RELATIVISTIC THEORY OF POLARIZATION EFFECTS

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A relation is obtained which connects the product of the wave functions of two free relativistic particles of given masses and spins with the wave function which describes the free motion of the system as a whole. This relation is the analogue of the Clebsch-Gordan relation for the composition of angular momenta. By means of this relation one can obtain formulas for relativistic polarization and correlation effects. Lorentz transformations are obtained for arbitrary tensor moments.*

1. In recent years a number of papers (for example references 2 and 3) have given general formulas expressing the cross section, the polarization, and other quantities characterizing the scattering of spinning particles, in terms of the matrix elements of the S matrix. Analogous formulas also exist for the correlation effects. The purpose of the present paper is to give the relativistic generalization of these results to the case of arbitrary spins of the colliding (or correlated) particles.

In the theory of angular momenta a large part is played by the Clebsch-Gordan expansion, which connects the product of the wave functions of two physical systems (with given angular momenta) with the wave function relating to the total angular momentum of the system as a whole. In dealing with collisions of two relativistic particles one has also to compound four-momenta and four-dimensional angular momenta. We shall show that a relation analogous to the Clebsch-Gordan expansion can be derived for the simultaneous composition of these quantities. For this we require the wave functions describing the free motion of relativistic particles, and the Lorentz transformation of these functions.

2. A nonrelativistic particle of mass κ and spin i is described by a wave function

$$\psi_{m_i}^{\varkappa i \alpha}(\mathbf{p}), \tag{1}$$

in which the arguments are the momentum $\mathbf{p}$ ($-\infty < p_1, p_2, p_3 < \infty$) and the spin component m_i ($m_i = -i, -i+1, \ldots i$). Besides its mass and spin, the particle can possess other invariant characteristics (for example charge), which we shall identify by the index α. On going over to the relativistic case we first have the question of what variables must serve as the arguments of the wave function of the particle. A unique answer (apart from the possibility of a similarity transformation) to this question is found in the theory of the representations of the inhomogeneous Lorentz group: the free motion of a particle of mass κ and spin i is described by a wave function (1) that depends on the same kinematic variables $\mathbf{p}$, m_i as in the nonrelativistic case. The transformations for space and time displacements and for spatial and Lorentz rotations are given by the infinitesimal transformations with the respective operators $\hat{\mathbf{p}}$, $\hat{p}_0$, $\hat{\mathbf{M}}$, $\hat{\mathbf{N}}$. For the wave function (1), according to references 4 and 5, these operators have the forms

$$\hat{\mathbf{p}} = \mathbf{p}, \quad \hat{p}_0 = e_p \equiv \sqrt{p^2 + \varkappa^2}, \tag{2}$$

$$\mathbf{M} = i\left[\mathbf{p}\frac{\partial}{\partial \mathbf{p}}\right] + \mathbf{i}, \quad \mathbf{N} = i\sqrt{e_p}\,\frac{\partial}{\partial \mathbf{p}}\sqrt{e_p} - \frac{[\mathbf{i}\times\mathbf{p}]}{e_p + \varkappa}, \tag{3}$$

where

$$[\mathbf{i}_1\mathbf{i}_2]_- = i\mathbf{i}_3, \ \ldots\ldots \ \mathbf{i}^2 = i(i+1). \tag{4}$$

Square brackets denote the vector product, and square brackets with minus sign as subscript denote the commutator. The slight difference between the forms of the operator $\mathbf{N}$ in Eq. (3) and in Eq. (11) of reference 5 comes from the fact that the corresponding wave functions differ by a factor $e_p^{1/2}$. In the present paper the wave functions $\psi_{m_i}^{i}(\mathbf{p})$ are so chosen that the invariant normalization integral has the form

$$\sum_{m_i}\int d^3p\,\psi_{m_i}^{*\varkappa i}(\mathbf{p})\,\psi_{m_i}^{\varkappa i}(\mathbf{p}).$$

The operators defined in Eqs. (2) and (3) satisfy

*The formula for the relativistic polarization effects has also been obtained independently by Chou Kuang Chao and M. I. Shirokov at the Joint Institute for Nuclear Research.[1]

the well known commutation relations, which secure relativistic invariance,[6] and this proves the correctness of the choice of the wave functions (1). The uniqueness and completeness of the treatment is assured by the fact that the operators (2) and (3), taken for $0 < \kappa < \infty$ and for all integral and half-integral non-negative i, exhaust all the possible irreducible representations of the Lorentz group that are suitable for the description of free particles[7] (expect particles with zero rest mass, which require some special consideration).

3. Let us construct for the wave function (1) the operator of a finite Lorentz transformation to a system moving with the four-velocity $u_\mu = (\mathbf{u}, iu_0)$ relative to the initial system. This transformation has the form:

$$\mathbf{x} = \mathbf{x}' + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{x}')}{u_0+1} + \mathbf{u}t', \qquad t = \mathbf{u}\mathbf{x}' + u_0 t'. \tag{5}$$

To the transformation (5) there will correspond a transformation U of the wave function,

$$\psi = U(u)\psi'. \tag{6}$$

The explicit form of the transformation (6) is uniquely determined by the relations (2), (3) and can be found, for example, by the method indicated in reference 8, which gives

$$\psi_{m_i}(\mathbf{p}) = \int d^3p'\delta\{\mathbf{p}' - \mathbf{p}'(\mathbf{p})\}\sqrt{e_{p'}/e_p}D^i_{m_i m_i'}(\mathbf{p},\mathbf{u})\psi_{m_i}(\mathbf{p}') = \sqrt{e_{p'}/e_p}D^i_{m_i m_i'}(\mathbf{p}\cdot\mathbf{u})\psi'_{m_i'}\{\mathbf{p}'(\mathbf{p})\}, \tag{7}$$

where

$$\mathbf{p}'(\mathbf{p}) = \mathbf{p} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{p})}{u_0+1} - \mathbf{u}e_p \tag{8}$$

is the transformation for the four-momentum, inverse to the transformation (5), and $D^i_{m_i m_i'}(\mathbf{p},\mathbf{u})$ is the matrix of the three-dimensional rotation for the spin. (We remark that the indices $m_i m_i'$ of this D^i are transposed as compared with the corresponding indices in reference 9). This rotation can be specified by an orthogonal matrix a_{ij} which defines a rotation for an arbitrary three-vector b_i:

$$b_i = a_{ij}b_j';$$

the matrix a_{ij} turns out to be given by

$$a_{ij} = \delta_{ij} + \frac{p_i p_j(u_0-1)}{e_p + \varkappa(e_p u_0 - \mathbf{pu} + \varkappa)} + \frac{u_i p_j}{e_p u_0 - \mathbf{pu} + \varkappa} + \frac{p_i u_j(-e_p u_0 + 2\mathbf{up} - u_0\varkappa - e_p - \varkappa)}{(e_p+\varkappa)(e_p u_0 - \mathbf{pu} + \varkappa)(u_0+1)} - \frac{u_i u_j(e_p - \varkappa)}{(u_0+1)(e_p u_0 - \mathbf{pu} + \varkappa)} \tag{9}$$

It is obvious that the rotation is around the axis perpendicular to the vectors $\mathbf{p}$, $\mathbf{u}$. From the matrix a_{ij} one can determine the matrix $D^i_{m_i m_i'}$ for an arbitrary spin by the well-known relation

$$i_j = a_{jk}(D^i)^{-1}i_k D^i. \tag{10}$$

For spin $\frac{1}{2}$ the matrix $D^{1/2}$ has been determined in reference 8 and is given by

$$D^{1/2} = \frac{(e_p+\varkappa)(u_0+1) - (\mathbf{u}\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma})}{\sqrt{2(u_0+1)(e_p+\varkappa)(u_0 e_p - \mathbf{up} + \varkappa)}}. \tag{11}$$

For spin 1 the matrix D^1 differs from a_{ij} only by the similarity transformation W:

$$W = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{pmatrix}, \tag{12}$$

$$D^1 = WaW^{-1}. \tag{13}$$

For higher spins (for example $\frac{3}{2}$) it is convenient to calculate the matrix D^i not from the relation (10) but from the matrices D^j for lower spins:

$$\sum_{Jm_J}(iIm_i m_I \mid iIJm_J')D^J_{m'_J m_J} = \sum_{m'_i m'_I} D^i_{m'_i m_i}D^I_{m'_I m_I}(iIm'_i m'_I \mid iIJm_J), \tag{14}$$

where $(iIm_i m_I \mid iIJm_J')$ are the vector-composition coefficients.

The above-described matrix D^i for the rotation of the spin in a Lorentz transformation is a purely relativistic effect (Thomas precession) without any nonrelativistic analogue.

4. The Lorentz transformation (7) for the wave function (1), as developed in the preceding section, provides a possibility for obtaining the basic formula of the present paper, the transformation from the wave functions of two free particles to the wave function of the system as a whole.

The wave function describing a system of two particles with masses κ_1 and κ_2 and spins i and I has the form

$$\psi^{\varkappa_1\varkappa_2 iI}_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2). \tag{15}$$

The wave function Φ of the system as a whole must depend on the total mass κ, the total intrinsic angular momentum J, the component of the total intrinsic angular momentum, m_J, and the total momentum $\mathbf{K}$; these quantities are defined by the relations:

$$\mathbf{K} = \mathbf{p}_1 + \mathbf{p}_2, \qquad \varkappa^2 = (e_{p_1} + E_{p_2})^2 - \mathbf{K}^2, \tag{16}$$

$$e_p = \sqrt{p^2 + \varkappa_1^2}, \quad E_p = \sqrt{p^2 + \varkappa_2^2}, \tag{17}$$

$$\mathbf{J} = \mathbf{M}_1 + \mathbf{M}_2, \qquad \mathbf{J}^2\Phi = J(J+1)\Phi. \tag{18}$$

The relations (18) are prescribed in the center-of-mass system (c.m.s.), in which $\mathbf{p}_1 + \mathbf{p}_2 = 0$.

The six variables κ, $\mathbf{K}$, J, m_J are not, however, sufficient for the description of the system (15), since the wave function (15) depends on the eight variables $\mathbf{p}_1$, $\mathbf{p}_2$, m_i, m_I. The two missing variables can of course be chosen to be invariants. For these two variables it is convenient to choose the orbital angular momentum l and the sum of spins s, taken in the c.m.s.

$$\mathbf{J} = \mathbf{l} + \mathbf{s}, \tag{19}$$

$$\mathbf{l} = \mathbf{l}_1 + \mathbf{l}_2, \qquad \mathbf{l}^2\Phi = l(l+1)\Phi, \tag{20}$$

$$\mathbf{s} = \mathbf{i} + \mathbf{I}, \qquad \mathbf{s}^2\Phi = s(s+1)\Phi. \tag{21}$$

All the quantities in Eqs. (19) to (21) are also taken in the c.m.s. We now have everything required to obtain the transition from the two-particle function $\psi_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2)$ to the function $\Phi^{\kappa J l s}_{m_J}(\mathbf{K})$ describing the system as a whole. This transition will consist of the following transformations: By means of the respective transformations D^i, D^I defined in Eqs. (9) and (10) one translates the two spins into the c.m.s.; one then has further to go over from the momenta $\mathbf{p}_1$, $\mathbf{p}_2$ to the total momentum $\mathbf{K}$ and the half-difference of momenta $\mathbf{p}$ in the c.m.s. Since in this system the momenta of the particles are equal in magnitude and opposite in direction, the momentum $\mathbf{p}$ is equal to the momentum of the first particle transformed to the c.m.s. by means of Eq. (8):

$$\mathbf{p} = \mathbf{p}_1 + \frac{\mathbf{K}(\mathbf{K}\cdot\mathbf{p}_1)}{e_p + \varkappa} - \frac{\mathbf{K}}{\varkappa} e_{p_1}, \tag{22}$$

where the quantities $\mathbf{K}$, e_{p_1}, κ are determined in accordance with Eqs. (16) and (17). The matrix of this transformation is

$$\langle \mathbf{p}_1 | \mathbf{p} \rangle = \sqrt{e_{p_1}/e_p} \tag{23}$$

in consequence of the well known relation

$$d^3p_1/e_{p_1} = d^3p/e_p = \text{inv}. \tag{24}$$

If we write the momentum $\mathbf{p}$ in spherical coordinates p, ϑ, φ, then the angular variables will describe the relative orbital motion and will be related to the angular momentum l defined in Eq. (20) and its component m_l,

$$(\vartheta\varphi \mid l m_l) = Y_{l m_l}(\vartheta\varphi), \tag{25}$$

while the absolute value p can be related to the total mass κ:

$$\varkappa = e_p + E_p. \tag{26}$$

From Eq. (26) it follows that

$$p^2 dp = d\varkappa\, p e_p E_p / (e_p + E_p), \tag{27}$$

so that

$$(\mathbf{p} \mid \varkappa l m_l) = \sqrt{\frac{p e_p E_p}{e_p + E_p}}\, Y_{l m_l}(\vartheta\varphi). \tag{28}$$

Now the spins i, I and the orbital angular momentum l have been transformed to the c.m.s. We must compound these three angular momenta according to the relation

$$\mathbf{J} = (\mathbf{i} + \mathbf{I}) + \mathbf{l} \tag{29}$$

by means of the proper Clebsch-Gordan coefficients. On combining the transformations calculated above, we can write the desired relation for the transition from ψ to Φ:

$$\begin{aligned} \psi^{\varkappa_1\varkappa_2 iI}_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2) &= \sum D^i_{m_i m_i'}(\mathbf{p}_1, \mathbf{K})\, D^I_{m_I m_I'}(\mathbf{p}_2\mathbf{K})\, Y_{l m_l}(\vartheta, \varphi) \\ &\times (iIm_i'm_I' \mid iIsm_s)(lsm_lm_s \mid lsJm_J) \\ &\times \sqrt{p e_{p_1} E_p/(e_p + E_p)}\, \Phi^{\varkappa J l s \varkappa_1\varkappa_2 iI}_{m_J}(\mathbf{K}) \\ &\equiv (\mathbf{p}_1\mathbf{p}_2 m_i m_I \mid V(\varkappa_1\varkappa_2 iI) \mid \varkappa J l s \mathbf{K} m_J)\, \Phi^{\varkappa J l s \varkappa_1\varkappa_2 iI}_{m_J}(\mathbf{K}). \end{aligned} \tag{30}$$

The summation is taken over the indices m_i', m_I', m_l, m_s, m_J, l, s, J. The law of conservation of the four-momentum is taken into account by the relations (16) and (20). Mathematically the relation (30) is the expansion of the direct product of two irreducible representations of class P^s_{+m} (cf. reference 5) of the inhomogeneous Lorentz group in terms of irreducible representations. It is entirely analogous to the Clebsch-Gordan expansion for the three-dimensional rotation group. By means of the expansion (30) the angular distributions for arbitrary polarization and correlation effects can be expressed in terms of the S matrix parametrized in an invariant way.

5. As an example let us consider the case in which a particle of mass κ_1 and spin i is incident on a particle of mass κ_2 and spin I, and the result is the emergence of particles with masses κ_1', κ_2' and spins i′, I′. It is obvious that the wave function $\psi^{\kappa_1\kappa_2 iI}_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2)$ of the initial state will be related to the wave function $\chi^{\kappa_1\kappa_2 i'I'}_{m_i' m_I'}(\mathbf{p}_1', \mathbf{p}_2')$ of the final state in the following way:

$$\begin{aligned} \chi^{\varkappa_1'\varkappa_2' i'I'\alpha'}_{m_i' m_I'}(\mathbf{p}_1'\cdot\mathbf{p}_2') &= (\mathbf{p}_1'\cdot\mathbf{p}_2' m_i' m_I' \mid V(\varkappa_1'\varkappa_2' i'I') \mid \varkappa J l' s' \mathbf{K} m_J) \\ &\times (\alpha' l' s' \varkappa_1'\varkappa_2' i'I' \mid S(\varkappa J) \mid \alpha l s \varkappa_1\varkappa_2 iI) \end{aligned}$$

$$(\varkappa J l s \mathbf{K} m_J \mid V^{-1}(\varkappa_1\varkappa_2 iI) \mid \mathbf{p}_1\mathbf{p}_2 m_i m_I)\, \psi^{\varkappa_1\varkappa_2 iI\alpha}_{m_i m_I}(\mathbf{p}_1\mathbf{p}_2), \tag{31}$$

where α, α' are indices for the channels, $(\ldots\ldots V^{-1}\ldots\ldots)$ is the matrix inverse to the corresponding matrix in Eq. (30), and $(\alpha' l' s' \kappa_1'\kappa_2' i'I' \mid S(\kappa J) \mid \alpha l s \kappa_1\kappa_2 iI)$ is the S matrix in invariant form. In virtue of the conservation laws the S matrix does not depend on **K**, m_J and is diagonal in the total mass κ and the total intrinsic angular momentum J. In practice it is of course more convenient to consider instead of the variable κ the energy of the incident particle, and this changes the normalization factor. The formula (28) differs in the following essential ways from the corresponding nonrelativistic expression: firstly, in the transformations V, V^{-1} the momenta are changed according to the relativistic relation (22), which is different from the nonrelativistic relation

$$\mathbf{p} = \mathbf{p}_1 - (\mathbf{K}/\varkappa)\varkappa_1 \quad \text{(nonrel.)} \qquad (32)$$

and secondly, the transformation V contains the matrices D^i, D^I of the relativistic spin rotations. Finally, in Eq. (31) all the energy dependences are expressed in terms of masses.

According to Eq. (9) the rotation D^i (or D^I) becomes a unit matrix if the three-momentum of the particle is parallel to the three-momentum of the system as a whole. Therefore, in particular, the rotations D^i, D^I are absent from the matrix V^{-1} in Eq. (31) if in the initial state one of the particles is at rest, or if the momenta of the two particles are parallel.

Let us now go over from the wave function $\psi^{\kappa_1\kappa_2 iI}_{m_i m_I}(\mathbf{p}_1, \mathbf{p}_2)$ to the corresponding density matrix $(m_i' m_I' | \rho^{\kappa_1\kappa_2 iI}(\mathbf{p}_1, \mathbf{p}_2) | m_i m_I)$. For simplicity we give the further developments in symbolic form, without writing out the indices of the matrices. The relation (31) thus takes the form

$$\chi = VSV^{-1}\psi. \qquad (33)$$

Corresponding to this the relation between the density matrices ρ of the incident wave and ρ_{sc} of the scattered wave is written in the form

$$\rho_{sc} = V(S-1)V^{-1}\rho V(S^+ - 1)V^{-1}. \qquad (34)$$

As is well known,[3] the relation (34) contains a complete description of the scattering of polarized particles. For example, the angular distribution, with a suitable normalization, is given by

$$d\sigma / d\Omega = \mathrm{Sp}\,\rho_{sc} = \mathrm{Sp}\,\{V(S-I)V^{-1}\rho V(S^+ - I)V^{-1}\}. \qquad (35)$$

The polarization **P** is given by

$$\mathbf{P} = \mathrm{Sp}\,\{\mathbf{i}\rho_{sc}\} / i\,\mathrm{Sp}\,\rho_{sc}. \qquad (36)$$

It can be seen from (34) that the relativistic effect of rotation of the spin does not appear in the scattering of unpolarized particles. In fact, in this case the matrix ρ of the initial state is a unit matrix with respect to the spin indices, so that the matrices D and D^{-1} acting on it from the left and the right cancel each other, and the matrices D and D^{-1} occurring in the V and V^{-1} at the beginning and end of the product in Eq. (35) cancel, because the trace is not affected by cyclic permutation of the factors. Nor does the use of a polarized target lead to the appearance of the relativistic effect, since the matrix D^I is unity for a particle at rest in the laboratory system. The relativistic spin effects can influence the cross section only in multiple scattering, for which the spin state obtained in the c.m.s. of the incident particle and the scatterer has to be transformed to the c.m.s. of the scattered particle and the next scatterer. Only in this case is the momentum of the particle not parallel to the relative velocity of the reference system, so that the corresponding spin rotation matrix D^I can be different from unity. We emphasize that all the above considerations relate to effects connected with the spin rotation.

The relativistic effects connected with the transformation of the momenta (forward bias of the cross section) occur in any scattering at sufficiently high energy.

6. For the study of the effect of the relativistic corrections on the polarization effects it is helpful to obtain a general formula for the transformation of the density matrix (i.e., the cross-section and all the tensor moments) in the passage from one Lorentz frame to another. The density matrix $\rho^{\kappa i}_{m_i m_i'}(\mathbf{p})$ for the scattered particle transforms like the outer product of a wave function by its conjugate, taken for the same value of the momentum:

$$\rho^{\varkappa i}_{m_i m_i'}(\mathbf{p}) \sim (-1)^{i-m}\, \psi^{\varkappa i}_{m_i}(\mathbf{p})\, \psi^{*\varkappa i}_{-m_i'}(\mathbf{p}). \qquad (37)$$

The factor $(-1)^{i-m_i'}$ and the minus sign on the m_i' in $\psi^{*\kappa i}_{-m_i}(\mathbf{p})$ are introduced so that the spin matrices can act on $\rho_{m_i m_i'}$ from the left not only for the index m_i but also for the index m_i'. From this we can get the transformation for $\rho_{m_i m_i'}$ from the known transformations (2) and (3) for ψ_{m_i} and $\psi_{m_i'}$.

By simple calculations one can show that the operators of infinitesimal transformations given by Eq. (37) for the function ρ have the forms

$$\mathbf{p} = 0, \qquad p_0 = 0, \qquad (38)$$

$$\mathbf{M} = -i\left[\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}}\right] + \mathbf{i} + \mathbf{i}' \qquad (39)$$

$$\mathbf{N} = i\sqrt{e_p}\,\frac{\partial}{\partial \mathbf{p}}\,\sqrt{e_p} - \frac{[(\mathbf{i} + \mathbf{i}') \times \mathbf{p}]}{e_p + \varkappa}. \qquad (40)$$

i′ denotes a spin operator composed of the same matrices as **i**, but acting on the variable m_i'. From the relation (40) it follows that the density matrix behaves under Lorentz transformations like the wave function of a particle with the spin $\mathbf{i} + \mathbf{i}$. Therefore a finite Lorentz transformation for the matrix $\rho_{m_i m_i'}(\mathbf{p})$ is given by a formula analogous to Eq. (7):

$$\rho(\mathbf{p}) = D^i D^{i'} \sqrt{e_{p'} / e_p}\, \rho'(\mathbf{p}'). \tag{41}$$

In particular, a rotation of the polarization vector $\mathbf{P}$ is determined by the matrix a_{ij} of Eq. (9) and is given by

$$\mathbf{P} = \mathbf{P}' - \frac{\mathbf{p}(\mathbf{p}\,\mathbf{P}')(u_0 - 1)}{(e_p + \varkappa)(e_p u_0 - \mathbf{pu} + \varkappa)} + \frac{\mathbf{u}(\mathbf{p}\cdot\mathbf{P}')}{e_p u_0 - \mathbf{pu} + \varkappa} + \frac{\mathbf{p}(\mathbf{u}\cdot\mathbf{P}')(-e_p u_0 + 2\mathbf{up} - u_0\varkappa - e_p - \varkappa)}{(e_p + \varkappa)(e_p u_0 - \mathbf{pu} + \varkappa)(u_0 + 1)} - \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{P}')(e_p - \varkappa)}{(u_0 + 1)(e_p u_0 - \mathbf{pu} + \varkappa)} \tag{42}$$

We note that it is clear from Eq. (42) that the relativistic effects manifest themselves not only in single scattering but also in double scattering from unpolarized targets, since the polarization appearing after the first scattering is perpendicular to the plane of the scattering and consequently is not changed by the rotation (42). For the spin $1/2$ this result was obtained in reference 10.

7. The exposition of the formal theory of scattering usually begins with the step of representing the wave function as a superposition of incident and scattered waves, $e^{ikz} + e^{ikr} f(\vartheta)/r$.

In the relativistic case, however, it is scarcely worthwhile to use this method, since in the coordinate representation the wave functions for a particle with spin as a rule have redundant components, which complicate the treatment, especially for higher spins. Furthermore a separate treatment has to be carried through for each value of the spin. The case of spin $1/2$ has been treated in this way in reference 10. It must also be noted that the question of coordinates in relativistic quantum theory is not a simple one.[11]

The impression may arise that the discussion that has been given has not been formulated in a relativistically invariant way. This is untrue, however. The point is that four-dimensional tensors and spinors do not exhaust the possibilities for covariant physical quantities. Moreover, relativistic tensors and spinors transforming by nonunitary representations of the Lorentz group are, strictly speaking, in general not very suitable for the wave functions of relativistic quantum systems, since their norms are not definite. On the other hand, there exist unitary representations of the inhomogeneous Lorentz group, whose wave functions have positive definite norms. The functions (1), transforming according to Eqs. (2) and (3), are just such functions. These wave functions are just as covariant as tensors, and the part of the tensor indices is played by the set of variables $\mathbf{p}$, m_i. An essential point is that these functions are covariant only with respect to the set of spin and momentum variables, and not with respect to each of them separately.

The density matrix (37) is also a covariant quantity. According to Eq. (38) displacement transformations are unit operators for this matrix, since it transforms by a unitary representation of the homogeneous Lorentz group. A point of interest is the expansion of this representation in terms of irreducible representations, which will give a relativistically invariant classification of the possible angular distributions for particles with spin in an arbitrary reference system.

Finally, we shall make one further remark concerning the invariance of the parametrization of the S matrix in Eq. (31). The quantity

$$(\mathbf{i} + \mathbf{l})^2, \tag{43}$$

defined in an arbitrary coordinate system, is of course not an invariant. One finds, however, by direct verification that the quantity

$$\{(D^i)^{-1}\mathbf{i}D^i + (D^l)^{-1}\mathbf{l}D^l\}^2 = s(s+1) \tag{44}$$

is an invariant, and in the rest system the expression (44) is the same as (43). In a similar way one can give an invariant definition of the intrinsic orbital angular momentum l.

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[1] Chou Kuang Chao and M. I. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1230 (1958), Soviet Phys. JETP **7**, 851 (1958).

[2] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

[3] A. Simon, Phys. Rev. **92**, 1050 (1953).

[4] Iu. M. Shirokov, Dokl. Akad. Nauk SSSR **94**, 857 (1954).

[5] Iu. M. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1196 (1957), Soviet Phys. JETP **6**, 919 (1958).

[6] V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. **34**, 211 (1948). P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[7] E. P. Wigner, Ann. of Math. **40**, 149 (1939).

[8] Iu. M. Shirokov, Dokl. Akad. Nauk SSSR **99**, 737 (1954).

[9] E. P. Wigner, Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, p. 165. Braunschweig 1931.

[10] H. P. Stapp, Phys. Rev. **103**, 425 (1956).

[11] Iu. M. Shirokov, Dokl. Akad. Nauk SSSR **111**, 1223 (1956), Soviet Phys. "Doklady" **1**, 777 (1956).

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