OSCILLATIONS IN THE CONDUCTIVITY OF METALLIC FILMS IN MAGNETIC FIELDS

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Oscillations of the conductivity of metallic films in magnetic fields are considered for arbitrary dispersion law of the electron energy and for arbitrary orientation of the field. The oscillations are not a quantum effect and are related to the finite thickness of the film and to the diffusion character of the reflection of electrons from its surface. In contrast to the Shubnikov-de Haas effect, the conductivity in the given case oscillates as a function of H and not of H^{-1}. Measurements of the amplitudes and periods of oscillation allow us to draw conclusions on the shape of the Fermi surface in metals.

1. Oscillations of the conductivity of a metallic film due to the size of the magnetic field H perpendicular to it have been carried out by Sondheimer[1] under the assumption of an isotropic dispersion law for the electron energy. They are determined by the diffusion character of the reflection of the electrons from the surface of the film. Thanks to this, the helical electron trajectory bounded by these surfaces (the electrons move in the magnetic field **H**) undergoes simultaneously the effect of the electric field **E** and is cut off at some fraction of a turn, dependent on **H**. The value of this fraction determines the oscillating component of the current, while the complete turns give the monotonically-varying component.

In the present paper it is shown that oscillations also ought to exist for an arbitrary dependence of the electron energy ϵ on the quasi-momentum p and for an arbitrary direction of the field H, not parallel to the surface of the film. Measurements of the amplitude and period for various orientations of the field permit us to make judgments on the shape of the Fermi surface in the metal. The problem of the possibility of complete establishment of the Fermi surface by means of the oscillations is an independent problem which we shall not consider in the present paper.

Let us estimate the magnitude of the period of oscillation. The electron travels from one surface of the film to the other in the time $t \approx a/|\bar{v}_\zeta|$, where a is the thickness of the film, the ζ axis is perpendicular to the surface of the film, and v_ζ is the time average of the ζ component of the electron velocity. The period or rotation of the electron in its orbit is $T = 2\pi m^*c/eH$, where e is the electronic charge, and m^* is the effective electronic mass, introduced by Lifshitz, Azbel', and Kaganov in reference 2. It depends on ϵ and p_z — quantities which are conserved in the presence of a magnetic field (z axis directed along **H**). In this case, electrons with energies close to the Fermi energy ϵ_0 are necessary for conduction. In a time t, an electron completes $n = t/T = aeH/2\pi m^*c|\bar{v}_\zeta|$ round trips. For this number to be changed by one, the magnetic field must change by the quantity $\Delta H = 2\pi m^*c|\bar{v}_\zeta|/ea$. This also is the period of oscillation of the current created by electrons with given p_z (we consider ϵ to be equal to ϵ_0). It depends on p_z, since groups of electrons with different p_z correspond to different periods. The total current was observed experimentally; it is obvious that, generally speaking, the oscillations connected with a given group of electrons will be "smeared out" by oscillations of other groups with neighboring values of p_z, corresponding to slightly longer and slightly shorter periods. The total effect observed includes oscillations whose periods are determined by certain isolated values of p_z. They belong to two basic types.

Oscillations of the first type are connected with p_z close to p_z^*, the maximum value of p_z on the Fermi surface. They are sinusoidal, and their amplitudes fall off as H^{-4}; this result agrees with that obtained by Sondheimer.[1] However, their period is determined not by a single parameter (the limiting Fermi momentum, as in reference 1), but by the relation $\sqrt{K}/\cos\theta$, where K is the Gaussian curvature of the Fermi surface at the point $p_z = p_z^0$, and θ is the angle between the z and ζ axes. Thus, it is possible, by means of the oscillations of the first type, to determine the Gaussian curvature of the Fermi surface at all its elliptic points. This possibility of the determination of the Gaussian curvature, although it does not give very great accuracy (the amplitudes of oscillation fall off rapidly with increasing field), can be shown to be suitable, since it does not involve the use of complicated high-frequency apparatus. In this case, all the data relative to the Gaussian curvature can be obtained in principle by investigating a single specimen, cut out in a suitable manner relative to the crystallographic axes.

Oscillations of the second type are connected with such $p_z = p_{z_i}$ for which the function $\Delta H(p_z)$ has an extremum. Then, approximately the same periods ΔH correspond to a rather large group of electrons with p_z close to p_{zi}. These oscillations are not sinusoidal, and their amplitudes fall off with the field as $H^{-5/2}$.

In order that the oscillations take place and be observable, it is obvious that the inequality

$$t_0 \gtrsim a/|\bar{v}_\zeta| \gg T, \tag{1.1}$$

must be satisfied, where t_0 is the time of volume relaxation of the conduction electrons. It should be noted that it is not easy to obtain experimental conditions under which $t_0 \gtrsim a/|\bar{v}_\xi|$. For this, we need monocrystalline films, free from stresses, and low temperatures. The difference in the film thickness at different points must be less than $|\bar{v}_\xi| T$. However, at the present time, these difficulties are surmountable, since we now can obtain metallic samples in which the mean free path l is sufficiently large at liquid-helium temperatures. For example, in the work of Kunzler and Renton,[3] $l = 0.45$ mm (tin), while monocrystalline films 0.1 mm thick (and in some cases less) can be obtained. The magnitude of the magnetic field necessary for the observation of oscillations is determined by the thickness of the film compatible with the inequality (1.1); for $a \approx 0.1$ mm, it amounts to thousands of oersteds.

We note that the oscillations of similar origin were recently observed by Babiskin and Siebenmann[4] on a sodium specimen which, it is true, had the shape not of a film but of a thin wire. In these experiments, $l = 0.12$ mm.

2. We now return to a rigorous solution of the problem. The linearized kinetic equation for electrons in a metal in the shape assumed by Lifshitz, Azbel', and Kaganov[2] has the form (see also reference 5):

$$v_\zeta \frac{\partial f}{\partial \zeta} + \Omega_0 \frac{\partial f}{\partial \tau} + \frac{1}{t_0}\hat{S}f = e\frac{\partial F_0}{\partial \varepsilon}\sum_{\nu=1}^{3} v_\nu E_\nu. \tag{2.1}$$

Here $\Omega_0 = eH/m_0c$, τ is the dimensionless coordinate introduced in reference 2, which determines the position of the electron in the orbit ($\tau = \Omega_0 t$, where t is the time of motion of the electron along the orbit, determined from an arbitrary initial time), F_0 is the equilibrium distribution function, f is the nonequilibrium addition to it, $\hat{S}/t_0$ is the collision operator, m_0 is a quantity with the dimensions of mass, which drops out of the final results; for intermediate estimates, it is appropriate to consider it to be of the order of m^*. The function f satisfies the conditions

$$f|_{\zeta=0,\, v_\zeta>0} = f|_{\zeta=a,\, v_\zeta<0} = 0 \tag{2.2}$$

(diffusivity of reflection) and in the case of a closed trajectory,[2] which is also of interest to us, it must be a periodic function of τ with period $\theta = 2\pi m^*/m_0$ [$m^* = (2\pi)^{-1}\partial S/\partial\epsilon$, S is the area cut out of the surface $\epsilon(p) = \text{const}$ by the surface $p_z = \text{const}$]. The ξ component of the current-density vector satisfies the conditions $j_\xi(0) = j_\xi(a) = 0$, but, inasmuch as $\operatorname{div} j = 0$ in a stationary case, then, generally, $j_\xi(\zeta) = 0$.

We set $f = e(\partial F_0/\partial\epsilon)\sum_{\nu=1}^{3} f_\nu$, where the functions f_ν are solutions of the equation

$$v_\zeta \partial f_\nu/\partial\zeta + \Omega_0 \partial f_\nu/\partial\tau + \hat{S}f_\nu/t_0 = v_\nu E_\nu \tag{2.3}$$

with the boundary conditions (2.2). The Fourier coefficients in the expansion

$$f(\zeta) = a^{-1}\sum_k \varphi_k^{(\nu)} \exp(2\pi i k\zeta/a) \tag{2.4}$$

satisfy the equation

$$\begin{aligned} &\partial\varphi_k^{(\nu)}/\partial\tau + (ik\alpha + \gamma\hat{S})\varphi_k^{(\nu)} \\ &= -\Omega_0^{-1}|v_\zeta|[f(0) + \underline{f}(a)] + \Omega_0^{-1} v_\nu \mathscr{E}_k^{(\nu)}. \end{aligned} \tag{2.5}$$

This relation is easily obtained by substituting (2.4) in (2.3) and taking (2.2) into account. Here

$$\alpha = 2\pi v_\zeta/a\Omega_0, \quad \gamma = 1/\Omega_0 t_0,$$

$$\mathscr{E}_k^{(\nu)} = \int_0^a \exp(-2\pi i k\zeta/a)\, E_\nu(\zeta)\, d\zeta.$$

We carried out further calculations under the assumption that $\hat{S} = 1$, i.e., that the collision operator can be described with the help of the relaxation time $t_0(p)$. Such a simplification is valid since (as we shall see below) for $t_0 \gg a/|\bar{v}_\xi|$ (and it is principally this case that is of interest to us), the time t_0 does not usually enter into the expression for the oscillating part of the conductivity tensor. This is natural, because the oscillations are due to those electrons which move from one surface of conductor to the other without undergoing collisions in the interior of the conductor. In this case the term which describes the volume collisions is necessary only for the intermediate calculations. If $t_0 \sim a/|\bar{v}_\xi|$, then the relaxation time enters into the final result, since it determines the fraction of electrons which do not reach the other surface of the film and, consequently, do not take part in the oscillations. In this case the assumption on the relaxation time permits us to determine the order of magnitude of the amplitudes of the oscillations and their exact period, since the period depends only on the peculiarities of the spectrum and the relaxation time does not enter into it.

The solution of (2.5) periodic in τ has the form:

$$\varphi_k^{(\nu)}(\tau) = \Omega_0^{-1}\mathscr{E}_k^{(\nu)}\int_{-\infty}^{\tau} v_\nu(\tau)\exp\left[\int_{\tau}^{\tau'}(\gamma + ik\alpha)\,d\tau''\right]d\tau' - a^{-1}\Omega_0^{-2}\int_{-\infty}^{\tau}|v_\zeta(\tau')|\,g_\nu(\tau')\exp\left[\int_{\tau}^{\tau'}(\gamma + ik\alpha)\,d\tau''\right]d\tau', \tag{2.6}$$

where

$$g_\nu(\tau) = a\Omega_0[f_\nu(0) + f_\nu(0)].$$

In turn,

$$g_\nu(\tau) = 2\Omega_0\sum_k \varphi_k^{(\nu)} = 2\Phi_\nu(\tau) - (2/a\Omega_0)\sum_k\int_{-\infty}^{\tau}|v_\zeta(\tau')|\,g_\nu(\tau')\exp\left[\int_{\tau}^{\tau'}(\gamma + ik\alpha)\,d\tau''\right]d\tau', \tag{2.7}$$

where

$$\Phi_\nu(\tau) = \sum_k \mathscr{E}_k^{(\nu)}\int_{-\infty}^{\tau} v_\nu(\tau)\exp\left[\int_{\tau}^{\tau'}(\gamma + ik\alpha)\,d\tau''\right]d\tau'. \tag{2.8}$$

The series on the right-hand side of (2.7) can be transformed with the aid of the identity

$$\sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi\sum_{n=-\infty}^{\infty}\delta(x - 2\pi n).$$

Integrating the δ-function in (2.7), we obtain the equation for the function $g_\nu(\tau)$:

$$\sum_{\tau_n \leqslant \tau} g_\nu(\tau_n)\exp\left(-\int_{\tau_n}^{\tau}\gamma d\tau'\right) = \Phi_\nu(\tau), \tag{2.9}$$

where the functions $\tau_n(\tau)$ are determined from the condition

$$\left|\int_{\tau_n}^{\tau}\alpha d\tau'\right| = 2\pi n \quad \text{or} \quad \int_{\tau_n}^{\tau} v_\zeta d\tau' = na\Omega_0\bar{v}_\zeta/|\bar{v}_\zeta| \equiv nq\bar{v}_\zeta \tag{2.10}$$

(the bar denotes averaging over the period of θ). It is not difficult to establish that, if the velocity $v_\zeta(\tau)$ has one and the same sign for all τ, Eq. (2.10) has a unique solution $\tau_n(\tau)$ for a given n. In particular, its unique solution for $n = 0$ is $\tau_0(\tau) = \tau$. It is then quite simple to find the solution of (2.9). From the general equality

$$\int_{\tau_{n+1}}^{\tau} v_\zeta d\tau' = (n+1)q\bar{v}_\zeta$$

we subtract its value for $n = 1$. As a result we get the identity

$$\tau_n[\tau_1(\tau)] = \tau_{n+1}(\tau). \tag{2.11}$$

From this it easily follows that

$$g_\nu(\tau) = \Phi_\nu(\tau) - \Phi_\nu(\tau_1)\exp\left(-\int_{\tau_1}^{\tau}\gamma d\tau'\right) = \Phi_\nu(\tau) - \Phi_\nu(\tau_1)\exp\{-\bar{\gamma}(\tau - \tau_1) - \Gamma(\tau) + \Gamma(\tau_1)\}, \tag{2.12}$$

(where $\gamma(\tau) = \bar{\gamma} + d\Gamma/d\tau$) is a solution of Eq. (2.9). The second component in (2.12) is an oscillating function of the variable q with period θ, i.e., an oscillating function of the field H. Indeed, let us set $\tau_1 = \tau - q + \Delta\tau$ and $v_\zeta = \bar{v}_\zeta + dW_\zeta/d\tau$. The function $W_\zeta(\tau)$ is a periodic function of τ with period θ. The function $\Delta\tau$ satisfies the equation

$$\Delta\tau(\tau, q) = W_\zeta(\tau)/\bar{v}_\zeta - W_\zeta(\tau - q + \Delta\tau)/\bar{v}_\zeta \tag{2.13}$$

and is consequently a periodic function of the second argument. But $\Phi_\nu(\tau_1) = \Phi_\nu(\tau - q + \Delta\tau)$, i.e., $\Phi_\nu(\tau_1)$ [and also $\Gamma(\tau_1)$], together with $g_\nu(\tau)$, are periodic functions of q. It is clear that the functions $\varphi_k^{(\nu)}(\tau)$ determined by Eq. (2.6) are, generally speaking, oscillating functions of q with the same period. This dependence is important and leads to the observed effects if the exponent in (2.12) is $\lesssim 1$. But $\tau - \tau_1 \approx q$, and this condition takes the form $a/|\bar{v}_\zeta|t_0 \lesssim 1$, i.e., it coincides with the inequality (1.1) given originally. The period of oscillation $\theta = 2\pi m^*/m_0$ depends on ϵ and p_z. Therefore, it will frequently be of use to us to replace q by the variable $u = qm_0/m^* = aeH/m^*c|\bar{v}_\zeta|$. The quantities $\Delta\tau$ and $\Phi(\tau_\zeta)$ are oscillating functions of this variable with period 2π.

For certain values of ϵ and p_z, the function $v_\zeta(\tau)$ can be shown to be fluctuating in sign. Then for some τ, Eq. (2.10) at $n = 0$ can have several solutions, i.e., the function $\tau_0(\tau) \leq \tau$ can be nonunique. Reasoning as above, it is not difficult to show that for such τ the solution of (2.9) has the form

$$g_\nu(\tau) = \Phi_\nu(\tau) - \Phi_\nu(\tau_0)\exp\left(-\int_{\tau_0}^{\tau}\gamma d\tau'\right), \tag{2.14}$$

where τ_0 denotes the maximum value of the function $\tau_0(\tau) \leq \tau$, different from τ. For τ such that the function $\tau_0(\tau)$ is single valued Eq. (2.12) holds as before, but now we mean by τ_1 the maximum value, generally speaking, of the multiply-valued function $\tau_1(\tau)$.

We note that (2.14) contains no oscillating dependence on the magnetic field H, inasmuch as the magnetic field does not enter at all into the equation for $\tau_0(\tau)$. The reason for this is the following. If the velocity $v_\zeta(\tau)$ is a sign-alternating function of τ, then an electron leaving the surface of the film at some value of τ can, by describing a fraction of a turn, return to the same surface and experience diffuse reflection from it. For such τ, the function $\tau_0(\tau) \le \tau$ is non-unique, but, inasmuch as the value of this fraction is independent of H, $g_\nu(\tau)$ has a non-periodic dependence on H. However, even in this case there are electrons which make a contribution to the oscillations. The fact is that if the field H is not parallel to the film surface then, in general, for arbitrary ϵ and p_z, there exist initial values of τ for which the electron reaches the opposite surface of the film. For such τ, the function $\tau_0(\tau) \le \tau$ is single-valued and consequently, the oscillating function (2.12) is valid for $g_\nu(\tau)$.

3. The experimentally-measured components of the mean current density are*

$$\bar{j}_\mu = a^{-1}\int_0^a j_\mu(\zeta)\,d\zeta = -\frac{2e^2m_0}{ah^3}\sum_{\nu=1}^{3}\int v_\mu\varphi_0^{(\nu)}\frac{\partial F_0}{\partial\varepsilon}\,d\tau\,dp_z\,d\varepsilon$$

$$= -\frac{2e^2m_0}{ah^3}\sum_{\nu=1}^{3}\int dp_z\,d\varepsilon\frac{\partial F_0}{\partial\varepsilon}$$

$$\times\left\{\frac{1}{\Omega_0}\mathscr{E}_0^{(\nu)}\int_0^\theta v_\mu(\tau)\int_{-\infty}^{\tau}d\tau'\exp\left(\int_\tau^{\tau'}\gamma d\tau''\right)v_\nu(\tau')\right. \quad (3.1)$$

$$-\frac{1}{a\Omega_0^2}\sum_k\int_0^\theta d\tau v_\mu(\tau)\int_{-\infty}^{\tau}d\tau'\,|v_\zeta(\tau')|\int_{\tau'^{(1)}}^{\tau'}d\rho$$

$$\left.\times\exp\left[ik\int_{\tau'}^{\rho}\alpha d\rho' + \int_\tau^\rho\gamma d\rho'\right]v_\nu(\rho)\right\} \equiv -\frac{2e^2m_0}{ah^3}$$

$$\times\sum_{\nu=1}^{3}\int dp_z\,d\varepsilon\frac{dF_0}{d\varepsilon}\left(\frac{1}{\Omega_0}\mathscr{E}_0^{(\nu)}A_{\mu\nu} + \frac{1}{a\Omega_0^2}\sum_k B^k_{\mu\nu}\mathscr{E}_k^{(\nu)}\right),$$

where $\tau^{(1)}$ is equal to τ_1 or τ_0, depending on whether Eq. (2.12) or Eq. (2.14) holds. The condition $j_\zeta(\zeta) = 0$ permits us to establish $E_\zeta(\zeta)$ as a linear function of E_ξ and E_η, and to obtain an expression for the current density in the form

$$\bar{j}_\mu = \sum_{\nu=1}^{2} s_{\mu\nu}E_\nu.$$

The effective-conductivity tensor of the film $s_{\mu\nu}$ (the inverse of the tensor $r_{\mu\nu}$ above) is also determined by experiment.

In order to compute its oscillating part $\Delta s_{\mu\nu}$, we find the derivatives

$$\partial^2 B^k_{\mu\nu}/\partial u^2 = (m^*/m_0)^2\,\partial^2 B^k_{\mu\nu}/\partial q^2,$$

from a knowledge of which we can easily find the oscillating parts of the coefficients $B^k_{\mu\nu}$. Let us first consider the case in which the function $\tau_0(\tau) \le \tau$ is single valued for all τ. Then

$$\frac{\partial}{\partial u}B^k_{\mu\nu} = \frac{m^*}{m_0}\frac{\partial}{\partial q}B^k_{\mu\nu} =$$

$$-\frac{m^*\bar{v}_\zeta}{m_0}\int_0^\theta d\tau v_\mu(\tau)\int_{-\infty}^{\tau}d\tau'\frac{|v_\zeta(\tau')|}{v_\zeta(\tau_1')}\times\exp\left(\int_\tau^{\tau'}\gamma d\tau''\right)v_\nu(\tau_1')$$

$$= -\frac{m^*|\bar{v}_\zeta|}{m_0}\int_0^\theta d\tau v_\mu(\tau)\int_{-\infty}^{\tau_1}v_\nu(\tau')\exp\left(\int_\tau^{\tau'}\gamma d\tau''\right)d\tau'. \quad (3.2)$$

Here we have used the identities $\partial\tau_1/\partial q = -\bar{v}_\zeta/v_\zeta(\tau_1)$ and $v_\zeta(\tau)d\tau = v_\zeta(\tau_1)d\tau_1$, which follow from the definition of the function $\tau_1(\tau)$. Then,

$$\frac{\partial^2}{\partial u^2}B^k_{\mu\nu} \equiv C_{\mu\nu}$$

$$= \bar{v}_\zeta^2\left(\frac{m^*}{m_0}\right)^2 e^{-\bar{\gamma}q}\int_0^\theta d\tau\frac{v_\mu(\tau)v_\nu(\tau_1)}{|v_\zeta(\tau_1)|}\times\exp\{\gamma\Delta\tau + \Gamma(\tau_1) - \Gamma(\tau)\}$$

$$\approx \bar{v}_\zeta^2\left(\frac{m^*}{m_0}\right)^2 e^{-\bar{\delta}u}\int_0^\theta d\tau\frac{v_\mu(\tau)v_\nu(\tau_1)}{|v_\zeta(\tau_1)|}. \quad (3.3)$$

We have left here only the zero-order terms in the expansion in $\gamma \ll 1$; $\delta = (m^*/m_0)\gamma$. It is clear that these terms are oscillating functions of u with period 2π; consequently, they are periodic functions of the magnetic field H. It is then not difficult to prove that the oscillating parts of the coefficients $B_{\mu\nu}$ computed to this degree of accuracy have the form $e^{-\bar{\delta}u}b_{\mu\nu}(\epsilon, p_z, u)$, where $b_{\mu\nu}$ are periodic functions of u. If $\nu = \zeta$, then the $v_\zeta(\tau_1)$ in the numerator and denominator of (3.3) cancel and the zero-order term (in the expansions in powers of γ) does not oscillate. Therefore, if $E_\zeta(\zeta)$ is of the same order as E_ξ and E_η, the contributions of the coefficients $B_{\mu\nu}$ ($\nu \ne \zeta$) in the oscillating part of the current density $\bar{j}_\mu$ is, roughly speaking, t_0/T times larger than the contribution of the coefficients $B_{\mu\zeta}$, and the latter can be neglected. Then the sum (3.1) is simplified, since only terms with $k = 0$ enter into it, inasmuch as E_ξ and E_η do not depend on ζ. It is not difficult to carry out all the calculations in the general case; however, henceforth we shall assume this simplification. We note that we can neglect the oscillations connected with E_ζ in every case if $t_0 \sim a/v_0$, where v_0 is of the order of the velocity on the Fermi surface. Such an estimate is easily obtained if we consider that the tensor $s_{\nu\mu}$ is of the order

$$\sigma^{(\infty)}_{\mu\nu} - \sigma^{(\infty)}_{\mu\zeta}\sigma^{(\infty)}_{\zeta\nu}/\sigma^{(\infty)}_{\zeta\zeta},$$

*After the present research was completed, a paper was published by Kaner[6] on the conductivity of metallic films in a magnetic field, in which the same expression is obtained for the current density in the film. However, the problem of the oscillating conductivity of the film was not considered there.

in this case, where $\sigma_{\mu\nu}^{(\infty)}$ is the conductivity tensor of a macroscopic sample, as computed in reference 2.

Now, if ϵ and p_z are such that the function $\tau_0(\tau) \leq \tau$ is not single valued for all τ, we can obtain Eq. (3.3) for $C_{\mu\nu}$ in the same way; in such a case, we integrate over that part of the interval from 0 to θ, where this function is single valued. We note that if $t_0 \gg a/|\bar{v}_\xi|$, then $\bar{\delta} u \ll 1$, and the relaxation time does not enter into (3.3), nor into the expression for the oscillating part of the conductivity tensor.

With the help of the identity $v_\xi(\tau)d\tau = v_\xi(\tau_1)d\tau_1$, we obtain

$$\int \frac{v_\mu(\tau)\, v_\nu(\tau_1)}{|v_\xi(\tau_1)|} d\tau = \int \frac{v_\mu(\tau_{-1})\, v_\nu(\tau)}{|v_\xi(\tau_{-1})|} d\tau. \quad (3.4)$$

Here $\tau_1(\tau)$ is the solution of (2.10) for $n = -1$ [obviously, it is its own inverse function relative to $\tau_1(\tau)$]; the first integral extends over the region of single-valuedness of the function $\tau_0(\tau) \geq \tau$, while the second extends over the region of single-valuedness of $\tau_0(\tau) \geq \tau$. It is easily verified that, for a reversal of the direction of the magnetic field, we can obtain an expression for the oscillating part of the current, replacing $\tau_1(\tau)$ in (3.3) by $\tau_{-1}(\tau)$. It therefore follows from (3.4) that $\Delta s_{\mu\nu}(-\mathbf{H}) = \Delta s_{\nu\mu}(\mathbf{H})$.

4. If $u \ll \epsilon_0/kT$, we can set $\partial F/\partial\epsilon = -\delta(\epsilon - \epsilon_0)$, and then

$$\Delta s_{\mu\nu} = (2e^2 m_0 / ah^3\Omega_0^2) \int_0^{p_z^0} ap_z L_{\mu\nu}(p_z, u)$$
$$= (2e^2 m_0 / ah^3\Omega_0^2) \int_0^{p_z^0} dp_z e^{-Q} l_{\mu\nu}(p_z, u), \quad (4.1)$$

where

$$l_{\mu\nu} = b_{\mu\nu}(\varepsilon_0, p_z, u) + b_{\mu\nu}(\varepsilon_0, -p_z, u), \quad Q = a\bar{t}_0^{-1}/|\bar{v}_\xi|.$$

The quantity $L_{\mu\nu}$ depends smoothly on p_z, and oscillates like the function $u \gg 1$. This means that $L_{\mu\nu}(p_z, u)$ oscillates as a function of the field H, while the period of oscillation $\Delta H = 2\pi m^* c|\bar{v}_\xi|/ea$ depends on p_z, i.e., as noted earlier, different periods of oscillation correspond to groups of electrons with different p_z. Oscillations of different types entering into the total effect described by Eq. (4.1) are connected with different portions of the integration inverval. Oscillations of the first type are determined by p_z close to p_z^0. Oscillations of the second type are connected with such $p_z = p_{zi}$, for which $du/dp_z = 0$. For such p_z, the function $\Delta H(p_z)$ has an extremum.

Let us begin with the consideration of oscillations of the first type. In the vicinity of the point p_z^0, the equation of the Fermi surface $\epsilon(\mathbf{p}) = \epsilon_0$ has the form

$$v_z^0 \Delta p_z + \frac{1}{2} \sum_{i,k=1}^{3} \beta_{ik} \Delta p_i \Delta p_k = 0, \quad (4.2)$$
$$\beta_{ik} = (\partial^2\varepsilon / \partial p_i \partial p_k)_0; \quad \Delta p_i = p_i - p_i^0.$$

We consider the point p_z^0 to be elliptic; consequently, the quadratic form $\sum_{i,k=1}^{2} \beta_{ik}\Delta p_i \Delta p_k$ is positive definite. The equations of motion of the electron in the zero approximation in the electric field have the form[2]

$$\frac{1}{m_0}\frac{dp_x}{d\tau} = -v_y, \quad \frac{1}{m_0}\frac{dp_y}{d\tau} = v_x. \quad (4.3)$$

As will be seen later [see Eq. (4.6)], the region of integration important to the determination of the first type is the one for which $|\Delta p_z|/p_z^0 \sim 1/u \ll 1$. It is obvious that $\Delta p_x \sim \Delta p_y \sim u^{-1/2} p_z^0$ in this region. Substituting into (4.3) the expressions for the velocities v_x and v_y [obtained from (4.2)] we get (with accuracy up to quantities of order higher than $u^{-1/2}$):

$$v_x = \sqrt{2v_z^0(p_z^0 - p_z)}\,(\beta_1^{1/2}\cos\omega\tau\cos\varphi + \beta_2^{1/2}\sin\omega\tau\sin\varphi);$$

$$v_y = \sqrt{2v_z^0(p_z^0 - p_z)}\,(-\beta_1^{1/2}\cos\omega\tau\sin\varphi + \beta_2^{1/2}\sin\omega\tau\cos\varphi);$$

$$v_z = v_z^0 + \sqrt{2v_z^0(p_z - p_z)}\,(\beta_{x'z}\beta_1^{-1/2}\cos\omega\tau + \beta_{y'z}\beta_2^{-1/2}\sin\omega\tau). \quad (4.4)$$

Here β_1 and β_2 are the principal values of the tensor β_{ik} $(i, k = 1, 2)$, φ is the angle between the xy axis and the principal axes x′, y′, and $\omega = m_0\sqrt{\beta_1\beta_2} = m_0/m^*$. It then follows that $u = \omega q = aeH\sqrt{\beta_1\beta_2}/c|\bar{v}_\xi|$. Now, $v_\xi = -v_y \sin\vartheta + v_z\cos\vartheta$ (we assume that the axes x and ξ coincide); therefore, if $\cos\vartheta > u^{-1/2}$, then $\tau_1 = \tau - q$ with accuracy up to quantities of the order $u^{-1/2}$, inasmuch as $w_\xi \sim v_\xi^0 u^{-1/2}$. If $\cos\vartheta > u^{-1/2}$, the function $\tau_0(\tau)$ is single valued for all significant values of p_z.

It follows from (3.3) and (4.4) that the oscillating part of the coefficients $C_{\mu\nu}$, computed with relative accuracy up to $1/u$, is proportional to the sum of integrals of the form

$$\int_0^\theta \cos\omega\tau\cos\omega(\tau - q)\,d\tau = \pi\omega^{-1}\cos u$$

and identical expressions with two sines or sine and cosine. It is therefore not difficult to demonstrate that the coefficients $b_{\mu\nu}$ have the form

$$b_{\mu\nu} = c_{\mu\nu} m_0^{-3} (\beta_1\beta_2)^{-1} (v_z^0)^2 |p_z^0 - p_z| \cos(u + \lambda_{\mu\nu}), \quad (4.5)$$

where $c_{\mu\nu}$ is a constant on the order of unity. Inasmuch as $b_{\mu\nu}(-p_z, u) = b_{\mu\nu}(p_z, u)$ in the given case, the part of the conductivity tensor connected with oscillations of the first type is

$$\Delta s_{\mu\nu}^{(1)} = \frac{4c_{\mu\nu}e^2 (v_z^0)^2}{ah^3\Omega^2} \int^{p_z^0} e^{-Q} (p_z^0 - p_z) \cos (u + \lambda_{\mu\nu}), \quad (4.6)$$

where $\Omega = eH/m^*c = eh\sqrt{\beta_1\beta_2} / c$. We expand Q and u in powers of $p_z - p_z^0$, limiting ourselves to the zero term in the former case and the linear term in the latter, and integrate by parts. We get

$$\Delta s_{\mu\lambda}^{(1)} = -\frac{4c_{\mu\nu}e^2 (v_z^0)^2}{a^3h^3\Omega^4\left(\frac{d}{dp_z}\frac{1}{|\bar{v}_\zeta|}\right)_0^2} e^{-Q_0} \cos\left[\frac{aeH\sqrt{\beta_1\beta_2}}{c\,|v_z^0|\cos\vartheta} + \lambda_{\mu\nu}\right]. \quad (4.7)$$

We note that $\sqrt{\beta_1\beta_2}/|v_z^0| = \sqrt{K}$, where K is the Gaussian curvature of the Fermi surface.

Thus, the oscillations of the first type are sinusoidal; their amplitudes fall off as H^{-4}, while their period is

$$\Delta H = 2\pi c \cos\vartheta / ae\sqrt{K}. \quad (4.8)$$

Its measurement determines the Gaussian curvature of the Fermi surface at the elliptic point, the normal to which is parallel to the direction of the magnetic field.

If the Fermi surface is nonconvex, then there can be several such points, and to each of them there will generally correspond a period of oscillation. If the Fermi surface is open (closed trajectory), then there can exist a maximum momentum p_z^0 for certain directions of H; oscillations of the first type are then lacking. In the case of closed Fermi surfaces, these oscillations are always present.

5. We now consider oscillations of the second type. For simplicity, we shall assume that there is only one point $p_z^* = p_z$ at which the derivative du/dp_z vanishes. In order to determine the oscillating part of the integral (4.1) connected with the behavior of the integrand $L_{\mu\nu}(p_z, u)$ near $p_z = p_{z1}$, we expand its second argument in powers of $p_z - p_{z1}$ as far as the quadratic term, while we set the first argument equal to p_{z1}, since $L_{\mu\nu}$ is a smooth function:

$$L_{\mu\nu}(p_z, u) \approx L_{\mu\nu}[p_{z1}, u_1 + u_1''(p_z - p_{z1})^2/2]. \quad (5.1)$$

In this case, inasmuch as we are only interested in the oscillating component in (4.1), we can consider that

$$\bar{L}_{\mu\nu}(p_{z1}) = (2\pi)^{-1}\int_0^{2\pi} L_{\mu\nu}(p_{z1}, u)\,du = 0.$$

Then the part of the conductivity tensor connected with oscillations of the second type is

$$\Delta s_{\mu\nu}^{(2)} = \frac{2e^2m_0}{ah^3\Omega_0^2}\int_{p_1}^{p_2} L_{\mu\nu}\left[p_{z1}, u_1 + \frac{1}{2}u_1''(p_z - p_{z1})^2\right] dp_z. \quad (5.2)$$

Here the upper and lower limits of the integral are so chosen that for $p_z = p_1$ and $p_z = p_2$, we can, with sufficient accuracy, limit ourselves to the quadratic terms in the expansion of (5.1). We introduce a new integration variable $\xi = \sqrt{|u_1''|} \times (p_z - p_{z1})$. If the magnetic field is large enough that $|\xi_i| = \sqrt{|u_1''|}\,|p_i - p_{z1}| \gg 1$, it is easy to show that oscillations of the second type are present, and that their amplitudes fall off as $H^{-5/2}$. Actually, limiting ourselves in accuracy to $1/|\xi_i|$, we can integrate in ξ over infinite limits and get

$$\Delta s_{\mu\nu}^{(2)} = \frac{2e^2m_0}{ah^3\Omega_0^2\sqrt{eaH/c}}\left|\frac{d^2}{dp_z^2}\frac{1}{|\bar{v}_\zeta|\,m^*}\right|_{(1)}^{-1/2} \times \int_{-\infty}^{\infty} L_{\mu\nu}(p_{z1}, u \pm \xi^2/2)\,d\xi. \quad (5.3)$$

The period of oscillation is $\Delta H = 2\pi c\,|\bar{v}_{\zeta_1}|\,m_1^*\,|\,ea$. We emphasize that these oscillations are absent in every case when the derivative does not generally vanish in the interval $0 < p_z \le p_z^0$. In particular, we can show that this derivative does not vanish if the Fermi surface is an ellipsoid. Of, course, the presence of oscillations of the second type is evidence of a significant deviation in the shape of the Fermi surface from ellipsoidal. Therefore, the shape of these oscillations should generally be quite different from sinusoidal.

6. In addition to the two types of oscillations considered here, there is a third type connected with those values of p_z for which a non-single valuedness arises in the function $\tau_0(\tau) \le \tau$. These oscillations are generally non-sinusoidal and their amplitudes fall off with increasing field more rapidly than was the case with the first two types (as H^{-5}). If the Fermi surface has a complicated structure (for example, if it is self-intersecting) then still other types of oscillations can exist.

It is not difficult to estimate the order of magnitude of the amplitudes of the oscillations

$$\Delta s_{\mu\nu} \approx e^2an(m^*v_0)^{-1}(v_0T/a)^k \sim H^{-k},$$

where n is the density of electrons and k = 4 or 5/2 for oscillations of the first or second type, respectively. Estimates in such form are suitable in that they do not depend on the number of components of the tensor $\Delta s_{\mu\nu}$ and on the magnitude of the ratio t_0v_0/a (it is assumed only that $t_0v_0/a \gtrsim 1$). It is then easy to obtain estimates for the relative magnitude of the oscillating part of the specific resistance. For example, if $t_0v_0/a \sim 1$, then

$$s_{\mu\nu} \sim \sigma_{\mu\nu}^{(\infty)} - \sigma_{\mu\zeta}^{(\infty)}\sigma_{\zeta\nu}^{(\infty)}/\sigma_{\zeta\zeta}^{(\infty)},$$

and we find that when $\sin\theta \gg v_0T/a$, while open trajectories are absent,

$$\Delta r_{\mu\nu}/r_{\mu\nu} \approx (v_0T/a)^{k-2}.$$

We note in conclusion that the study of oscillations allows us, in a number of cases, to proceed directly to qualitative conclusions regarding the shape of the Fermi surface. The presence of oscillations of the second type points up a significant departure of its shape from the ellipsoidal. The presence of several component oscillations of the first type with different periods demonstrates that the Fermi surface is nonconvex or, that there are several surfaces. Complete disappearance of the oscillations of the first type for certain directions of the magnetic field, not lying in the plane of the film, in the case in which the inequality (1.1) is satisfied, means that the Fermi surface is open in these directions.

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