one can define a scalar product

$$(f_1, f_2) \equiv \int f_1^*(\mathbf{n}) f_2(\mathbf{n})\, d\Omega(\mathbf{n}),$$

which remains invariant under (1.7), i.e., which is equal to

$$(f_1', f_2') = \int \left[\frac{k(\mathbf{n})}{k(S^{-1}\mathbf{n})}\right]^2 d\Omega(\mathbf{n}) f_1^*(S^{-1}\mathbf{n}) f_2(S^{-1}\mathbf{n}).$$

Indeed, we note that

$$[k(\mathbf{n})]^2 d\Omega(\mathbf{n}) = [k(S^{-1}\mathbf{n})]^2 d\Omega(S^{-1}\mathbf{n}).$$

Then writing $S^{-1}\mathbf{n} = \mathbf{m}$, we obtain $(f_1' f_2') = (f_1 f_2)$. These unitary representations form the so-called first class of unitary representations of the Lorentz group.

The Lorentz group also has another set of unitary representations (the second class), but for our purposes only the first will be necessary.

2. THE TRANSFORMATION OF THE SPIN UNDER PROPER LORENTZ TRANSFORMATIONS

Let L_p (or L_p) be a pure Lorentz transformation such that

$$p = L_p p_0, \quad (2.1)$$

where $p = (\mathbf{p}, E)$, and $p_0 = (0, M)$.

Let S be any Lorentz transformation. We shall write the transformation SL_p as a product of a pure Lorentz transformation and a rotation, i.e., $SL_p = LR$. It is easily seen that here $L = L_{Sp}$. One can thus consider the formula

$$SL_p = L_{sp} R(S, p) \quad (2.2)$$

to be a definition of the rotation $R(S, p)$.

It is easily shown that

$$R(S_2 S_1, (S_2 S_1)^{-1} p)$$
$$= R(S_2, S_2^{-1} p) R(S_1, (S_2 S_1)^{-1} p). \quad (2.3)$$

From this one easily finds that the association

$$\Psi_{sM}(\mathbf{p}\sigma) \xrightarrow{S} \Psi'_{sM}(\mathbf{p}\sigma) = R^s(S, S^{-1}p)_{\sigma\sigma'} \Psi_{sM}(S^{-1}\mathbf{p}, \sigma') \quad (2.4)$$

(where R^s is the matrix which represents the rotation $R(S, S^{-1}p)$ in the $(2s+1)$-dimensional irreducible representation of the three-dimensional rotation group[4]) gives a representation of the Lorentz group. This is the representation according to which the wave-function $\Psi_{sM}(\mathbf{p}, \sigma)$ of a particle with mass M and spin s transforms.

3. INTEGRAL TRANSFORMATION

Let us now break up the representation sM defined by (2.4) into irreducible representations ρm [see (1.7) and (1.10)].

For the representation sM we can define the invariant scalar product

$$\sum_\sigma \int \Psi^*(\mathbf{p}, \sigma)\Psi(\mathbf{p}, \sigma)d^3\mathbf{p}/E_p,$$

so that in decomposing sM we can obtain only unitary representations.* Thus the desired decomposition leads to the integral transformation

$$\Psi_{\mathbf{p}\sigma} = \sum_{m=-s}^{s}\int_0^\infty d\rho \int d\Omega(\mathbf{n}) Y_{\rho m\mathbf{n}}(\mathbf{p}, \sigma) f_{\rho m\mathbf{n}}, \tag{3.1}$$

$$f_{\rho m\mathbf{n}} = \sum_{\sigma=-s}^{s}\int \frac{d^3\mathbf{p}}{E_p} Y'_{\rho m\mathbf{n}}(\mathbf{p}, \sigma)\Psi_{\mathbf{p}\sigma}. \tag{3.1a}$$

The kernels $Y_{\rho m\mathbf{n}}(\mathbf{p}, \sigma)$ and Y' must satisfy the following condition. If $\Psi_{\mathbf{p}\sigma} \rightleftarrows f_{\rho m\mathbf{n}}$ according to (3.1) and (3.1a), and if $\Psi_{\mathbf{p}\sigma} \xrightarrow{S} \Psi'_{\mathbf{p}\sigma}$ according to (2.4) and $f_{\rho m\mathbf{n}} \xrightarrow{S} f'_{\rho m\mathbf{n}}$ according to (1.7), then $\Psi'_{\mathbf{p}\sigma} \rightleftarrows f'_{\rho m\mathbf{n}}$ according to (3.1) and (3.1a), for all S. We may write this as the following condition on $Y_{\rho m\mathbf{n}}(\mathbf{p}, \sigma)$:

$$\begin{aligned} &R^s(S, S^{-1}p)_{\sigma\sigma'} Y_{\rho m S^{-1}\mathbf{n}}(S^{-1}\mathbf{p}, \sigma') \\ &= [k(\mathbf{n})/k(S^{-1}\mathbf{n})]^{-1-i\rho/2} e^{im\varphi(S,\mathbf{n})} Y_{\rho m\mathbf{n}}(\mathbf{p}, \sigma). \end{aligned} \tag{3.2}$$

The inverse kernel $Y'_{\rho m\mathbf{n}}(\mathbf{p}, \sigma)$ satisfies a similar condition if we write

$$Y'_{\rho m\mathbf{n}}(\mathbf{p}, \sigma) = C_{m\rho}\overline{Y}_{\rho m\mathbf{n}}(\mathbf{p}, \sigma),$$

and if (3.2) is satisfied.

Let us first set $\mathbf{p} = 0$ and $S = R$ in (3.2). Then $R^s(S, S^{-1}p) = R^s(R)$ is simply the matrix corresponding to the rotation R. In this case (3.2) gives

$$R^s(R)_{\sigma\sigma'} Y_{\rho m R^{-1}\mathbf{n}}(0, \sigma') = e^{im\varphi(R,\mathbf{n})} Y_{\rho m\mathbf{n}}(0, \sigma).$$

Comparing this with (1.9b) and recalling that $R_3(\varphi)_{\sigma\sigma'} = e^{-i\sigma\varphi}\delta_{\sigma\sigma'}$, we find that

$$Y_{\rho m\mathbf{n}}(0, \sigma) = R^s(\mathbf{n})_{\sigma m}. \tag{3.3}$$

Further, let us set $S = L_p$ in (3.2). Then

$$R(S, S^{-1}p)_{\sigma\sigma'} = \delta_{\sigma\sigma'},\ k(\mathbf{n})/k(S^{-1}\mathbf{n}) = M/(E_p - \mathbf{p}\cdot\mathbf{n}),$$

and we obtain

$$Y_{\rho m\mathbf{n}}(\mathbf{p}, \sigma) = \left(\frac{E_p - \mathbf{p}\cdot\mathbf{n}}{M}\right)^{-1-i\rho/2} e^{-im\varphi(L_p,\mathbf{n})} Y_{\rho m L_p^{-1}\mathbf{n}}(0, \sigma). \tag{3.4}$$

Noting that in (1.9c) $R(L_p, \mathbf{n})\mathbf{n} = L_p^{-1}\mathbf{n}$, we can rewrite (3.4), using (3.3), in the form

$$Y_{\rho m\mathbf{n}}(\mathbf{p}\sigma) = \left(\frac{E_p - \mathbf{p}\cdot\mathbf{n}}{M}\right)^{-1-i\rho/2} R^s_{\sigma\sigma'}(L_p, \mathbf{n}) R_{\sigma' m}(\mathbf{n}). \tag{3.5}$$

Let us now prove that the Y kernel as defined by (3.4) satisfies (3.2). We do this by inserting (3.4) into (3.2). The left side then becomes

$$\begin{aligned} &R(S, S^{-1}p)_{\sigma\sigma'} R_{\sigma' m}(L^{-1}_{S^{-1}p}, S^{-1}\mathbf{n}) \exp\{-im\varphi(L_{S^{-1}p}, S^{-1}\mathbf{n})\} \\ &\times \{k(S^{-1}\mathbf{n})/k(L^{-1}_{S^{-1}p}, S^{-1}\mathbf{n})\}^{1+i\rho/2}, \end{aligned}$$

while the right side becomes

$$\begin{aligned} &\exp\{im\varphi(S, \mathbf{n})\}[k(\mathbf{n})/k(S^{-1}\mathbf{n})]^{-1-i\rho/2}[k(\mathbf{n})/k(L_p^{-1}\mathbf{n})]^{1+i\rho/2} \\ &\times \exp\{-im\varphi(L_p, \mathbf{n})\} R(L_p^{-1}\mathbf{n})_{\sigma m} \end{aligned}$$

According to (2.2), $SL_{S^{-1}p} = L\ R(S, S^{-1}p)$. Therefore

$$k(L^{-1}_{S^{-1}p}S^{-1}\mathbf{n}) = k[R^{-1}(S, S^{-1}p)L_p^{-1}\mathbf{n}] = k(L_p^{-1}\mathbf{n}),$$

since a rotation does not change the magnitude k. Thus the factors k on both sides cancel. Further,

$$\begin{aligned} &R(S, S^{-1}p)_{\sigma\sigma'} R(L^{-1}_{S^{-1}p}S^{-1}\mathbf{n})_{\sigma' m} \\ &= R(S, S^{-1}p)_{\sigma\sigma'} R[R^{-1}(S, S^{-1}p)L_p^{-1}\mathbf{n}]_{\sigma' m} = R(L_p^{-1}\mathbf{n})_{\sigma m} \\ &\times \exp\{-im\varphi[R^{-1}(S, S^{-1}p), R^{-1}(S, S^{-1}p)L_p^{-1}\mathbf{n}]\}. \end{aligned}$$

Thus the rotations on the right and left are also equal; we are left with the angles φ. We use (2.2) to write the last angle obtained in the form

$$\varphi[R^{-1}(S, S^{-1}p), L^{-1}_{S^{-1}p}S^{-1}\mathbf{n}].$$

To this we must add $\varphi(L_{S^{-1}p}, S^{-1}\mathbf{n})$; according to (1.8) and (2.2), their sum is

$$\varphi[L_{S^{-1}p}R^{-1}(S, S^{-1}p), S^{-1}\mathbf{n}] = \varphi[S^{-1}L_p, S^{-1}\mathbf{n}].$$

Taking the angle $\varphi(S, \mathbf{n})$ from the right to the left and adding it here, we obtain $\varphi(L_p, \mathbf{n})$; the same angle remains on the right. This completes the proof.

4. CALCULATION OF $C_{m\rho}$

We have thus arrived at the mutually reciprocal integral equations

$$\begin{aligned} \Psi_{sM}(\mathbf{p}, \sigma) = &\sum_m \int d\rho d\Omega(\mathbf{n})[(E_p - \mathbf{p}\cdot\mathbf{n})/M]^{-1-i\rho/2} \\ &\times R_{\sigma\sigma'}(L_p, \mathbf{n}) R_{\sigma' m}(\mathbf{n}) f_{\rho m}(\mathbf{n}); \end{aligned} \tag{4.1}$$

*In fact, only representations of the first class.[1]

$$f_{\rho m}(\mathbf{n}) = C_{m\rho}\sum_{\sigma}\int\frac{d^3\mathbf{p}}{E_p}[(E_p-\mathbf{pn})/M]^{-1+i\rho/2}$$
$$\times \bar{R}_{\sigma\sigma'}(L_{\mathbf{p}},\mathbf{n})\bar{R}_{\sigma'm}(\mathbf{n})\Psi_{sM}(\mathbf{p}\sigma) \qquad (4.2)$$

[see Eqs. (3.1) to (3.5)].

To find the still undetermined factor $C_{m\rho}$, let us rewrite (4.1) and (4.2) in the form

$$\delta(\rho-\rho')\delta(\mathbf{n}-\mathbf{n}')\delta_{mm'} = C_{m\rho}\int\frac{d^3\mathbf{p}}{E_p}[(E_p-\mathbf{p}\cdot\mathbf{n})/M]^{-1+i\rho/2}$$
$$\times[(E_p-\mathbf{p}\cdot\mathbf{n}')/M]^{-1-i\rho'/2}$$
$$\times\bar{R}_{\sigma\sigma'}(L_p,\mathbf{n})\bar{R}_{\sigma'm}(\mathbf{n})R_{\sigma\sigma''}(L_p,\mathbf{n}')R_{\sigma''m'}(\mathbf{n}'). \qquad (4.3)$$

We now multiply both sides of (4.3) by $\bar{R}_{am'}(\mathbf{n}')$, sum over m', integrate over $d\Omega(\mathbf{n}')$, and choose $\mathbf{n}=\mathbf{n}_0$ along the 3-axis. We thus obtain

$$\delta(\rho-\rho')\delta_{ma} = C_{m\rho}\int\frac{d^3\mathbf{p}}{E_p}[(E_p-\mathbf{p}\cdot\mathbf{n}_0)/M]^{-1+i\rho/2}$$
$$\times[(E_p-\mathbf{p}\cdot\mathbf{n}')/M]^{-1-i\rho'/2}\bar{R}_{\sigma m}(L_p,\mathbf{n}_0)R_{\sigma a}(L_p,\mathbf{n}')d\Omega(\mathbf{n}').$$

In this expression we write $R(L_p,\mathbf{n}') = R(\mathbf{p}/p)\times R(L_{p_0},\mathbf{n}'')R^{-1}(\mathbf{p}/p)$. Here $\mathbf{p}_0$ is directed along the 3-axis, and $\mathbf{n}''$ is obtained from $\mathbf{n}'$ by the same transformation which carries $\mathbf{p}$ into $\mathbf{p}_0$. Since $\mathbf{p}\cdot\mathbf{n}' = \mathbf{p}_0\cdot\mathbf{n}''$ and $d\Omega(\mathbf{n}') = d\Omega(\mathbf{n}'')$, we have

$$\delta(\rho-\rho')\delta_{ma} = C_{m\rho}\int\frac{d^3\mathbf{p}}{E_p}[(E_p-\mathbf{p}\cdot\mathbf{n}_0)/M]^{-1+i\rho/2}$$
$$\times[(E_p-\mathbf{p}_0\cdot\mathbf{n}'')/M]^{-1-i\rho'/2}$$
$$\times\bar{R}_{\sigma m}(L_p,\mathbf{n}_0)R_{\sigma\alpha}(\mathbf{p}/p)\bar{R}_{\alpha\beta}(\mathbf{p}/p)R_{\alpha\beta}(L_{p_0}\mathbf{n}'')d\Omega(\mathbf{n}'').$$

We shall first perform the integration over the azimuth angles, writing

$$d\Omega(\mathbf{p}) = \sin\theta\, d\theta\, d\varphi,\quad d\Omega(\mathbf{n}'') = \sin\theta''\, d\theta''\, d\varphi''.$$

According to (1.1), $R(\mathbf{p}/p) = R_3(\varphi+\pi/2)R_1(\theta)$. According to Appendix A

$$R(L_p,\mathbf{n}_0) = R_3(\varphi-\pi/2)R_1(\alpha(p,t))R_3^{-1}(\varphi-\pi/2),$$
$$R(L_{p_0}\mathbf{n}'') = R_3(\varphi''+\pi/2)R_1(\alpha(p,t''))R_3^{-1}(\varphi''+\pi/2).$$

Here $t = \mathbf{p}\cdot\mathbf{n}/p = \cos\theta$ and $t'' = \mathbf{p}\cdot\mathbf{n}''/p = \cos\theta''$, while the angle $\alpha(p,t)$ is defined [see Eqs. (A.2) to (A.4)]. Since $R_3(\varphi)_{nm} = \delta_{nm}\exp\{-in\varphi\}$,

$$\frac{1}{(2\pi)^2}\delta(\rho-\rho')\delta_{ma} = C_{m\rho}$$
$$\times\int_0^\infty\frac{p^2dp}{E_p}\int_{-1}^{1}dt\int_{-1}^{1}dt''[(E_p-pt)/M]^{-1+i\rho/2}$$
$$\times[(E_p-pt'')/M]^{-1-i\rho'/2}(-1)^{m-\sigma}\sum_{\alpha,\sigma}\bar{u}^s_{\sigma m}(\alpha(p,t))$$
$$\times u^s_{\alpha\alpha}(\alpha(p,t''))u^s_{\sigma\alpha}(\theta)\bar{u}^s_{a\alpha}(\theta)\delta_{am};$$

here the $u^s_{mn}(x)$ are functions defined by Gel'fand and Shapiro.[4] By making use of the properties of these functions,* this equation is easily transformed to the form

$$(1/2\pi)^2\delta(\rho-\rho') = C_{m\rho}\int_0^\infty\frac{p^2dp}{E_p}\int_{-1}^{1}dt\int_{-1}^{1}dt''[(E_p-pt)/M]^{-1+i\rho/2}$$
$$\times[(E_p-pt'')/M]^{-1-i\rho'/2}$$
$$\times\sum_\alpha u^s_{m\alpha}[\alpha(p,t)+\theta]u^s_{\alpha\alpha}(\alpha(p,t''))\bar{u}^s_{m\alpha}(\theta), \qquad (4.4)$$

or

$$(2\pi)^{-2}\delta(\rho-\rho') = C_{m\rho}\int_0^\infty\frac{p^2dp}{E_p}\sum_\alpha F_{\alpha\rho's}(p)f_{m\alpha\rho s}(p),$$

where

$$F_{\alpha\rho's}(p) = \int_{-1}^{1}[(E_p-pt'')/M]^{-1-i\rho'/2}u^s_{\alpha\alpha}(\alpha(p,t''))dt'';$$

$$f_{m\alpha\rho s}(p) = \int_{-1}^{1}[(E_p-pt)/M]^{-1+i\rho/2}u^s_{m\alpha}[\alpha(pt)+\theta]\bar{u}^s_{m\alpha}(\theta)dt.$$

We now calculate $F_{\alpha\rho s}(p)$. We do this by using the power series

$$u^s_{\alpha\alpha}(\alpha) = \sum_q a^q_{s\alpha}x^q,$$

where $x = 1+\cos\alpha$, and

$$a^q_{s\alpha} = \frac{(-)^{s-q}(s+q)!}{2^q(q-\alpha)!(q+\alpha)!(s-q)!}.$$

The calculation gives

$$F_{\alpha\rho s}(p) = \frac{1}{p}\left\{\sum_q a^q_{s\alpha}2^{q+1}\frac{q\cos(\lambda\rho/2)+(\rho/2)\sin(\lambda\rho/2)}{q^2+(\rho/2)^2}\right.$$
$$\left.+Q_{\alpha\rho s}(p)\right\}. \qquad (4.6)$$

Here $\cosh\lambda = E_p/M$, and $Q_{\alpha\rho s}(p)$ can be represented in the form $A(p,\rho)\cos(\lambda\rho/2) + B(p,\rho)\sin(\lambda\rho/2)$, where A and B are functions bounded for all $p>0$ and approaching zero as $p\to\infty$ as $1/p$ uniformly in ρ.

We now evaluate $f_{m\alpha\rho s}(p)$. To do this we make use of the series

$$u^s_{m\alpha}(\theta) = (1+x)^{|(m+\alpha)/2|}(1-x)^{|(m-\alpha)/2|}$$
$$\times\sum_0^{n_{\max}}A_n(1+x)^n, \qquad x=\cos\theta.$$

Using (A.5) and (A.6), we find that for $|m\pm\alpha|\neq 0$

$$^*\quad u^s_{mn}(\theta) = (-1)^{s-m}i^{n-m}[2^s(s-m)!]^{-1}$$
$$\times[(s+n)!(s-m)!/(s-n)!(s+m)!]^{1/2}(1-x)^{(m-n)/2}$$
$$\times(1+x)^{-(m+n)/2}(d/dx)^{s-n}(1-x)^{s-m}(1+x)^{s+m},\quad x=\cos\theta;$$
$$u^s_{mn}(\theta) = u^s_{nm}(\theta) = u^s_{-n-m}(\theta);$$
$$\sum_\alpha u^s_{m\alpha}(\theta_1)u^s_{\alpha n}(\theta_2) = u^s_{mn}(\theta_1+\theta_2).$$

$$|f_{m\alpha\rho s}(p)| < \text{const}/p^{1+a}, \qquad (4.7)$$

where a is the smaller of $|m \pm \alpha|$. We rewrite the sum in (4.4) in the form

$$\sum_{\alpha} u^s_{m\alpha}[\alpha(p,t)+\theta]\,\bar{u}^s_{m\alpha}(\theta)\,u^s_{\alpha\alpha}(\alpha')$$

$$= \sum_{\alpha} u^s_{m\alpha}[\alpha(pt)+\theta]\,\bar{u}^s_{m\alpha}(\theta)\,u^s_{mm}(\alpha')$$

$$+ \sum_{\alpha\neq m} u^s_{m\alpha}(\alpha+\theta)\,\bar{u}^s_{m\alpha}(\theta)\,[u^s_{\alpha\alpha}(\alpha')-u^s_{mm}(\alpha')].$$

Because $u_{mm} = u_{-m-m}$, we see that $\alpha = -m$ does not enter into the last sum. Using the summation theorem for the u_{mm} functions, we can write (4.4) in the form

$$\delta(\rho-\rho') = (2\pi)^2 C_{m\rho}\int_0^\infty \frac{p^2dp}{E_p}\Big\{F_{m\rho's}(p)\,F_{m-\rho s}(p)$$

$$+ \sum_{|\alpha|\neq|m|} f_{m\alpha\rho s}(p)\,[F_{\alpha\rho's}(p)-F_{m\rho's}(p)]\Big\}. \qquad (4.8)$$

It follows from (4.6) and (4.7) that only the product of the principal parts of $F_{m\rho s}$ and $F_{m-\rho s}$ contributes to the δ-function. All the other parts of the integral in (4.8) can only give a finite contribution (which must cancel in the sum). Thus

$$\delta(\rho-\rho') \sim (2\pi)^2 C_{m\rho}\int_0^\infty d\lambda \sum_q \frac{q\cos(\lambda\rho/2)+(\rho/2)\sin(\lambda\rho/2)}{q^2+(\rho/2)^2}$$

$$\times \sum_{q'} \frac{q'\cos(\lambda\rho'/2)+(\rho'/2)\sin(\lambda\rho'/2)}{(q')^2+(\rho'/2)^2}\,2^{q+q'+2}a^q_{sm}a^{q'}_{sm}.$$

Since

$$\frac{1}{\pi}\int_0^\infty \cos(\lambda\rho/2)\cos(\lambda\rho'/2)\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \sin(\lambda\rho/2)\sin(\lambda\rho'/2)\,d\lambda = \delta(\rho-\rho')$$

and

$$\int_0^\infty \cos(\lambda\rho/2)\sin(\lambda\rho'/2)\,d\lambda \approx 0,$$

we arrive at

$$\frac{1}{C_{m\rho}} = 16\pi^3 \sum_{qq'} \frac{[qq'+(\rho/2)^2]\,a^q_{sm}a^{q'}_{sm}2^{q+q'}}{[q^2+(\rho/2)^2]\,[(q')^2+(\rho/2)^2]}.$$

It follows from this that (see Appendix B)

$$C_{m\rho} = \frac{\rho^2+(2m)^2}{(4\pi)^3}. \qquad (4.9)$$

5. CONCLUSION

Thus our final result is that if a relation of the form (4.1) exists, the inverse relation is given by (4.2) with $C_{m\rho}$ given by (4.9). In these equations $E_p = (M^2+p^2)^{1/2}$, the rotation R(n) is defined by (1.1), and $R(L_p, \mathbf{n})$ is defined by (1.9c) and by specifying the direction of its axis perpendicular to both **p** and **n.** By R^s_{ab} we denote the matrix elements of the (2s + 1)-dimensional irreducible representation of the three-dimensional rotation group which correspond to the rotation R^4

Equation (4.1) gives the expansion of the wave function $\Psi_{sM}(\mathbf{p}, \sigma)$, transforming like a wave function of a particle with spin s and mass M according to the s, M representation [see (2.4)] in terms of the functions $f_{\rho m}(\mathbf{n})$, transforming according to the irreducible representations of the proper Lorentz group. Equation (4.2) gives these irreducible components of $\Psi_{sM}(\mathbf{p}, \sigma)$ in terms of the function itself.

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APPENDIX A

Let $\boldsymbol{\alpha}$ be the vector of the rotation $R(L_p, \mathbf{n})$, so that

$$R(L_p, \mathbf{n})\,\mathbf{n} = \cos\alpha\cdot\mathbf{n} + [\boldsymbol{\alpha}\times\mathbf{n}]\sin\alpha/\alpha + \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\mathbf{n})(1-\cos\alpha)/\alpha^2.$$

From the conditions

$$L_p^{-1}\mathbf{n} = R(L_p, \mathbf{n})\,\mathbf{n} \quad\text{and}\quad \boldsymbol{\alpha}\sim[\mathbf{p}\times\mathbf{n}], \qquad (A.1)$$

which determine $R(L_p, \mathbf{n})$, and from the formula

$$L_p^{-1}\mathbf{n} = [\mathbf{n}M + \mathbf{p}(\mathbf{p}\cdot\mathbf{n})/(E_p+M) - \mathbf{p}]\,[E_p - (\mathbf{p}\cdot\mathbf{n})]^{-1},$$

which is implied by (1.2) and (2.1), we easily arrive at

$$\frac{\boldsymbol{\alpha}}{\alpha}\sin\alpha = \frac{[\mathbf{p}\times\mathbf{n}]}{E_p+M}\left[1+\frac{M}{E_p-(\mathbf{p}\cdot\mathbf{n})}\right]. \qquad (A.2)$$

This leads to

$$1+\cos\alpha = \frac{(E_p-\mathbf{p}\cdot\mathbf{n}+M)^2}{(E_p+M)(E_p-\mathbf{p}\cdot\mathbf{n})}, \qquad (A.3)$$

$$1-\cos\alpha = \frac{p^2-(\mathbf{p}\cdot\mathbf{n})^2}{(E_p+M)(E_p-\mathbf{p}\cdot\mathbf{n})}. \qquad (A.4)$$

It is now easily shown that

$$1+\cos(\alpha+\theta) = (E_p-p)(1+t)/(E-pt), \qquad (A.5)$$

$$1-\cos(\alpha+\theta) = (E_p+p)(1-t)/(E-pt), \qquad (A.6)$$

where $t = \cos\theta = \mathbf{p}\cdot\mathbf{n}/p$.

APPENDIX B

We shall here prove (4.9). We have

$$\sum_{qq'} a^q a^{q'} 2^{q+q'} \frac{qq'+(\rho/2)^2}{[q^2+(\rho/2)^2]\,[q'^2+(\rho/2)^2]} = 2\sum_q \frac{q2^q a^q}{q^2+(\rho/2)^2}\sum_{q'}\frac{2^{q'}a^{q'}}{q+q'}.$$

Comparing the equation $A_q \equiv \sum_{q'} 2^{q'}a^{q'}/(q+q')$

with $u^s_{mm} = \sum_{q'} a^{q'} x^{q'}$, we see that

$$A_q = \frac{1}{2^q} \int_0^2 dx u^s_{mm} x^{q-1}$$

$$= \frac{(-)^{s-m}}{2^{s+q}(s-m)!} \int_0^2 x^{q-m-1} dx \, (d/dx)^{s-m} (2-x)^{s-m} x^{s+m}.$$

When $q > m$, integration by parts shows simply that $A_q = 0$. When $q = m$, we easily obtain

$$A_m = (-)^{s-m} \frac{(s-m)!\,(2m-1)!}{(s+m)!} = \frac{1}{2m a^m_{sm} 2^m}.$$

Thus

$$\sum_{q,q'} = \frac{1}{m^2 + (\rho/2)^2}.$$

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