BINDING ENERGY OF LIGHT HYPERNUCLEI ACCORDING TO MESON THEORY

V. A. LIUL'KA and V. A. FILIMONOV

Moscow State University

Submitted to JETP editor May 31, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 1026-1030 (October, 1958)

The binding energy of Λ particles in the hypernuclei H^3_Λ, H^4_Λ, He^4_Λ, and He^5_Λ has been computed in the second and fourth orders of perturbation theory. The theoretical results are in satisfactory agreement with experiment.

THE intensified study of hyperons in recent years has led to the discovery of many of their properties, such as the fact that the binding energy increases linearly with the atomic number,[1] and, in particular, to a more accurate measurement of the binding energy of the Λ particle in light hypernuclei. This opens new possibilities for the theoretical analysis of hyperon properties from the point of view of elementary particle interactions, say in terms of the π and K meson fields. Several papers[2-4] have investigated the binding energy of hypernuclei from field-theoretical considerations. Filimonov[4] tried to study the binding energy of the Λ particle in hypernuclei by using the elementary-particle systematics of Gell-Mann and Nishijima in the second and fourth orders of perturbation theory with a cutoff of the Chew type. This was done on the assumption that the K meson — baryon interaction is pseudoscalar and that g_Λ, the coupling constant for the interaction between the K meson and the Λ particle, is the same as g_Σ, the coupling constant for the interaction between the K meson and the Σ particle.

There exists at present no definitive data on the parity of K mesons or on the relative parity of baryons. For this reason theoretical investigations have assumed both scalar and pseudoscalar interactions between K mesons and baryons. Further, the study of K^+ meson scattering by nucleons leads to the conclusion that the interaction between the K^+ and the nucleon in the state with isotopic spin $T = 1$ is much stronger than the interaction in the state with $T = 0$. It has been suggested[5] that this is because $g_\Lambda \neq g_\Sigma$.

We shall use meson theory to study the binding energy of Λ particles in light hyperons, assuming that the K meson — baryon interaction is scalar, but making no assumptions as to the equality of g_Λ and g_Σ. The method used will be that of Filimonov.[4] We start with the interaction Hamiltonian[6]

$$\begin{aligned}\mathscr{H} = {} & iG_\pi \{\bar{N}\gamma_5\boldsymbol{\tau\pi}N + \bar{\Lambda}\gamma_5\boldsymbol{\pi\Sigma} + \bar{\Sigma}\boldsymbol{\pi}\gamma_5\Lambda + [\bar{\Sigma}\gamma_5\ \Sigma]\boldsymbol{\pi} \\ & + \bar{\Xi}\gamma_5\boldsymbol{\tau\pi}\Xi\} + G_\Lambda(\bar{N}K\Lambda + \textbf{Herm. conj.}) \\ & + G_\Sigma(\bar{N}\boldsymbol{\tau}\Sigma K + \textbf{Herm. conj.}) + G_{\Xi\Lambda}(\bar{\Xi}\tau_2K^*\Lambda + \textbf{Herm. conj.}) \\ & + G_{\Xi\Sigma}(\bar{\Xi}\tau_2\boldsymbol{\tau}\Sigma K^* + \textbf{Herm. conj.}).\end{aligned} \tag{1}$$

We assume that the coupling constants for the π meson — baryon interaction are equal, in agreement with symmetry considerations for strong interactions.[7] For baryons fixed at the points x_n, the interaction Hamiltonian becomes

$$\begin{aligned}\mathscr{H} = {} & \sqrt{4\pi}\, g_\pi \frac{i}{2V^{1/2}} \sum_{n=1}^{N}\sum_k\sum_{i=1}^{3} v(k)(\sigma_n\cdot\mathbf{k})\, T_i^{(\pi)(n)} \\ & \times \hat{a}^{(n)}(2\omega^{(\pi)})^{-1/2}(a_{ik} + \overset{*}{a}_{i,-k})\, e^{i\mathbf{k}\cdot\mathbf{x}_n} \\ & + \sqrt{4\pi}V^{-1/2}\Big\{\sum_{n=1}^{N}\sum_k\sum_{i=1}^{2} v(k)\, T_i^{(K)(n)}(2\omega^{(K)})^{-1/2} \\ & \times (c_{ik} + \overset{*}{b}_{ik})\, e^{i\mathbf{k}\cdot\mathbf{x}_n} + \textbf{Herm. conj.}\Big\}.\end{aligned} \tag{2}$$

On going from (1) to (2) we have transformed from pseudoscalar coupling to pseudovector coupling; $v(k)$ is a function which gives the cutoff for the virtual meson momenta, and $\hat{a}^{(n)}$ is a diagonal matrix whose components are the reciprocals of the baryon masses.[4] The eight-by-eight matrices $T_i^{(\pi)}$ and $T_i^{(K)}$ have the following nonzero matrix elements:

$$\begin{aligned} & T_1^{(\pi)}:\ a_{12} = a_{21} = a_{34} = a_{43} = -ia_{56} = ia_{65} = a_{78} = a_{87} = 1; \\ & T_2^{(\pi)}:\ ia_{12} = -ia_{21} = a_{35} = a_{53} = ia_{46} \\ & \quad = -ia_{64} = ia_{78} = -ia_{87} = 1; \\ & T_3^{(\pi)}:\ a_{11} = -a_{22} = a_{36} = a_{63} = -ia_{45} \\ & \quad = ia_{54} = a_{77} = -a_{88} = 1; \end{aligned} \tag{3}$$

$$\begin{aligned} & T_1^{(K)}:\ a_{13} = g_\Lambda;\ a_{24} = -ia_{25} = a_{16} = g_\Sigma;\ a_{38} = -ig_{\Xi\Lambda}; \\ & \quad -ia_{47} = a_{57} = -a_{68} = g_{\Xi\Sigma}; \\ & T_2^{(K)}:\ a_{23} = g_\Lambda;\ a_{14} = ia_{15} = -a_{26} = g_\Sigma;\ a_{37} = ig_{\Xi\Lambda}; \\ & \quad ia_{48} = a_{58} = a_{67} = g_{\Xi\Sigma}; \end{aligned} \tag{4}$$

$g^2 = G^2/4\pi$ (we set $\hbar = 1$ and $c = 1$), and a_i, c_i, and b_i, and a_i^*, c_i^*, and b_i^* are annihilation and creation operators for the π_i, K_i, and $\widetilde{K}_i$ mesons respectively.

Equation (2) can be used to obtain the Λ-nucleon interaction potentials in the second and fourth orders of perturbation theory by a previously developed method.[4,8] We call V^{1K} the potential due to exchange of a single virtual K meson, $V^{K\pi}$ the potential due to exchange of a π and K meson, V^{2K} the potential due to exchange of two K mesons, and $V^{2\pi}$ the potential due to exchange of two π mesons. These potentials are

$$V^{1K} = -g_\Lambda^2 \frac{4\pi}{(2\pi)^3}(\lambda_1\lambda_2^* + \lambda_2\lambda_1^*)\int\left\{v^2(k)\, e^{i\mathbf{k}(\mathbf{x}_1-\mathbf{x}_2)} / \omega^{(K2)}\,(d\mathbf{k})\right\}; \quad (5)$$

$$V^{K\pi} = -g_\pi^2 \frac{3(4\pi)^2}{4(2\pi)^6 M_\Lambda M_N}(\lambda_1\lambda_2^* + \lambda_2\lambda_1^*)$$
$$\times \iint v_1^2(k)\, v_2^2(\mathbf{k})\,(\boldsymbol{\sigma}_1\cdot\mathbf{k}_2)(\boldsymbol{\sigma}_2\cdot\mathbf{k}_2)\, e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{x}_1-\mathbf{x}_2)}$$
$$\times\Big\{\frac{1}{2}(g_\Lambda^2 + g_\Sigma^2)\,(\omega_1^{(\pi)-3}\omega_2^{(K)-2} + \omega_2^{(\pi)-2}\omega_1^{(K)-3})$$
$$-\frac{1}{2}(g_\Lambda - g_\Sigma)^2 / \omega_2^{(\pi)2}(\omega_2^{(\pi)} + \omega_1^{(K)})\,\omega_1^{(K)2}\Big\}(d\mathbf{k}_1)(d\mathbf{k}_2); \quad (6)$$

$$V^{2K} = -\frac{(4\pi)^2}{4(2\pi)^6}\iint v_1^2(k)\, v_2^2(k)\, e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{x}_1-\mathbf{x}_2)}$$
$$\times\{4g_{\Xi\Lambda}^2(g_\Lambda^2 + 3g_\Sigma^2)\,(\omega_1^{(K)3}\,(\omega_1^{(K)} + \omega_2^{(K)})\,\omega_2^{(K)})^{-1}$$
$$+[2g_{\Xi\Lambda}^2(g_\Lambda^2 + 3g_\Sigma^2) + 2g_\Lambda^2(g_\Lambda^2 + 3g_\Sigma^2)]$$
$$\times(\omega_1^{(K)2}(\omega_1^{(K)} + \omega_2^{(K)})\,\omega_2^{(K)2})^{-1}\}(d\mathbf{k}_1)(d\mathbf{k}_2). \quad (7)$$

The operators λ and λ^* transform the Λ particle into a nucleon, and the nucleon into a Λ particle, respectively. The expression for V^{2K} is simplified if we set $g_\Lambda = g_\Xi$, which is not in contradiction with the presently available data.

The potential due to two π mesons has been studied previously by one of the authors.[4] In order to calculate the interaction between the Λ particle and a nucleon in the core nucleus, we choose the nucleon and Λ particle wave functions in the form

$$\varphi_N(r) = (2\pi\gamma^2 r_c^2)^{-3/4}\exp[-r^2/4\gamma^2 r_c^2],$$
$$\varphi_\Lambda(r) = \eta^{3/2}(2\pi\gamma^2 r_c^2)^{-3/4}\exp[-r^2\eta^2/4\gamma^2 r_c^2], \quad (8)$$
$$r_c = \hbar/m_\pi c = 1.4\cdot 10^{-13}\,\text{cm},$$

where η is a parameter of variation.

Equation (8) can be used to calculate the potential energy in the singlet and triplet state of the Λ-nucleon pair for the forces given by Eqs. (5) to (7). We obtain

$$E_{\mathbf{singl}}^{1K} = -E_{\mathbf{tripl}}^{1K}$$
$$= -g_\Lambda^2 m_\pi c^2\left\{\frac{1}{\sqrt{\pi}\gamma_1} + [\Phi(\gamma_1\nu) - 1]\,\nu e^{\gamma_1^2\nu^2}\right\}, \quad (9)$$

$$E_{\mathbf{singl}}^{K\pi} = 3E_{\mathbf{tripl}}^{K\pi} = g_\pi^2\frac{m_\pi^2}{M_\Lambda M_N}\frac{3}{4\pi^2\gamma_1^2}$$
$$\times m_\pi c^2\left\{\frac{g_\Lambda^2 + g_\Sigma^2}{2}F_1^{K\pi}(\gamma_1) - \frac{(g_\Lambda - g_\Sigma)^2}{2}F_2^{K\pi}(\gamma_1)\right\}, \quad (10)$$

$$F_1^{K\pi}(\gamma_1) = \int_0^{x_m'}\int_0^{x_m''} x_1x_2^3\varphi(\gamma_1, x_1, x_2)[(x_1^2+\nu^2)^{-1}(x_2^2+1)^{-3/2}$$
$$+(x_1^2+\nu^2)^{-3/2}(x_2^2+1)^{-1}]\,dx_1\,dx_2, \quad (11)$$

$$F_2^{K\pi}(\gamma_1) = \int_0^{x_m'}\int_0^{x_m''}\frac{x_1x_2^3\,\varphi(\gamma_1, x_1, x_2)}{(x_1^2+\nu^2)(x_2^2+1)(\sqrt{x_2^2+1}+\sqrt{x_1^2+\nu^2})}\,dx_1\,dx_2, \quad (12)$$

$$\varphi(\gamma_1, x_1, x_2) = \exp\{-\gamma_1^2(x_1-x_2)^2\} - \exp\{-\gamma_1^2(x_1+x_2)^2\}, \quad (13)$$

$$E_{\mathbf{singl}}^{2K} = E_{\mathbf{tripl}}^{2K} = -g_\Lambda^2\frac{g_\Lambda^2 + 3g_\Sigma^2}{\pi^2}\frac{m_\pi c^2}{\gamma_1^2}F^{2K}(\gamma_1), \quad (14)$$

$$F^{2K}(\gamma_1) = \int_0^{x_m'}\int_0^{x_m''}\frac{x_1x_2\varphi(\gamma_1, x_1, x_2)}{(x_1^2+\nu^2)(x_1^2+\nu^2)^{3/2}}\,dx_1\,dx_2. \quad (15)$$

The expression for $E^{2\pi}$ was calculated previously,[4] and it is

$$x = r_ck,\quad x_m = r_ck_m,\quad \gamma_1^2 = \frac{1+\eta^2}{2\eta^2}\gamma^2,\quad \nu = \frac{m_K}{m_\pi};$$

where k_m is the cutoff momentum. We are using a rectangular cutoff. The second term in (10) vanishes if $g_\Lambda = g_\Sigma$. When $g_\Lambda^2 = 3g_\Sigma^2$, it is about 1/30 of the first term and is therefore omitted in the calculations. The integrals $F_1^{K\pi}(\gamma_1)$ and $F_1^{2K}(\gamma_1)$ are found by numerical integration for $x_m = 6$.

For several values of γ_1^2 these integrals are

γ_1^2	0.75	1.00	1.25	1.50	1.75
$F^{K\pi}$	1.730	1.524	1.375	1.268	1.181
F^{2K}	0.0318	0.0279	0.0253	0.0232	0.0216

The kinetic energy of the Λ particle is

$$E_{\mathbf{kin}} = {}^3/_8\,\eta^2(m_\pi / M_\Lambda'\gamma^2)\, m_\pi c^2, \quad (16)$$

where M_Λ' is the reduced mass of the Λ particle and the core.

A direct calculation of the potential energy of interaction in the singlet and triplet states of the Λ-nucleon pair shows that the singlet state is preferred. As in the case of pseudoscalar coupling, therefore, light hypernuclei in the ground state have the lowest possible spin.

Although experiments have been suggested for finding the spin of hypernuclei,[9,10] there exists as yet no relevant experimental data.

If the spins of H_Λ^3, H_Λ^4, He_Λ^4, and He_Λ^5 are $\frac{1}{2}$, 0, 0, and $\frac{1}{2}$, respectively, we obtain the following expressions for the energies of the Λ particles in the hypernuclei, as functions of the parameter of variation η:

$$H_\Lambda^3: E = E_{\mathbf{kin}} + 2E_c^{2\pi} + E_c^{1K} + \frac{5}{3}E_c^{K\pi} + 2E_c^{2K},$$
$$H_\Lambda^4,\ He_\Lambda^4: E = E_{\mathbf{kin}} + 3E_c^{2\pi} + 2E_c^{K\pi} + 3E_c^{2K}, \quad (17)$$
$$He_\Lambda^5: E = E_{\mathbf{kin}} + 4E_c^{2\pi} - 2E_c^{1K} + 2E_c^{K\pi} + 4E_c^{2K}.$$

In $E^{2\pi}$ we have neglected the part depending on the spin of the hypernuclei.

The binding energy of the Λ particle is $B_\Lambda = -E_{min}$, the minimum of the values in (17). We have found B_Λ for the four above hypernuclei using $g_\pi^2 = 16$, a value obtained from nucleon-nucleon forces, π meson photoproduction, etc.[11]

We shall make the two assumptions that $g_\Lambda^2 = g_\Sigma^2$ and $g_\Lambda^2 = 3g_\Sigma^2$. With the second of these assumptions the scattering amplitude of K^+ mesons on nucleons vanishes in the second order of perturbation theory for $T = 0$, as is required by experiment.[5] Further, Minami[13] has studied the relation between the K^+-proton scattering cross section with charge exchange and the sum of the scattering cross sections without charge exchange. If the ratio between these values is taken from experiment (about 0.2), we find that $x = g_\Sigma^2/g_\Lambda^2 \approx 0.36$, which is in good agreement with the calculations.[5]

The results of the calculations are shown in the table. We have used the same values of γ^2 as before.[4]

Hypernucleus		H_Λ^3	H_Λ^4, He_Λ^4	He_Λ^5
B_Λ, Mev	$g_\Lambda^2=1.0$	<0	0.8	3.0
$g_\Lambda^2=g_\Sigma^2$	$g_\Lambda^2=1.2$	<0	0.9	2.7
B_Λ, Mev	$g_\Lambda^2=1.0$	~ 0	1.2	2.2
$g_\Lambda=3g_\Sigma^2$	$g_\Lambda^2=1.2$	0.1	1.1	2.1
B_Λ, Mev, exp.[12]		0.25 ± 0.31	1.44 ± 0.20 1.70 ± 0.24	2.56 ± 0.17

It is seen from the table that if we assume that $g_\Lambda^2 = 3g_\Sigma^2$, we obtain agreement with the experimental hypernucleon binding energies for g_Λ^2 between about 1.0 and 1.2, the range given by K^+-nucleon scattering experiments.[5] The assumption that $g_\Lambda^2 = g_\Sigma^2$ leads to a binding energy which increases more rapidly with the number of nucleons in the nucleus than does the result obtained for $g_\Lambda^2 = 3g_\Sigma^2$, and thus does not agree with the experimental data.

With our simplifying assumptions we cannot, of course, expect complete quantitative agreement with experiment; the overall agreement is, however, quite satisfactory.

The authors express their gratitude to Professor D. D. Ivanenko for encouragement during the performance of this work.

[1] D. D. Ivanenko and N. N. Kolesnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 800 (1956), Soviet Phys. JETP **3**, 955 (1956); R. Gatto, Nuovo cimento **3**, 499 (1956); G. Wentzel, Phys. Rev. **101**, 835 (1956).

[2] D. Lichtenberg and M. Ross, Phys. Rev. **103**, 1131 (1956).

[3] M. Dallaporta and F. Ferrari, Nuovo cimento **5**, 123 (1957).

[4] V. A. Filimonov, Научн. докл. Высш. школы (Sci. Papers of the Higher Schools) (in press).

[5] C. Ceolin and L. Teffaro, Nuovo cimento **5**, 435 (1957).

[6] A. Salam, Nuclear Phys. **2**, 173 (1956).

[7] M. Gell-Mann, Phys. Rev. **106**, 298 (1957).

[8] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).

[9] K. Dalitz, Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956 (Interscience, New York, 1956).

[10] Chou Kuang-Chao and M. I. Shirokov, Nuclear Phys. **6**, 10 (1958).

[11] G. Chew, Proceedings of the Seventh Annual Rochester Conference on High-Energy Physics, 1957 (Interscience, New York, 1957).

[12] V. Telegdi, Proceedings of the Seventh Annual Rochester Conference on High-Energy Physics, 1957 (Interscience, New York, 1957).

[13] S. Minami, Progr. Theoret. Phys. **16**, 66 (1956).

Translated by E. J. Saletan
214