ON AXIALLY ASYMMETRIC NUCLEI

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The possibility of the existence of nuclei lacking axial symmetry is demonstrated within the framework of the generalized model.

DAVYDOV and Filippov[1] have assumed that medium and heavy nuclei may possess axial asymmetry and have calculated the energy levels of the nucleus and the probabilities of radiative transitions on the basis of this assumption. However, the theoretical possibility of the existence of axially-asymmetric nuclei is doubtful, since in A. Bohr's model the equilibrium value for the axial asymmetry of nuclei turns out to be equal to zero.[2*] It will be shown below that a consistent investigation within the framework of the generalized model leads to the result that in the general case the axial asymmetry of nuclei differs from zero.

We consider several nucleons outside a filled shell, which move within the field of the core and which do not interact with each other. The field of the core can be represented by an oscillator field which at a certain energy U_0 goes over into a horizontal straight line (if U_0 is significantly greater than the energy of the nucleon the horizontal part of the curve need not be taken into account). In the adiabatic approximation we neglect at first the kinetic energies of the "slow" collective degrees of freedom that determine the shape of the nucleus and evaluate the energies of the "fast" nucleon degrees of freedom corresponding to arbitrary values of the parameters that describe the shape of the nucleus. Therefore all three oscillator frequencies must at first be taken as arbitrary. Then the energy E_{nuc} of m nucleons is given by:

$$E_{\text{nuc}}=\hbar\sum_{s=1}^{m}\left[\left(n_{xs}+\frac{1}{2}\right)\omega_1+\left(n_{ys}+\frac{1}{2}\right)\omega_2+\left(n_{zs}+\frac{1}{2}\right)\omega_3\right];\quad n_{xs}+n_{ys}+n_{zs}=n. \tag{1}$$

Since each frequency ω_i is inversely proportional[3] to the square of the corresponding semi-axis of the ellipsoid R_i we can assume that:

$$\begin{aligned}(\omega/\omega_1)^{1/2}&=R_1/R=1+b_{00}-b_0+b_2\sqrt{6};\\(\omega/\omega_2)^{1/2}&=R_2/R=1+b_{00}-b_0-b_2\sqrt{6};\\(\omega/\omega_3)^{1/2}&=R_3/R=1+b_{00}+2b_0.\end{aligned} \tag{2}$$

We find b_{00} from the condition that the nuclear volume remains constant $\omega_1\omega_2\omega_3=\omega^3=\text{const}$ $(R_1R_2R_3=R^3)$: $b_{00}\approx b_0^2+2b_2^2$, while the parameters $a_0=4(\pi/5)^{1/2}b_0=\beta\cos\gamma$ and $a_2=4(\pi/5)^{1/2}b_2=\beta\sin\gamma/\sqrt{2}$ have for small β the same meaning as in Bohr's paper;[2] β is the parameter describing the deviation of the nucleus from spherical shape, while γ is the axial asymmetry parameter (in the case of $\gamma=n\pi/3$; $n=0,\mp 1,\ldots$ the nucleus is axially symmetric). To E_{nuc} we should add the energy of the core which depends on b_0 and b_2: $E_{\text{core}}=c_n\hbar\omega(b_0^2+2b_2^2)/2=5c_n\hbar\omega\beta^2/32\pi$. We substitute (2) into (1) and keep only terms linear in b_0 and b_2; the quadratic terms can be neglected in comparison with E_{core}, since their ratio to E_{core} is, evidently, of the order of $\sim m/(A-m)\ll 1$ (A is the nuclear mass number). In order to find the equilibrium shape of the nucleus we shall find the minimum with respect to b_0 and b_2 of the energy $E=E_{\text{nuc}}+E_{\text{core}}=m(n+3/2)\hbar\omega+\Delta E$;

$$\Delta E/\hbar\omega=\sum_s^m\left[n_{xs}(b_0-b_2\sqrt{6})/2+n_{ys}(b_0+b_2\sqrt{6})/2-n_{zs}b_0\right]+c_n(b_0^2/2+b_2^2). \tag{3}$$

We then obtain

$$b_0=\frac{1}{2c_n}\sum_s(2n_{zs}-n_{xs}-n_{ys});\quad b_2=\frac{1}{c_n}\sqrt{\frac{3}{8}}\sum_s(n_{xs}-n_{ys});$$

$$\beta=\frac{1}{c_n}\sqrt{\frac{\pi}{5}}\left\{3\left[\sum_s(n_{xs}-n_{ys})\right]^2+\left[\sum_s(2n-3n_{xs}-3n_{ys})\right]^2\right\}^{1/2};$$

$$\gamma=\tan^{-1}\left\{\sum_s(n_{xs}-n_{ys})\left[\sum_s(2n-3n_{xs}-3n_{ys})\right]^{-1}\right\}. \tag{4}$$

On substituting (4) into (3) we obtain

$$\frac{\Delta E}{\hbar\omega}=-\frac{1}{8c_n}\left\{3\left[\sum_s(n_{xs}-n_{ys})\right]^2+\left[\sum_s(2n-3n_{xs}-3n_{ys})\right]^2\right\}.$$

*Brief indications of the possibility of the existence of axially asymmetric nuclei can be found only in unpublished numerical calculations of Gursky.[4]

It is easily seen that for a given principal quantum number n the lowest value of ΔE for one nucleon, equal to $\Delta E_{\min} = -n^2\hbar\omega/2c_n$, is given in three cases by:

$$1)\ n_x = n,\ n_y = n_z = 0;\ 2)\ n_y = n,\ n_z = n_x = 0;\ 3)\ n_z = n,\ n_x = n_y = 0.$$

Here $\gamma = 0$ or $\mp\pi/3$, i.e., the nucleus is axially symmetric; $\beta = (2/c_n)\sqrt{\pi/5}\,n$.

We can show that $\beta > 0$, i.e., that the nucleus is elongated, by calculating R_i from (2). As a result of this calculation we find that two of the ratios R_i/R are equal to $1 - n/2c_n$, while the third ratio is equal to $1 + n/c_n$. The second nucleon may have the same quantum numbers but for the third nucleon only the following values of n_x, n_y, n_z are possible as a result of Pauli's exclusion principle: $n_x = n-1$, $n_y = 1$, $n_z = 0$ or $n_y = 0$, $n_z = 1$ (if for the first pair of nucleons $n_x = n$) and two similar combinations involving $n_y = n-1$ and $n_z = n-1$, if $n_y = n$ or $n_z = n$ for the first pair. At the same time $\gamma = \tan^{-1}[\sqrt{3}/(6n-3)]$ (if we measure γ from zero) i.e., the nucleus turns out to be axially asymmetric. As further nucleons are added γ increases (cf. below; here $m \ll (n+1)(n+2)$ and $n > 1$, for $n = 1$, $\gamma = 0$).

Number of nucleons, m	$\leqslant 2$	3	4	5	6
$\tan\gamma$	0	$\frac{\sqrt{3}}{6n-3}$	$\frac{\sqrt{3}}{4n-3}$	$\frac{\sqrt{3}}{2.5n-3}$	$\frac{\sqrt{3}}{2n-3}$

We now consider an almost completely filled shell. In this case the energy is given by

$$\frac{\Delta E}{\hbar\omega} = c_{n+1}(b_0^2/2 + b_2^2) - \sum_s^m [n_{xs}(b_0 - b_2\sqrt{6})/2 + n_{ys}(b_0 + b_2\sqrt{6})/2 - n_{zs}b_0],$$

m is the number of holes. The minimum in ΔE corresponds to

$$b_0 = \frac{1}{2c_{n+1}}\sum (n_{xs} + n_{ys} - 2n_{zs});$$

$$b_2 = \frac{1}{c_{n+1}}\sqrt{\frac{3}{8}}\sum_s (n_{ys} - n_{xs}).$$

As in a previous case the lowest value of ΔE for one hole is obtained for $n_x = n$, or $n_y = n$ or $n_z = n$, but $\beta = -(2/c_{n+1})\sqrt{\pi/5}\,n$ (two of the ratios R_i/R are equal to $1 + n/2c_{n+1}$, while the third is equal to $1 - n/c_{n+1}$), i.e., the nucleus is flattened. The values of γ are the same ones as in the first case (cf. above). Thus, γ reaches its greatest values in the middle of the shell. The average value of γ for a given shell decreases as the shell number n increases.

In Bohr's paper[2] the equilibrium value of γ was zero because the existence of axial symmetry was assumed from the outset in the expression for E_n.

The model in which the shells are filled in succession, starting with the first one, reduces to the model discussed above, which has several nucleons outside a filled shell. Indeed, if we use oscillator functions, we can easily show that the energy of each filled shell is equal to:

$$E_n/\hbar = {}^1/_3(n+1)(n+2)(n+{}^3/_2)(\omega_1+\omega_2+\omega_3) \approx (n+1)(n+2)(n+{}^3/_2)\,\omega(1+5\beta^2/64\pi). \quad (5)$$

There are no terms linear in β in this expression. Thus, the filled shells, just as in the liquid drop model, determine only the value of the stiffness for the parameter β. We note that in the shell model we can determine the shape of the nucleus exactly, without having to restrict ourselves to small values of β. In order to do this it is necessary only, as was done earlier, to assume that levels with different values of n do not cross. The energy of a nucleus having m nucleons outside a filled shell is given by: $E = E_{nuc} + E_{core}$; E_{nuc} is determined for formula (1) as before; E_{core}, as is seen from (5), is of the form: $E_{core}/\hbar = c'_n(\omega_1 + \omega_2 + \omega_3 - 3\omega)$. The value of c'_n is smaller than the value given by (5), owing to Coulomb repulsion.

By finding directly the minimum of E with respect to ω_1, ω_2, ω_3, subject to the condition $\omega_1\omega_2\omega_3 = \text{const}$, we obtain:

$$\omega_i = \omega(\varepsilon_{i+1}\varepsilon_{i+2}\varepsilon_i^{-2})^{1/3};\quad E_{\min}/\hbar = 3\omega[(\varepsilon_1\varepsilon_2\varepsilon_3)^{1/3} - c'_n];$$

here $\varepsilon_\alpha = \sum_s^m (n_{\alpha s} + {}^1/_2) + c'_n$.

For $m = 1$ or $m = 2$ the lowest value $E_{\min}$ is obtained, as before, for $n_x = n$, or $n_y = n$, or $n_z = n$; in the case of $m = 1$ two of the frequencies are equal to $\omega(n + c'_n + \frac{1}{2})^{1/3}/(c'_n + \frac{1}{2})^{1/3}$, while the third is equal to $\omega(c'_n + \frac{1}{2})^{2/3}/(n + c'_n + \frac{1}{2})^{2/3}$ and the nucleus is axially symmetric (small values of β correspond to $mn \ll c'_n$). For the third nucleon $n_x = n-1$ (if for the first pair $n_x = n$) etc. In this case

$$\omega_1/\omega = (c'_n + {}^5/_2)^{1/3}(c'_n + {}^3/_2)^{1/3}(3n + c'_n + {}^1/_2)^{-2/3};$$

$$\omega_2/\omega = (c'_n + {}^3/_2)^{1/3}(3n + c'_n + {}^1/_2)^{1/3}(c'_n + {}^5/_2)^{-2/3};$$

$$\omega_3/\omega = (3n + c'_n + {}^1/_2)^{1/3}(c'_n + {}^5/_2)^{1/3}(c'_n + {}^3/_2)^{-2/3},$$

i.e., the nucleus is asymmetric.

The model considered by us is too crude and simple to permit us on its basis to draw any quantitative conclusions with respect to real nuclei, but it is quite sufficient to establish the possibility in principle of the existence of axially asymmetric nuclei.

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[1] A. S. Davydov and G. F. Filippov, J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 440 (1958), Soviet Phys. JETP **8**, 303 (1959).

[2] A. Bohr, Kong. Dansk. Vid. Selsk. Mat.-fys. Medd., **26**, 14 (1952); **Проблемы современной физики** (Problems of Modern Physics) No. 9 (1955).

[3] W. Heisenberg, The Physics of Atomic Nuclei (Russian transl.) IIL, 1953.

[4] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956); M. Gursky, Phys. Rev. **98**, 1205 (1955).

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