# ENERGY DEPENDENCE OF MULTIPLE-PRODUCTION REACTION CROSS SECTIONS NEAR THRESHOLD

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The energy dependence of multiple-production reaction cross sections near threshold is obtained, with account taken of effects of the Pauli principle and of the Coulomb interaction of the emitted particles with the residual nucleus. In the general case the recoil of the nucleus and the Coulomb interactions between the emitted particles were neglected. For the case in which not more than one charged particle is among those emitted the nuclear recoil is taken into account and the formulas obtained are exact.

THE functional dependence of the cross section on the energy near threshold for the reactions in question can be found in general form, independent of the concrete interaction mechanism. This has been shown by Wigner for reactions with two particles in the final state.[1] It is a very simple matter to obtain the threshold dependence for reactions in which the final stage is that of the simultaneous emission of N neutral particles with zero orbital angular momenta from an infinitely heavy complex (nucleus), i.e., under the conditions of short-range interaction and symmetry of the wave function in the coordinates of the particles.[2] Hart and others[3] have studied the special case $N = 2$ (emission of a charged particle and a neutral one) with arbitrary orbital angular momenta and with inclusion of effects of the nuclear recoil.

We shall consider the case of emission of N arbitrary particles with masses $\mu_\alpha \ll M_{nuc}$ ($M_{nuc}$ is the mass of the residual nucleus), with inclusion of effects of the Pauli principle and of the Coulomb interactions of the emitted particles with the nucleus (of charge $Ze \gg e$).

It must be pointed out that for $N \geq 2$ the problem is considerably simplified by the use of the momentum representation, since this removes the necessity of finding the explicit form of the Green's functions in the coordinate representation.[2,3] Even the calculation of the asymptotic behavior of the Green's functions, which is needed for the construction of the reaction amplitude in the coordinate representation, involves considerable difficulty.[3] If there are groups of identical particles among those emitted, then in the determination of the threshold energy dependence the classification in terms of the permutation symmetry is often entirely sufficient for the choice of the allowed channels (including those in terms of the total angular momentum), and one avoids the necessity of examining arbitrary orbital angular momenta of the emitted particles.

## 1. NEUTRAL PARTICLES

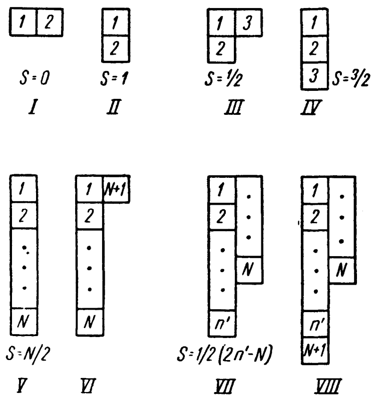
In the center-of-mass system and in the variables $\mathbf{r}_\alpha = \mathbf{x}_\alpha (\mu_\alpha/M)^{1/2}$, $\mathbf{k}_\alpha = \mathbf{p}_\alpha (M/\mu_\alpha)^{1/2}$ the reaction amplitude (with nuclear recoil neglected) has the form*

$$F(\mathbf{k}_1 \ldots \mathbf{k}_N) = \int \exp\left\{-i \sum_{\alpha=1}^{N} \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\right\} U(r, q)\, \Psi(r, q)\, \varphi^*_{nuc}(q)\, dr dq, \quad (1)$$

where $U(r, q)$ is the total interaction energy of the emitted particles and the nucleus (it is essential that all the interactions are short-range ones), $\Psi(r, q)$ is the exact wave function, including all the reaction channels, and $\varphi_{nuc}(q)$ is the final state of the nucleus; the integration is taken over the entire set of particle coordinates r and nuclear variables q. As can be seen from the notation, the coordinates and momenta $\mathbf{r}_\alpha$, $\mathbf{k}_\alpha$ of the particles (relative to the nucleus) are measured in certain conventional units that depend on the arbitrary auxiliary mass M.

The amplitude (1) must be symmetrized in the $\mathbf{k}_\alpha$ in a definite way. Since we are interested in the functional dependence, the proportionality coefficients in the equations will be dropped. The reaction cross section is expressed in the form of an integral over the possible final states of the

*We take throughout $h = c = 1$.




particles

$$\sigma_N = \int |F|^2 \prod_{\alpha=1}^{N} k_\alpha^2 dk_\alpha d\Omega_\alpha \delta(k^2 - K^2), \tag{2}$$

$$K^2 = 2ME, \; k^2 = \sum_{\alpha=1}^{N} k_\alpha^2,$$

where E is the energy of the exit channel, i.e., the total kinetic energy in the c.m. system.

To obtain the dependence $\sigma_N(K)$ for $K \to 0$ it is enough to take the lowest power of k in the expansion of the expression (1). If the expansion begins with the m-th power, then

$$\sigma_N = K^{3N-2+2m}. \tag{3}$$

It is obvious that $U(r, q)\Psi(r, q)\varphi^*_{\text{nuc}}(q)$ is of zeroth order in k, and we have only to calculate the lowest term in the expansion of $\exp\{-i\sum \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\}$ in powers of $\sum \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha$ which remains after the symmetrization in the $\mathbf{k}_\alpha$.

The symmetry of the wave function is given by the Young arrays (cf., e.g., reference 4); there exist several possible arrays for any prescribed set of N particles. For each Young array there is a definite lowest power $\left(\sum \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\right)^m$ which does not vanish identically through the symmetrization. Out of all the possible Young arrays for the given N particles there will be contributions to the reaction at threshold only from those which have the minimum value of m. We shall calculate this minimum exponent.

An identical vanishing of a symmetrized power term can occur only through alternations (antisymmetrizations) through the columns of an array, so that we shall not concern ourselves with symmetrization along the rows of an array. If an array is possible with a single row, not containing alternations, then m = 0. Thus for bosons or for identical fermions numbering $N \le 2s + 1$ (s being the spin of the fermions) we get the previous result of reference 2.

We shall now consider the case of N identical fermions with spin $s = \frac{1}{2}$, and shall show by mathematical induction that $m = N - 2$ for $N \ge 2$. For N = 2, 3 the possible arrays are I, II, III, IV (see figure). It is easy to verify that $m_I = 0$, $m_{II} = m_{III} = 1$, $m_{IV} = 2$. It remains to show that when we go from N to N + 1 the minimum exponent increases from m to m + 1. We first note that, as in the passage from II to III, in going from the one-column array V to the array VI the lowest exponent does not change, so that the minimum exponent comes from arrays with the maximum number of columns (two in the case considered), and in the proof it suffices to examine transitions of the type VII → VIII, in which the new cell that is added does not form a new column. To begin with we shall show that the m-th power vanishes for the case of N + 1 particles.

We denote the complete alternation through all N + 1 momenta $\mathbf{k}_\alpha$ (array VIII) by the symbol $\underset{\alpha}{A}$. An index (or several indices) under an alternation symbol A will indicate the set of corresponding momenta through which the alternation is carried out. We note that the original alternated function can depend on an altogether different set of momenta. For example, it can depend on only part of the momenta through which the alternation is to be taken, or can even be a constant. After the alternation it will in the general case depend on all the momenta of the alternation. In the proof we have to break up the complete alternation into two stages: we first carry out the alternation through a certain part of the set of momenta, and then the remaining part of the alternation with the remaining momenta. For example, the complete alternation can be broken up into the alternation through the momenta $\mathbf{k}_\beta$ of the N particles ($\beta = 1, 2, \ldots N$) and the subsequent alternation through $\mathbf{k}_\beta$, $\mathbf{k}_{N+1}$ without alternation of the $\mathbf{k}_\beta$. We denote the second stage by the symbol $\underset{N+1,(\beta)}{A}$. Now separating the term $\mathbf{k}_{N+1}\mathbf{r}_{N+1}$ from the sum $\sum_{\alpha=1}^{N+1} \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha$ and expanding the m-th power of this sum by the binomial theorem, we get

$$\underset{\alpha}{A}\left(\sum_{\alpha=1}^{N+1} \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\right)^m = \underset{N+1,(\beta)}{A}(\mathbf{k}_{N+1}\cdot\mathbf{r}_{N+1})^m \underset{\beta}{A}\, 1 + \ldots$$
$$\ldots + m \underset{N+1,(\beta)}{A}(\mathbf{k}_{N+1}\cdot\mathbf{r}_{N+1}) \underset{\beta}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta\cdot\mathbf{r}_\beta\right)^{m-1}$$
$$+ \underset{\alpha}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta\cdot\mathbf{r}_\beta\right)^m.$$

All the terms of this expansion except the last vanish already on the alternation through the $\mathbf{k}_\beta$, since this alternation is equivalent to array VII, for which all powers of the sum $\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta$ vanish up to the $(m-1)$ st power inclusive (case of N particles).

We transform the last term in the following way: we break up the m-th power into a sum of N terms of a single type,

$$(\mathbf{k}_1 \cdot \mathbf{r}_1)\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m-1} + (\mathbf{k}_2 \cdot \mathbf{r}_2)\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m-1} + \ldots + (\mathbf{k}_N \cdot \mathbf{r}_N)\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m-1}$$

In each term we single out from $\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta$ the term that appears as the coefficient of the $(m-1)$ st power, and corresponding to this we break up the alternation of each term into two stages:

$$\underset{\alpha}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m} = \underset{1,(\gamma,N+1)}{A}(\mathbf{k}_1 \cdot \mathbf{r}_1)\underset{\gamma,N+1}{A}\left(\mathbf{k}_1 \cdot \mathbf{r}_1 + \sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma\right)^{m-1} + \ldots$$
$$\ldots + \underset{N,(\gamma,N+1)}{A}(\mathbf{k}_N \cdot \mathbf{r}_N)\underset{\gamma,N+1}{A}\left(\mathbf{k}_N \cdot \mathbf{r}_N + \sum_{\gamma=1}^{N-1} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma\right)^{m-1}. \quad (4)$$

Let us take, for example, the first term of the sum (4) and show that it vanishes already in the alternation through $\gamma$, $N+1$ because of the vanishing of the factor

$$\underset{\gamma,N+1}{A}\left(\mathbf{k}_1 \cdot \mathbf{r}_1 + \sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma\right)^{m-1}. \quad (5)$$

We note that the expression (5) can be regarded as a homogeneous polynomial of degree $(m-1)$ in the coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N$ with coefficients depending on all $N+1$ momenta $\mathbf{k}_\alpha$. We now consider another polynomial

$$P_{m-1}(\mathbf{r}_1, \ldots \mathbf{r}_N, \mathbf{r}_{N+1}) = \underset{\gamma,N+1}{A}\left[\mathbf{k}_1 \cdot \mathbf{r}_1 + \left(\sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma + \mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1}\right)\right]^{m-1},$$

which includes (5) as part of itself. If we expand this latter quantity in powers of $\left(\sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma + \mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1}\right)$, it can be seen that it vanishes identically, since we arrive at the array VII with the numbering of the cells changed. This means that the factor (5) also vanishes, since it is a particular value of the other polynomial for $\mathbf{r}_{N+1} = 0$.

Similarly one can show that each of the successive terms of the sum (4) vanishes, by going over to one of the arrays for N particles.

Finally, we convince ourselves that the $(m+1)$ st power does not vanish:

$$\underset{\alpha}{A}\left(\sum_{\alpha=1}^{N+1} \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\right)^{m+1} = \underset{N+1,(\beta)}{A}(\mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1})^{m+1}\underset{\beta}{A}1 + \ldots$$
$$\ldots + \frac{(m+1)m}{2}\underset{N+1(\beta)}{A}(\mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1})^2\underset{\beta}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m-1}$$
$$+ (m+1)\underset{N+1,(\beta)}{A}(\mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1})\underset{\beta}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m}$$
$$+ \underset{\alpha}{A}\left(\sum_{\beta=1}^{N} \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^{m+1}. \quad (6)$$

All the terms of this expansion except the last two vanish on the alternation with respect to $\beta$. In analogy with Eq. (4) we break up the next to last term of the expansion (6) into terms of the type

$$\underset{1,N+1,(\gamma)}{A}(\mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1})(\mathbf{k}_1 \cdot \mathbf{r}_1)\underset{\gamma}{A}\left(\mathbf{k}_1 \cdot \mathbf{r}_1 + \sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma\right)^{m-1}.$$

After expanding the $(m-1)$ st power in this term by the binomial theorem and alternating with respect to $\gamma$ we have remaining

$$\underset{1,N+1,(\gamma)}{A}(\mathbf{k}_{N+1} \cdot \mathbf{r}_{N+1})(\mathbf{k}_1 \cdot \mathbf{r}_1)\underset{\gamma}{A}\left(\sum_{\gamma=2}^{N} \mathbf{k}_\gamma \cdot \mathbf{r}_\gamma\right)^{m-1}.$$

It is easy to see that the further alternation through 1, $N+1$, $(\gamma)$, and combination of all analogous terms together with the last term in Eq. (6) does not lead to cancellation.

It is obvious from the proof that all arrays of two columns give the minimun degree $m = N - 2$, since it is immaterial to which column one adds a new cell. This means that all total spins $S \leq N/2 - 1$ give the same dependence $\sigma(K)$, and the cross-section for the largest total spin $S = N/2$ vanishes more rapidly at the threshold by a factor $K^2$.

In order to go over to the case of an arbitrary set of N particles, we note that the result just established can be formulated in the following way:

> The exponent of the lowest power of $\left(\sum_{\alpha=1}^{N} \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha\right)$ that does not vanish on alternation through a prescribed Young array is equal to the total number of cells in all rows of the array except the first, and in going from N to $N+1$ it increases by unity if the new cell does not form a new column. (7)

Noting that no use was made in the proof of the possible number of columns, and that consequently the result (7) is valid for arrays with an arbitrary number of columns, in the case of arbitrary spin s we get: $m = N - (2s + 1)$ for $N \geq 2s + 1$. For an arbitrary set of N particles the alternation must be carried out through a single common array, made up of the arrays for the different groups of identical particles written side by side. This gives for the minimum exponent:

$$m = \sum_i m_i, \quad m_i = \begin{cases} 0 & \text{for } N_i \leqslant 2s_i + 1, \\ N_i - (2s_i + 1) & \text{for } N_i \geqslant 2s_i + 1, \end{cases} \tag{8}$$

where the sum is taken over all groups of identical fermions; $N_i$ is the number of fermions of the i-th group, and $s_i$ is their spin.

It must be emphasized that Eq. (8) gives the minimum exponent in the expansion of the reaction amplitude that is possible according to the Pauli principle for the given set of emergent particles. In principle it is possible that there could be additional degrees of forbiddenness according to symmetry type, which could exclude the channels with the minimum m. In this case the degree m in Eq. (3) is given by Eq. (7) for the allowed arrays.*

Equation (3) represents the first term of an expansion in powers of $kR_0$ ($R_0$ is the radius of the nucleus, or more precisely the radius of the reaction zone). With increase of N the errors accumulate not faster than $N^{1/2}$, so that the region of applicability of Eq. (3) is limited by the inequality $N^{1/2} KR_0 \lesssim 1$.

## 2. CHARGED PARTICLES

Neglecting the Coulomb interactions between the emitted particles, we get for the reaction amplitude

$$F(\mathbf{k}_1 \ldots \mathbf{k}_N) = \int \prod_{\alpha=1}^{N} \psi^*(\mathbf{k}_\alpha, \mathbf{r}_\alpha) U(r, q) \Psi(r, q) \varphi_я^*(q)\, dr\, dq;$$

$$\psi(\mathbf{k}_\alpha, \mathbf{r}_\alpha) = \exp\left(-\frac{\pi a_\alpha}{2k_\alpha}\right) \Gamma\left(1 - i\frac{a_\alpha}{k_\alpha}\right) \exp(i\mathbf{k}_\alpha \cdot \mathbf{r}_\alpha) F\left(i\frac{a_\alpha}{k_\alpha}, 1, -i(k_\alpha r_\alpha + \mathbf{k}_\alpha \cdot \mathbf{r}_\alpha)\right), \tag{9}$$

$$a_\alpha = \sqrt{M\mu_\alpha}\, Zz_\alpha e^2,$$

$z_\alpha e$ is the charge of the $\alpha$-th particle. The amplitude (9) differs from Eq. (1) only in that for the charged particles the plane waves are replaced by Coulomb solutions with the asymptotic forms made up of plane and converging waves. Using the expansion of the confluent hypergeometric function, we get for small momenta

$$\psi(\mathbf{k}_\alpha, \mathbf{r}_\alpha) = \psi(\mathbf{k}_\alpha, 0)[1 + i\mathbf{k}_\alpha \cdot \mathbf{r}_\alpha + \ldots][f_0(r_\alpha + \mathbf{n}_\alpha \cdot \mathbf{r}_\alpha) + k_\alpha f_1(r_\alpha + \mathbf{n}_\alpha \cdot \mathbf{r}_\alpha) + \ldots] \approx \psi(\mathbf{k}_\alpha, 0) f_{0\alpha}, \tag{10}$$

where $\mathbf{n}_\alpha = \mathbf{k}_\alpha / k_\alpha$, and $f_1, f_2, \ldots$ are certain functions of the quantity $(r_\alpha + \mathbf{n}_\alpha \cdot \mathbf{r}_\alpha)$, the expansions for which contain arbitrarily high powers.

When the amplitude (9) is alternated through the group of neutral particles, according to Sec. 1, the contribution they give in the integrand is $\underset{\beta}{A}\left(\sum_\beta \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^m$. The alternation through the charged particles does not change the functional dependence, since for an arbitrary Young array

$$\underset{\alpha}{A} \prod_\alpha \psi(\mathbf{k}_\alpha, 0) f_0(r_\alpha + \mathbf{n}_\alpha \cdot \mathbf{r}_\alpha) = \prod_\alpha \psi(\mathbf{k}_\alpha, 0) \underset{\alpha}{A} \prod_\alpha f_0(r_\alpha + \mathbf{n}_\alpha \cdot \mathbf{r}_\alpha) \not\equiv 0. \tag{11}$$

This means that arbitrary orbital angular momenta of the charged particles emitted give always the same threshold dependence.

If among the N particles there are n charged particles ($n_+$ positive and $n_-$ negative) and $n_0$ neutral particles, then near the threshold

$$\sigma_N = \int \prod_{\alpha=1}^{n} |\psi(\mathbf{k}_\alpha, 0)|^2 f(\mathbf{k}_\beta, \mathbf{n}_\alpha) \prod_{\gamma=1}^{N} k_\gamma^2\, dk_\gamma\, d\Omega_\gamma \cdot \delta(k^2 - K^2),$$

where for the negative particles

$$|\psi(\mathbf{k}_\alpha, 0)|^2 = \frac{2\pi |a_\alpha|}{k_\alpha [1 - \exp(-2\pi |a_\alpha| / k_\alpha)]} \approx \frac{2\pi |a_\alpha|}{k_\alpha},$$

for the positive ones

$$|\psi(\mathbf{k}_\alpha, 0)|^2 = \frac{2\pi a_\alpha}{k_\alpha [\exp(2\pi a_\alpha / k_\alpha) - 1]} \approx \frac{2\pi a_\alpha}{k_\alpha} \exp\left(-\frac{2\pi a_\alpha}{k_\alpha}\right),$$

and

$$f(\mathbf{k}_\beta, \mathbf{n}_\alpha) = \left| \int \underset{\beta}{A} \left(\sum_\beta \mathbf{k}_\beta \cdot \mathbf{r}_\beta\right)^m \underset{\alpha}{A} \prod_\alpha f_{0\alpha} \cdot U \Psi \varphi_я^*\, dr\, dq \right|^2$$

is a certain function that is homogeneous of degree 2m in the momenta $\mathbf{k}_\beta$ of the neutral particles.

In computing the integral over the hypersphere $\sum_{\gamma=1}^{N} k_\gamma^2 = K^2$ in Eq. (11) we need obtain only the main contribution (method of steepest descent). For the exponential factors (from the positive particles) we have to find the position of the extremal point on the hypersphere under the condition $\sum_{\alpha=1}^{n_+} k_\alpha^2 = K^2$. Finally, setting $K = (2ME)^{1/2}$, we get

*The degree m also has the following further meaning: in a given channel the orbital angular momenta of the emergent particles obey $l_\alpha \leq m$ (at threshold).

$$\sigma_N = E^L \exp\left\{-\frac{2\pi Z}{137\sqrt{2E}}\left(\sum_{i=1}^{p} n_i z_i^{2/3}\mu_i^{1/3}\right)^{3/2}\right\},$$

$$L = \frac{1}{2}\Big\{3N - 2 + 2m - n + \frac{\varepsilon(n_+)}{2}(2m + 2n_0 + n_- + N - 1)\Big\}, \tag{12}$$

$$\varepsilon(n_+) = \begin{cases} 0 & \text{for } n_+ = 0, \\ 1 & \text{for } n_+ \geqslant 1. \end{cases}$$

$n_i$ is the number of positive particles with mass $\mu_i$ and charge $z_i$, and p is the number of different masses of the positive particles. The exponent m from the group of neutral particles is determined as shown in Sec. 1.

The largest corrections to the threshold dependence (12) are those from the errors in the expansions (10), the subsequent terms of which give corrections $\sim KR_0$ and $\sim (KR_0)(Ze^2(M\mu)^{1/2}R_0)$. The main contribution to the integral is subject to corrections $\sim K(2\pi Ze^2(M\mu)^{1/2})^{-1}$. Smallness of all these parameters also limits the region of applicability of Eq. (12). If the particles have an actual or virtual energy level $\epsilon$ close to zero, there is the further condition* $E \lesssim |\epsilon|$.

Let us consider the special case $N = 2$, $n = 1$, treated in reference 3. The formula (12) gives

$$\sigma = E^{3/2+m} \quad \text{for } n_- = 1, \ n_+ = 0,$$

$$\sigma = E^{7/4+3m/2}\exp\{-2\pi Zze^2\sqrt{\mu/2E}\} \tag{13}$$

$$\text{for } n_- = 0, \ n_+ = 1.$$

The minimum exponent is $m = 0$. If there is an additional forbiddenness from the orbital angular momentum of the neutral particle, then the lowest degree from the expansion of $\exp(-i\mathbf{k}\cdot\mathbf{r})$ in powers of kr is $l$, where $l$ is the smallest allowed angular momentum. This means that in order to use the formulas (13) in the case of arbitrary orbital angular momentum $l$ of the neutral particle we must take† $m = l$.

*The effect of the interaction of the reaction products on the accuracy of the threshold dependence has been dealt with partly in reference 3.

†Similarly, we can take into account the raising of the lowest degree m owing to any additional forbiddenness in the group of neutral particles by finding the first term of the expansion of $\exp\{-i\sum \mathbf{k}_\beta \cdot \mathbf{r}_\beta\}$ in powers of kr.

The results of reference 3 differ from Eq. (13) only by the inclusion of the motion of the nucleus. A comparison of the results shows that inclusion of the nuclear recoil does not change the threshold dependence in the case $n_+ = 0$, and changes the exponent in the case $n_+ = 1$ in the following way: $\mu \rightarrow \mu M_{nuc}/(\mu + M_{nuc})$. It is not hard to show that for $n = 0, 1$ with an arbitrary number of neutral particles the formula (12) is exact if we replace the mass of the charged particle by its reduced mass, $\mu \rightarrow \mu M_{nuc}/(\mu + M_{nuc})$. To do this one must reduce the kinetic energy in the exact Hamiltonian with recoil terms to the canonical form by the method of Jacobi, leaving the coordinate of the charged particle unchanged.*

In the case $n \geq 2$ the formula (12) is already an approximate one, and it makes sense to use it only under the conditions $\sum \mu_\alpha \ll M_{nuc}$, $\left|\sum z_\alpha\right| \ll Z$. It can be expected that under these conditions inclusion of the motion of the nucleus and the Coulomb interaction between the emergent particles will not change the threshold dependence for $n_+ = 0$, and will shift the exponent by amounts $\sim\sum_\alpha \mu_\alpha/M_{nuc}$ and $\lesssim \sum_\alpha |z_\alpha|/Z$ for $n_+ \geq 1$.

In conclusion I express my gratitude to Professor I. Ia. Pomeranchuk for suggesting the problem and for a discussion.

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*For the special case $n = n_0 = 1$, for example, this transformation was used in reference 3.