GALVANOMAGNETIC CHARACTERISTICS OF METALS WITH OPEN FERMI SURFACES. I

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Submitted to JETP editor July 10, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1251-1264 (November, 1958)

Galvanomagnetic phenomena in strong fields are studied for metals with open Fermi surfaces. Characteristic features of the angular dependences of galvanomagnetic parameters on the magnetic field direction are investigated and their relation to the topology of an open surface is determined. In particular, the possibility of resistivity saturation, previously established in principle[1] for certain orientations of the magnetic field relative to the crystal axes, and its quadratic increase for other directions are analyzed in detail. It is found under what conditions the resistivity can increase linearly with the field as a result of averaging over the orientations of the crystallites in polycrystalline samples.

THE theory of galvanomagnetic phenomena in metals, developed in reference 1 for an arbitrary dispersion law $\epsilon = \epsilon(\mathbf{p})$ and an arbitrary collision integral, enable us to draw certain inferences regarding the topology of the Fermi surface from the experimentally determined galvanomagnetic characteristics. When the Fermi surface is open or when it is located near open constant-energy surfaces, the asymptotic behavior of the conductivity tensor σ_{ik} in strong magnetic fields can differ essentially from its asymptotic behavior when the constant-energy surfaces are closed. Thus, for example, the asymptotic Hall coefficient for metals with an open Fermi surface can depend on the orientation of the magnetic fields with respect to the crystallographic axes, whereas for closed surfaces the Hall coefficient in strong magnetic fields is isotropic (if the number of electrons does not equal the number of holes).

An even more important distinguishing characteristic of open surfaces is the fact that for some directions of the magnetic field the resistivity can approach saturation, while for other directions it can rise without limit ($\rho \sim H^2$).

In the present paper we investigate the galvanomagnetic properties of metals with different types of open constant-energy surfaces and determine the characteristics of the angular dependences of these properties.

1. GEOMETRY OF PLANE SECTIONS OF AN OPEN FERMI SURFACE

For convenience we shall discuss only the constant-energy surfaces which have the form of an "undulating cylinder" (Fig. 1) and the surface which is represented by a three-dimensional grid of undulating cylinders (Fig. 2).

Such surfaces are of the most common type. The existence of the three-dimensional grid has been detected experimentally by Pippard[2] for copper, while Verkin[3] has found the undulating cylinder type for zinc.

Further extension to arbitrary open constant-energy surfaces is not difficult, because the sections of such surfaces by a plane $p_z = \text{const}$ may contain momentum-space trajectories ($\epsilon = \text{const}$, $p_z = \text{const}$) of all possible types — closed and open trajectories, with or without the periodicity of the reciprocal lattice.

The direction of the magnetic field will always be taken along the z axis. The set of magnetic field directions for which it is possible to have a band of open momentum-space trajectories is determined by the structure of the constant-energy surface. In the case of the undulating-cylinder type, open trajectories are encountered only when the magnetic field is perpendicular to the cylinder axis. Open trajectories are encountered much more frequently in the case of the three-dimensional grid. Figure 3 is a stereographic projection of the magnetic field directions for which open plane sections appear in the case of the three-dimensional grid type of surface. There is a two-dimensional set (solid angle) of magnetic field directions for which open trajectories are possible (regions I in Fig. 3).

When the magnetic field direction coincides with one of the crystallographic axes open trajectories are possible only for certain isolated values of p_z and do not contribute to the tensor σ_{ik}.

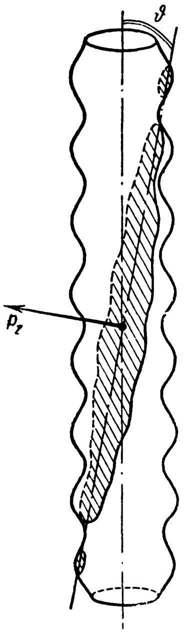

FIG. 1. Section of an "undulating cylinder" type of constant-energy surface by a plane p_z = const.

When the magnetic field is not along one of the crystallographic axes, we can have an entire band of open trajectories, integration along which contributes to σ_{ik}. A geometrical analysis shows that open trajectories appear, in particular, when the magnetic field direction is located in any principal crystallographic plane.

When the magnetic field direction is not located in any principal crystallographic plane, the condition for realization of a band of open trajectories (in the simplest case of constant-diameter tubes) is

$$b_1 \cos\varphi \tan\vartheta \leqslant d, \quad b_2 \sin\varphi \tan\vartheta \leqslant d; \qquad (1)$$

where d is the tube diameter, H, ϑ, φ are the polar coordinates of the vector H in the coordinate system where the polar axis coincides with the crystallographic axis that is closest to the magnetic field direction (axis 3 in Fig. 2), and b_1, b_2 are the periods of the reciprocal lattice in directions 1 and 2. In the general case of tubes with nonuniform diameter condition (1) is not exact and the boundaries of regions I and II (in Fig. 3) are deformed. When only one of conditions (1) is fulfilled, the closed trajectories pass through several cells of the reciprocal lattice (region III in Fig. 3).

A simple analysis shows that for a given magnetic-field direction open trajectories in all cross sections p_z = const have a common direction, which is the intersection of the xy plane perpendicular to the magnetic field with the nearest (in angle) principal crystallographic plane (open trajectories are always located in one of the principal crystallographic planes (1 0 0), (0 1 0) or (0 0 1). This direction will hereinafter be the x axis. Thus the projections of all trajectories will be of finite length along the y axis (see Fig. 4). It must always be kept in mind that for a suitable choice of axes the magnetic field direction completely determines the entire coordinate system.

2. ASYMPTOTIC BEHAVIOR OF GALVANOMAGNETIC CHARACTERISTICS IN STRONG FIELDS. TYPES OF SINGULAR MAGNETIC FIELD DIRECTIONS

In describing the motion of an electron we shall, as in reference 1, use the quantities ϵ and p_z, which are conserved in a magnetic field, as well as the dimensionless quantity $\tau = t/T_0$, which characterizes the position of the electron in the momentum-space trajectory ϵ = const and in p_z = const. Here t, the time of motion along a given trajectory, is given by $(c/eH)\int dp_x/v_y$ (see reference 1), and T_0 agrees in order of magnitude with the time required for the momentum projection (p_x or p_y) of an electron moving along a momentum-space trajectory either to return to its original value (when the trajectory is located entirely within a single cell of the reciprocal lattice) or to change by a quantity of the order of the reciprocal lattice. The latter case is possible for an electron moving along an open trajectory or along a closed trajectory which passes through a few cells of the reciprocal lattice.

To determine the conductivity tensor σ_{ik}, it is necessary to solve a kinematic equation which, after linearization for a weak electric field E, is of the form[1]

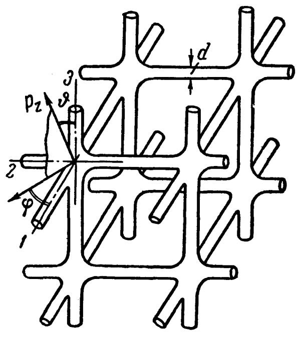

FIG. 2

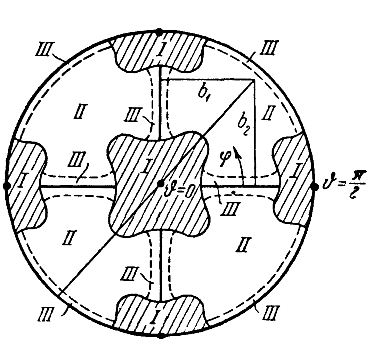

FIG. 3

FIG. 2. Constant-energy surface of the "three-dimensional grid" type.

FIG. 3. Stereographic projection of magnetic field directions. The shaded regions (I) give the field directions for which a band of open trajectories exists. Region II has no open trajectories. Region III, bounded by dashed lines, represents the field directions for which elongated closed trajectories exist (enclosing many cells).

$$\frac{t_0}{T_0}\frac{\partial \psi_i}{\partial \tau} + \widetilde{W}\{\psi_i\} = f_0'(\varepsilon)\, v_i, \qquad (2)$$

where ψ_i characterizes the addition to the equilibrium function f_0 of the electron distribution:

$$f = f_0 - e t_0 \mathbf{E}\cdot\boldsymbol{\psi};$$

$\widetilde{W}\{f\}/t_0$ is the collision integral, t_0 is the mean free electron time, and $|\widetilde{W}| \sim 1$.

The electrical conductivity tensor σ_{ik} is given by[1]

$$\sigma_{ik} = -2e^2 t_0 h^{-3} \int v_i \psi_k \, d\mathbf{p}. \qquad (3)$$

Strong magnetic fields are understood to be those which satisfy the condition

$$\gamma_0 = T_0/t_0 = H_0/H \ll 1. \qquad (4)$$

The characteristic field H_0 is the field for which the characteristic time T_0 equals the mean free time t_0. We shall hereinafter assume that condition (4) is always fulfilled.

When all closed trajectories are located within a single cell of the reciprocal lattice and the open trajectories possess the periodicity of the reciprocal lattice, Eq. (2) can be solved by successive approximations (as in reference 1), with the following representation of ψ_i:

$$\psi_i = \sum_{l=0}^{\infty} \gamma_0^l \psi_i^{(l)}. \qquad (5)$$

As will be seen below, the expansion (5) is not always permissible for open Fermi surfaces. Specifically, this can occur when the momentum-space trajectory extends over a few cells of the reciprocal lattice. This is associated with the fact that the characteristic time of electron motion in the magnetic field is the electron period of revolution along the trajectory (for a closed trajectory), which in the case of greatly extended closed trajectories that pass through a large number of reciprocal lattice cells can be comparable with or even much longer than the mean free time. Consequently, the character of the dependence of σ_{ik} on the magnetic field (that is, on $\gamma_0 = H_0/H$) varies greatly with the magnetic field orientation and near certain singular directions the expansion in powers of γ_0 is not permissible.

To simplify the following investigation, we illustrate the foregoing for the case $\widetilde{W} = 1$ (that is, for the case in which collisions can be described by introducing only the relaxation time t_0). Equation (2) then becomes

$$\frac{1}{\gamma_0}\frac{\partial \psi_i}{\partial \tau} + \psi_i = f_0'(\varepsilon)\, v_i. \qquad (6)$$

For closed trajectories, as in reference 1, we

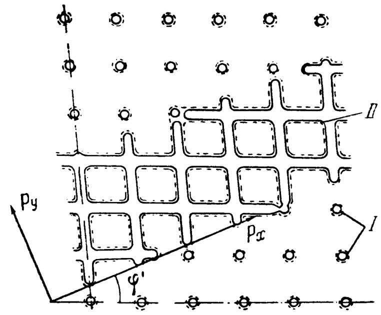

FIG. 4. Intersection of the plane p_z = const and the "three-dimensional-grid" constant-energy surfaces $\epsilon = \epsilon_0$ (solid curves) and $\epsilon = \epsilon_0 + \delta\epsilon$ (dashed curves). Closed trajectories of type I correspond to energy regions $\epsilon < \epsilon_0$; trajectories of type II correspond to energy regions $\epsilon > \epsilon_0$. Accordingly, opposite directions are taken along these trajectories. These two types of curves are separated by open trajectories. The direction of an open trajectory is given by the angle φ', for which $\cos\varphi' = \cos\varphi(1 + \sin^2\varphi\tan^2\vartheta)^{-1/2}$.

use the Fourier method to obtain

$$\sigma_{il} = -2e^2 t_0 h^{-3} \times \iint \sum_{k=1}^{\infty} f_0'(\varepsilon)\frac{eHT}{c}\frac{v_i^{-k}v_l^k + v_i^k v_l^{-k} + (ik/\gamma)(v_i^k v_l^{-k} - v_i^{-k}v_l^k)}{k^2/\gamma^2 + 1}\, d\varepsilon\, dp_z, \qquad (7)$$

where $\gamma = T/t_0$, and T is the period of revolution of an electron in a closed trajectory:

$$T = (c/eH)\oint dp_x/v_y. \qquad (8)$$

Near certain singular directions of the magnetic field (small ϑ in Fig. 1 and region III in Fig. 3) the trajectories are closed and greatly extended. To determine the basic angular dependence of σ_{ik} near these directions the angular dependence of γ must be taken into account explicitly. In the case of an undulating cylinder with $\vartheta \ll 1$ we have $T = \alpha T_0/\vartheta$, $\gamma = \alpha\gamma_0/\vartheta$, $\alpha \sim 1$ for most trajectories and $\gamma \sim \gamma_0$ for the remaining trajectories.

The contribution to σ_{il} from the "stretched" trajectories will be

$$\sigma_{il}' = \sum_{k=1}^{\infty}\frac{A_{il}^k\gamma_0 + B_{il}^k\vartheta}{k^2\vartheta^2 + \alpha^2\gamma_0^2}, \quad \gamma_0 = \frac{T_0}{t_0} = \frac{H_0}{H}; \qquad (9)$$

where α, A^k, B^k also depend on the angles. Thus σ_{il} is a function of γ_0 and also of γ_0/ϑ. When $\vartheta \sim \gamma_0 \ll 1$ it can be seen from (9) that the following expansion is possible:

$$\sigma_{il} = \sigma_{il}(\gamma_0, \gamma_0/\vartheta) = \sum_n \gamma_0^n \sigma_{il}^{(n)}(\gamma_0/\vartheta), \qquad (10)$$

whereas the expansion given in (5) is impermissible.

When $\widetilde{W} \neq 1$ the character of the angular dependence of σ_{ik} in (7) and (10) does not change qualitatively.

We have no further interest in the smooth angular dependence of σ_{ik}, which results from the specific form of the collision integral. We shall distinguish only the steep angular variation of σ_{ik} which is associated with the approach to the singular magnetic field directions.

When a two-dimensional region (solid angle) of directions of H exists for which there is a band of open trajectories, an analysis of the methods and equations of reference 1 shows that within this region it is reasonable to use the expansion of ψ_i in powers of γ_0 [expansion (5)] even when the open trajectories are aperiodic. The essential requirement for this purpose is that the mean velocities $(\overline{v}_i)_{\Delta\tau}$, during the mean free time on each portion of the trajectory (that is, in the interval $\Delta\tau \sim 1/\gamma_0$) should, to a very high degree of probability, differ very little from the mean value $\overline{v}_i$ over the entire trajectory* (it must be noted, of course, that this condition is not fulfilled for greatly extended closed trajectories).

By means of the given expansion we can see that in the most usual case when, for a given magnetic field direction, all open trajectories have a common direction (taken along the x axis) Eqs. (14), (21), (29) and (54) of reference 1 are valid and, specifically, the resistivity increases quadratically with the field, $\rho \sim H^2$. The foregoing statement refers to all surfaces considered in the present paper, for which more details are given in the following section. On the other hand, when open trajectories exist simultaneously for different directions (belonging to the same or to different surfaces if there are several of these) all components of σ_{ik} and ρ_{ik} approach saturation.

When the open trajectories have a common x direction, the asymptotic value of the transverse conductivity component σ_{yx} is of the order of $\gamma_0 \sim 1/H$ and, in virtue of Eqs. (21) and (23) of reference 1, can be written as

$$\sigma_{yx} = -2e^2h^{-3}T_0 \int \frac{\partial p_x}{\partial \tau} \left\{C_x^{(1)}(p_z,\varepsilon) - \frac{c}{eHT_0} f_0'(\varepsilon)\, p_y\right\} d\tau dp_z d\varepsilon, \tag{11}$$

where $C_x^{(1)}$ is obtained from the condition

$$\overline{\widetilde{W}\{\psi^1{}_x\}} = \overline{\widetilde{W}\left\{C^1{}_x - \frac{c}{eHT_0} f'_0(\varepsilon)\, p_y\right\}} = 0 \tag{12}$$

[see Eqs. (16) to (18) of reference 1].

The second term is independent of the collision integral and is determined only by the geometry of the surface. The first term disappears for closed trajectories in virtue of the condition $\oint (\partial p_x/\partial\tau)\, d\tau = 0$. However, for open trajectories

$$\bar{v}_y \sim \lim \frac{1}{T} \int_0^T \frac{\partial p_x}{\partial \tau} d\tau \sim b_x/T_0 \neq 0.$$

Consequently σ_{yx} together with the Hall coefficient depends on the form of the collision integral and the magnetic field direction if the trajectories are not straight lines (in which case $\psi_x^{(1)} = \psi_x^{(1)}(\epsilon, p_z) = 0$). In this case the singularity in the angular dependence of the galvanomagnetic characteristics is the point where open trajectories disappear.

In accordance with the above, the special magnetic field directions can be of at least three types:

(1) Magnetic field directions for which a band of open trajectories exists form a one-dimensional set. This occurs, specifically, if there is an isolated direction of open trajectories. Examples are directions perpendicular to the axis for an undulating cylinder, and continuous curves connecting regions I in the sterographic projection of directions of H for a three-dimensional grid (Fig. 3).

(2) There exists a two-dimensional region (solid angle) of magnetic field directions corresponding to open trajectories. The singular directions of the magnetic field in this case are the boundaries of this region. An example is given by the boundaries of the shaded regions I in Fig. 3.

(3) In the open trajectory region there exists an isolated magnetic field direction for which the band of open trajectories disappears. An example is given by the directions of the principal crystallographic axes for a three-dimensional grid ($\vartheta = 0$ and similar points in Fig. 3).

As has been shown above, in case (1) a strong angular dependence of σ_{il} appears in the closed trajectory region as the singular line of magnetic field directions is approached.

In case (2) when the width of the open trajectory band vanishes on the boundary of region I, then, as will be shown, a strong angular dependence of σ_{il} appears inside this boundary (that is, in the open trajectory region). The same occurs in case (3).

A more detailed discussion with specific examples will be given in the next section.

3. INVESTIGATION OF ASYMPTOTIC BEHAVIOR FOR SOME TYPES OF FERMI SURFACE

a. Surface of the Undulating Cylinder Type

When the magnetic field is not perpendicular to the axis of a constant-energy surface of the undulating cylinder type, all trajectories $\epsilon = \text{const}$ and $p_z = \text{const}$ are closed.

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*This requirement follows from the fact that the expansion of ψ_i in powers of γ_0 contains expressions such as $\int_0^\tau (v_i - \overline{v}_i)\, d\tau$ and their iterations.

When the angle ϑ between the magnetic field directions and the plane perpendicular to the cylinder axis is not small, the period T_ϑ of electron revolution through its entire momentum-space trajectory is much smaller than the characteristic mean free time t_0:

$$T_\vartheta/t_0 \approx T_0/t_0\vartheta \ll 1,$$

$\gamma_0 \ll 1$.

The conductivity tensor σ_{ik} and the resistivity tensor ρ_{ik} in strong magnetic fields have the same asymptotic form as in the case of closed constant-energy surfaces (1):

$$\sigma_{ik} = \begin{pmatrix} \gamma_0^2 a_{xx} & \gamma_0 a_{xy} & \gamma_0 a_{xz} \\ \gamma_0 a_{yx} & \gamma_0^2 a_{yy} & \gamma_0 a_{yz} \\ \gamma_0 a_{zx} & \gamma_0 a_{zy} & a_{zz} \end{pmatrix}; \qquad (13)$$

$$\rho_{ik} = \begin{pmatrix} b_{xx} & \gamma_0^{-1} b_{xy} & b_{xz} \\ \gamma_0^{-1} b_{yx} & b_{yy} & b_{y} \\ b_{zx} & b_{zy} & b_{zzz} \end{pmatrix}. \qquad (14)$$

For $\vartheta \gg \gamma_0$ the Hall coefficient is isotropic, independent of the form of collision integral, and inversely proportional to the volume between the cylinder surface and two adjacent crystallographic planes perpendicular to the cylinder axis.

When $\vartheta \sim \gamma_0 \ll 1$ the electron period of revolution in the largest trajectory (Fig. 1) is comparable with t_0 and σ_{ik} contains terms, which cannot be expanded in powers of γ_0 since these terms contain γ_0 in the combination γ_0/ϑ.

We shall continue to take the z axis in the direction of the magnetic field, with the x axis in the plane of the magnetic field, and cylinder axis and the y axis perpendicular to this plane.

Taking into account the angular dependence of the Fourier components of velocities $\mathbf{v}^k$ in different trajectories, we obtain for σ_{il}, in accordance with Sec. 2,

$$\sigma_{xx} = \sum_k \frac{A^1{}_k \vartheta^2 \gamma_0^2}{k^2\vartheta^2 + \alpha^2\gamma_0^2} + \alpha'_{xx}\gamma_0^2,$$
$$\sigma_{yy} = \sum_k \frac{B_k^1 \gamma_0^2}{k^2\vartheta^2 + \alpha^2\gamma_0^2} + a'_{yy}\gamma_0^2 \qquad (15)$$

with similar expressions for the other components (α, A_k^1, B_k^1 are coefficients independent of γ_0 and ϑ in first approximation). The second terms in (15) are associated with the trajectories which remain closed even when $\vartheta = 0$ (see Fig. 1).

For small γ_0, $\vartheta \ll 1$ but arbitrary γ_0/ϑ we can retain only the first nonvanishing terms in the expansion (10). A simple analysis then shows that the conductivity tensor can be represented asymptotically by the following general form:

$$\sigma_{il} = \begin{pmatrix} \gamma_0^2 a_{xx}(\eta) & \gamma_0 a_{xy}(\eta) & \gamma_0 a_{xz}(\eta) \\ \gamma_0 a_{yx}(\eta) & a_{yy}(\eta) & a_{yz}(\eta) \\ \gamma_0 a_{zx}(\eta) & a_{zy}(\eta) & a_{zz}(\eta) \end{pmatrix} \qquad (16)$$

where $\eta = \gamma_0/\vartheta$.

When $\vartheta \ll \gamma_0$ (i.e., $\eta \to \infty$) $a_{ik}(\eta)$ tends to the finite values $a_{ik}(\infty)$; this correspond to Eqs. (29) of reference 1. On the other hand, when $\vartheta \gg \gamma_0$ the tensor $a_{ik}(\eta)$ has the form

$$a_{ik} = \begin{pmatrix} a_{xx}(0) & a_{xy}(0) & a_{xz}(0) \\ a_{yx}(0) & \alpha_1\eta^2 & \alpha_2\eta \\ a_{zx}(0) & -\alpha_2\eta & a_{zz}(0) \end{pmatrix}, \qquad (17)$$

where α_1 and α_2 are constants. It is easily seen that in this case (16) reduces to (13).

Expressions of the form

$$a_{il}(\eta) = \frac{\alpha_{il}^{(0)} + \alpha_{il}^{(1)}\eta + \alpha_{il}^{(2)}\eta^2}{1 + \beta_{il}\eta^2} \qquad (18)$$

furnish a good extrapolation of $a_{il}(\eta)$ corresponding to the structure of σ_{il}. Here

$$\alpha_{yy}^{(0)} = \alpha_{yy}^{(1)} = \alpha_{yz}^{(0)} = \alpha_{zy}^{(0)} = 0.$$

In the special case of a right circular cylinder, $\epsilon = (p^2 - p_3^2)/2m_0$ (p_s is the cylinder axis) and $\tilde{W} = 1$, Eq. (18) is exact and

$$a_{xy}(\eta) = -a_{yx}(\eta) = a_{xx}(\eta) = \sigma_0/(1+\eta^2);$$
$$a_{zz}(\eta) = a_{yy}(\eta) = \sigma_0\eta^2/(1+\eta^2);$$
$$a_{yz} = a_{xz} = -a_{zy} = -a_{zx} = \sigma_0\eta/(1+\eta^2);$$
$$\sigma_0 = 2Vh^{-3}(e^2t_0/m_0).$$

It must be noted that the singular character of $a_{il}(\eta)$ in the case of a right cylinder (specifically, the fact that $a_{zz}(0) = 0$; $a_{xx}(\infty) = a_{xy}(\infty) = 0$) is associated with the absence of undulation; then $\bar{v}_z = 0$ on each trajectory and there is no velocity component along the cylinder axis.

For the resistivity tensor ρ_{ik} we obtain

$$\rho_{ik} = (\sigma^{-1})_{ik} = \begin{pmatrix} \gamma_0^{-2} b_{xx}(\eta) & \gamma_0^{-1} b_{xy}(\eta) & \gamma_0^{-1} b_{xz}(\eta) \\ \gamma_0^{-1} b_{yx}(\eta) & b_{yy}(\eta) & b_{yz}(\eta) \\ \gamma_0^{-1} b_{zx}(\eta) & b_{zy}(\eta) & b_{zz}(\eta) \end{pmatrix}, \qquad (19)$$

where $b_{ik} = (a^{-1})_{ik}$.

When $\vartheta \ll \gamma_0$ ($\eta \gg 1$), $b_{ik} \cong b_{ik}(\infty)$ are finite quantities, which are generally different from zero. In the limit $\vartheta = 0$ ($\eta = \infty$) Eq. (19) agrees with Eq. (54) of reference 1.

When $\vartheta \gg \gamma_0$ ($\eta \to 0$) the tensor $b_{ik}(\eta)$ becomes

$$b_{ik} = \begin{pmatrix} \eta^2\beta_1 & b_{xy}(0) & \eta\beta_2 \\ b_{yx}(0) & b_{yy}(0) & b_{yz}(0) \\ \eta\beta_2 & b_{zy}(0) & b_{zz}(0) \end{pmatrix}; \qquad (20)$$

and the expressions for ρ_{ik} reduce to (14). In virtue of $\rho_{ik}(H) = \rho_{ki}(-H)$ we then have $b_{xy}(\eta) = -b_{yx}(\eta)$; specifically, $b_{xy}(0) = -b_{yx}(0) = 1/a_{xy}(0)$.

Equations (16) to (20) remain valid even when the Fermi surface breaks up into an undulating cylinder and an arbitrary number of closed regions. When additional open surfaces exist these equations remain valid for those magnetic field directions for which the sections of the additional surfaces are closed.

Since in the expressions for σ_{ik} and ρ_{ik} the coordinate axes were selected in a special manner, it is of interest to write the expression for the resistivity with an arbitrary direction of the current (perpendicular to the magnetic field). This gives

$$\rho = \rho_{xx}\cos^2\alpha + \rho_{yy}\sin^2\alpha - (\rho_{xy} + \rho_{yx})\sin\alpha\cos\alpha = \gamma_0^{-2}b_{xx}(\eta)\cos^2\alpha - \gamma_0^{-1}(b_{xy}(\eta) + b_{yx}(\eta))\sin\alpha\cos\alpha + b_{yy}(\eta)\sin^2\alpha, \quad (21)$$

where α is the angle between the current direction and the x axis.

Taking into account the behavior of $b_{ik}(\eta)$ for $\eta \ll 1$ and $\eta \gg 1$, we represent b_{xx} in the form

$$b_{xx}(\eta) = \frac{\beta_1\eta^2}{1+\lambda^2\eta^2}c(\eta),$$

and for ρ we obtain

$$\rho = \frac{\beta_1\cos^2\alpha}{\vartheta^2 + \lambda^2\gamma_0^2}c(\eta) + A, \quad (22)$$

where A, β_1, λ are smooth functions of the angles and $c(\eta)$ is a smooth function of its argument $\eta = \gamma_0/\vartheta$, with $c(0) = c(\infty) = 1$.

Thus the basic dependence on angles and on the field as the singular line $\vartheta = 0$ is approached is given by (22), or explicitly

$$\rho = \frac{\beta_1 H^2\cos^2\alpha}{\vartheta^2H^2 + \lambda^2H_0^2}c(\eta) + A. \quad (23)$$

Thus as the direction $\vartheta = 0$ is approached there appears a quadratic rise of resistivity with the field, whereas in all other directions saturation is obtained in the fields $H \sim H_0/\vartheta$. For a fixed field H there is a sharp maximum in the angular dependence of resistivity, the width of which is inversely proportional to the field:

$$\Delta\vartheta \sim \gamma_0 = H_0/H.$$

Since this maximum is very sharp in large fields, any averaging over the angles in the interval $\delta\vartheta \gg \gamma_0$ (including $\vartheta = 0$) leads to a linear rise of resistivity with the field:

$$\overline{(\rho)}_{\delta\vartheta} = \frac{\varkappa H\cos^2\alpha}{\delta\vartheta H_0} + A; \qquad \varkappa = \frac{\beta_1}{\lambda}\int_0^\infty \frac{c(\eta/\lambda)}{1+\eta^2}d\eta. \quad (24)$$

In particular, when the resistivity is measured in a thin polycrystalline wire with diameter of the order of the crystallite size the averaged resistivity is observed:

$$\rho = (\bar{\varkappa}/2\pi)H/H_0 + \bar{A} \quad (25)$$

(the bar over κ and A denotes averaging over the smooth angular dependence in the equatorial plane $\vartheta = 0$).

The linear law discovered by Kapitza[4] is possibly associated with similar averaging. It must be emphasized again that because of the narrowness of the maximum the method of averaging does not play an important part; specifically, averaging of the conductivity σ_{ik} followed by a transformation to inverse quantities yields the same result.

It must be noted that conclusions regarding the growth of resistivity near singular field directions [Eq. (23)] remain valid in the presence of few surfaces of the undulating cylinder type with differently directed axes (in view of the fact that for each of the singular field directions open sections occur for only one of these surfaces, which thus determines the behavior of ρ and σ). This can occur for cubic crystals, in which case the singular directions lie in each of the three principal crystallographic planes and the averaging again gives the linear law $\rho \sim H$.

The transverse component of the resistivity for an arbitrary current direction x' will be

$$\rho_{y'x'} = \rho_{yx}\cos^2\alpha - \rho_{xy}\sin^2\alpha + (\rho_{xx} - \rho_{yy})\sin\alpha\cos\alpha.$$

Hence for the Hall coefficient with an arbitrary current direction we obtain

$$R = \frac{\rho_{y'x'}(H) - \rho_{y'x'}(-H)}{2H} = \frac{b_{yx}(\eta) + b_{yx}(-\eta)}{2H_0} = R_0 b(\eta), \quad (26)$$

where R_0, the Hall coefficient for $\vartheta \gg \gamma_0$, is independent of the angles and the field:

$$R_0 = b_{yx}(0)/H_0 = 1/nec \quad (27)$$

(n is the number of electrons per cell) and $b(\eta)$ is a smooth function of η with

$$b(0) = 1; \quad b(\infty) = b_{yx}(\infty)/b_{yx}(0).$$

The characteristic angular dependence of the Hall coefficient on field direction for a fixed H (the rotation diagram) is shown in Fig. 5. $R = R_0$ everywhere except for a narrow region near $\vartheta = 0$, the width of which decreases with the field: $\Delta\vartheta \sim \gamma_0 = H_0/H$. In this region the asymptotic behavior of the Hall coefficient depends on the specific form of the collision integral and cannot be expressed simply.

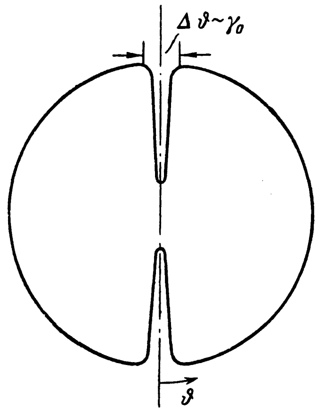


FIG. 5. Dependence of R on the angular coordinate ϑ of H (rotation diagram) for the undulating-cylinder type of surface.

b. "Three-dimensional Grid" Type of Surface

It has already been indicated that this surface (Fig. 2) possesses all of the types of singularities enumerated in Sec. 2. Specifically, the portions of the equatorial lines which do not belong to regions I in the stereographic projection of magnetic field directions (the heavy continuous lines in Fig. 3) are singular lines equivalent to the directions $\vartheta = 0$ for surfaces of the undulating cylinder type. Region III surrounding this line contains strongly stretched closed trajectories; the entire analysis and all results of the preceding section can be applied to this region. The part of ϑ is played by the angle between the magnetic field direction and the nearest principal crystallographic plane.

Singular directions of another type are found in the boundaries of regions I and II (Fig. 3) as well as the principal crystallographic axes. Before investigating these it must be noted that region II contains only closed trajectories located within the limit of a single cell while region I contains open trajectories in addition. Therefore the standard asymptotic forms (13) apply to region II up to its boundary and the steep angular dependence occurs only on the inner side of this boundary.

The asymptotic behavior of the conductivity within region I can be most conveniently investigated by considering separately the contribution to each component of the conductivity σ_{ik} (Eq. (3)) from the band of open trajectories and from all other simple closed lines. It will be rememb́ered that the contribution from simple closed trajectories is given by (13).

To determine the contribution from the band of open trajectories we first note that the magnetic field direction (inside region I) uniquely determines the single common direction of all open trajectories (the intersection of the plane perpendicular to the magnetic field with the nearest crystallographic plane). By selecting this direction as the x axis we obtain a picture similar to the previously discussed case of an undulating cylinder with the magnetic field perpendicular to its axis ($\vartheta = 0$). Therefore the corresponding contribution to σ_{ik} is given by (16), where, however, instead of $a_{ik}(\eta)$ there are constants which are generally proportional to the width of the band of open trajectories (when this is small). The width of the band of open trajectories becomes zero on the boundary between the regions I and II and also in the directions of the principal crystallographic axes. Near these singular directions the width is generally proportional to ϑ', which is the absolute value of the angle between the magnetic field and the I-II boundary (or the direction of the nearest crystallographic axis).

Therefore near the singular directions, retaining in σ_{ik} the principal terms in the small parameters γ_0 and ϑ', we obtain

$$\sigma_{ik} = \begin{pmatrix} \gamma_0^2 a_{xx} & \gamma_0 a_{xy} & \gamma_0 a_{xz} \\ \gamma_0 a_{yx} & \gamma_0^2 a_{yy} + \vartheta' c_2 & \gamma_0 a_{yz} + \vartheta' c_3 \\ \gamma_0 a_{zx} & \gamma_0 a_{zy} + \vartheta' c_3 & a_{zz} \end{pmatrix}; \qquad (28)$$

where a_{ik}, c_i are smooth functions of the angles and are generally quantities of the same order of magnitude.

For the reciprocal tensor ρ_{ik} we have with the same accuracy

$$\rho_{ik} = \begin{pmatrix} b_{xx} + b'_{xx}\vartheta' / \gamma_0^2 & \gamma_0^{-1} b_{xy} & b_{xz} + b'_{xz}\vartheta' / \gamma_0 \\ \gamma_0^{-1} b_{yx} & b_{yy} & b_{yz} \\ b_{zx} + b'_{zx}\vartheta' / \gamma_0 & b_{zy} & b_{zz} \end{pmatrix} \qquad (29)$$

(b_{ik} are smooth functions of the angles with $b_{xy} = 1/a_{xy}$).

To determine the resistivity for an arbitrary current direction and a perpendicular field direction, the components of the tensor ρ_{ik} must again be transformed as in the preceding section [Eq. (21)]. If α is the angle between the current direction and the x direction indicated above, the resistivity ρ is given by

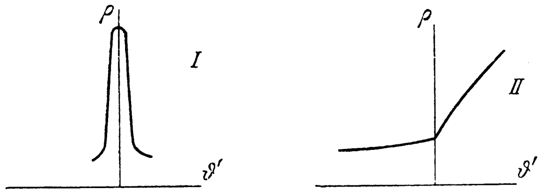
$$\rho = \gamma_0^{-2}\vartheta' b'_{xx} \cos^2\alpha + \gamma_0^{-1} \cos\alpha \sin\alpha \,(b_{yx} + b_{xy}) + b_{xx} \cos^2\alpha + b_{yy} \sin^2\alpha.$$

Assuming that with accuracy to terms $\sim \gamma_0$, $b_{xy} = -b_{yx}$, we have for $H \gg H_0$ ($\gamma_0 \ll 1$)

$$\rho = \vartheta' b_{xx} (H / H_0)^2 \cos^2\alpha + A \qquad (30)$$

(A is a smooth function of the angles which is independent of H).

Thus in regions I the resistivity grows quadratically, disappearing at the boundaries and at the centers of the regions. In the current direction $\cos\alpha = 0$ (perpendicular to the direction of



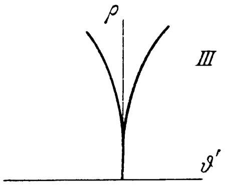


FIG. 6. Angular dependence of ρ near singular points of types I, II, and III.

open trajectories) the resistivity also tends to saturation.

It follows from Eqs. (22) and (30) that in very strong fields ($H \gg H_0$) the angular dependence of the resistivity near directions of the three singular types given at the end of the second section will have the form shown schematically in Fig. 6. A characteristic resistivity rosette containing all three types of singularities is obtained by rotating the field in a plane that passes through one of the crystallographic axes but does not lie too close to a crystallographic plane such as the (110) plane (Fig. 7). The experimental plotting of these angular dependences in different planes enables us to determine the nature of the singular points and thus to "probe" the entire Fermi surface.

We note that all the results obtained remain valid when a metal possesses an arbitrary number of closed Fermi surfaces in addition to one open surface.

As was shown in reference 1, the asymptotic behavior of σ_{yx} together with that of the Hall coefficient R in the case of simple closed trajectories is independent of the form of the collision integral and is associated only with the geometry of the Fermi surface. We are therefore able to calculate R in region II as well as close to singular points of the discussed type.

By means of Eqs. (11) and (12) and by taking into account the invariance of the distribution function under transformations associated with the translational symmetry of the lattice, we obtain for field directions close to the crystallographic axes

$$\sigma_{yx} = -\frac{2ec}{h^3H}\left\{V - d_{\min}b_1b_2\left[1 - \frac{b_1\cos\varphi + b_2\sin\varphi}{d_{\min}}\sin\vartheta\right]\cos\vartheta + u(\vartheta,\varphi)\right\} \tag{31}$$

where V is the total volume of the tubes in one cell, $d_{\min}$ is the minimum diameter of a tube in the direction of a crystallographic axis, ϑ, φ are the angular coordinates of H (see Fig. 2), and $u(\vartheta, \varphi)$ depends on the specific form of the collision integral $\widetilde{W}$, being given by the integral

$$u(\vartheta, \varphi) = -2b_1\int \bar{p}_y(p_z)\,dp_z,$$

taken only over open trajectories. The magnitude of $\bar{p}_y(p_z)$ is determined from the condition (12):

$$\overline{\widetilde{W}\{C_x^{(1)}\}} = \widetilde{W}\left\{f_0'(\varepsilon)\frac{c}{eHT_0}\bar{p}_y(p_z)\right\} = \overline{\widetilde{W}\left\{f_0'(\varepsilon)\frac{c}{eHT_0}p_y(\tau)\right\}}.$$

When $\vartheta = 0$ the open trajectories disappear and $u(\vartheta, \varphi) = 0$. For small ϑ we have $u(\vartheta, \varphi) \sim \vartheta$; for constant tube diameter u decreases more rapidly than ϑ and this term in (31) can be neglected.

We can therefore make the following statements regarding the asymptotic behavior of the Hall coefficient. $R = (\rho_{y'x'}(H) - \rho_{y'x'}(-H))/2H$ is independent of H but is a function of the angles and the form of the collision integral in regions I and III (Fig. 3). In region II

$$R = 1/nec, \tag{32}$$

where $n = 2Vh^{-3}$ is the number of electrons. When the magnetic field is directed along a crystallographic axis the asymptotic Hall coefficient is also determined only by the topology of the Fermi surface and is independent of the form of collision integral. R can be written formally, by analogy with (32):

$$R = 1/n_iec; \quad i = 1, 2, 3, \tag{33}$$

where the numbers n_i do not equal the number of electrons n and are given by

$$n_1 = n - 2h^{-3}d_1b_2b_3; \quad n_2 = n - 2h^{-3}d_2b_1b_3; \quad n_3 = n - 2h^{-3}d_3b_1b_2; \tag{34}$$

d_1, d_2, d_3 are here the minimum diameters of the tubes in the directions of the crystallographic axes 1, 2, 3.

An interesting fact must be noted. As shown in reference 1, the concepts of "electrons" and "holes" can be introduced only for closed surfaces. In the case of closed trajectories on open surfaces we

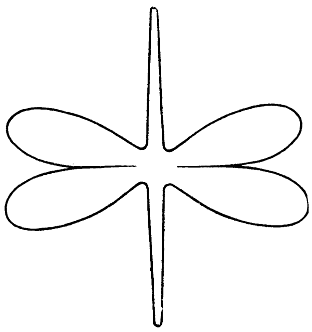


FIG. 7. Rotation diagram ($\rho = \rho(\vartheta)$) with three types of singular points.

can also speak in some sense of two types of motion depending on the direction in which a trajectory is traversed (for closed surfaces this direction differs for "electrons" and "holes"). On an open surface, however, the direction at a single point may depend on the magnetic field direction (that is, on the direction of the intersecting plane). This occurs in the case of the "three-dimensional grid" (Fig. 4). Therefore the "number of carriers" of types I and II depends on the magnetic field direction and is not invariant.

From this point of view the quantities n_i in (33) can be interpreted formally as differences between the numbers of such carriers, in the form $n_i = n_i^I - n_i^{II}$, although we naturally cannot speak of any division into "holes" and "electrons".

The authors are indebted to M. I. Kaganov for valuable discussions.

Note added in proof (October 7, 1958). An angular dependence of the resistivity which becomes linear as the result of averaging, has been determined experimentally very recently by Alekseevskii and Gaidukov [J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 554 (1958), Soviet Phys. JETP **8**, 383 (1959)].

[1] Lifshitz, Azbel', and Kaganov, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 63 (1956), Soviet Phys. JETP **4**, 41 (1957).

[2] A. B. Pippard, Phil. Trans. Roy. Soc. (London) **A250**, 325 (1957).

[3] B. I. Verkin, Doctoral Dissertation, Khar'kov State University, 1956.

[4] P. L. Kapitza, Proc. Roy. Soc. **A123**, 292 (1929).

Translated by I. Emin
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