To perform the last integration we introduce the cut-off parameter Λ, and then

$$I = \frac{i}{(4\pi)^6}\frac{i}{\lambda_{12}^2\lambda_{13}^2\lambda_{23}^2}[\ln\varphi\Lambda + 1 - \ln(\Box - i\varepsilon)]. \tag{A.8}$$

Noting that $\lambda_{12}\lambda_{13}\lambda_{23} = -\varphi$ and that the terms that do not depend on κ_{ik} make no contribution, we get the result (20) given in the text.

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PROPAGATION OF A NON-SELF-SIMILAR THERMAL WAVE

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The propagation of a thermal wave from an instantaneous point source in a gas is investigated with account of the temperature dependence of the internal energy of the gas. The case when the internal energy is associated not only with the matter but also with radiation is considered. The range of the radiation is assumed to depend on the temperature in accordance with a power law. An approximate method can also be used in the case of an arbitrary dependence of the internal energy and of the heat flux on the temperature.

LET a quantity of heat Q_0 be liberated at a given initial instant of time within a small volume (at a point). Then, if the density of the medium is constant, and the thermal conductivity and the specific heat are each proportional to the temperature raised to a certain power, the problem is a self-similar one and its solution can be obtained in closed form. Such a problem was investigated by Zel'dovich and Kompaneets.[1]

If a thermal wave propagates in a gas then, because of the high temperature, the molecules of the gas break up into atoms and the latter are ionized, and this leads to a temperature dependence of the internal energy of the gas. Calculations[2,3] show that the internal energy of a gas may be approximated over a wide range of temperatures by a power of the temperature ($\sim aT^{\lambda}$). However, at very high temperatures (on the order of several millions of degrees for air of normal density) it is necessary to take into account, in addition to the energy of the matter, also the radiation energy, which is proportional to the fourth power of the temperature $\sim bT^4$. Such a problem is no longer self-similar even if the radiation range is expressed by a power of the temperature. Another non-self-similar problem will be one in which the internal energy is given by a power of the temperature, but the range of radiation is given not by a single power, but involves two or more terms.

We shall discuss the problem of the propagation of a non-self-similar thermal wave by considering a special case when the internal energy is expressed by the following two term formula

$$E = aT^{\lambda} + bT^4 \tag{1}$$

(here $b = 4\sigma/c$, c is the velocity of light, σ =

5.67×10^{-5} ergs/cm^2 sec deg^4 is the Stefan-Boltzmann constant, and a and λ are arbitrary), while the mean free path of the radiation depends on the temperature in accordance with the power law $l = l_0 T^n$. The heat balance equation can be written in the form:

$$\frac{\partial E(T)}{\partial t} = \frac{c_0}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T^k}{\partial r}, \quad k = n + 4, \quad c_0 = 16\sigma l_0/3k. \tag{2}$$

We present an approximate method of solving Eq. (2), based on a number of physical considerations. Since the thermal conductivity depends strongly on the temperature ($\kappa \sim T^{k-1}$), then in the course of propagation of the thermal wave the temperature tends to a uniform distribution more rapidly in regions which have undergone greater heating, thus forming a "plateau." A sharp variation in the temperature occurs only in the narrow zone near the front, which in the first approximation need not be taken into account in the energy balance if the temperature of the "plateau" is ex pressed in terms of the radius of the wave front as follows:

$$Q_0 = 4\pi \int_0^{r_f} (aT_*^\lambda + bT_*^4) r^2 dr. \tag{3}$$

In order to determine $r_f(t)$, we shall make use of the integral relationship between the moments, which was investigated by Barenblatt.[4] If we multiply Eq. (2) by r^m ($m = 2, 3, 4 \ldots$) and integrate with respect to r between the limits from 0 to r_f, taking into account the conditions that the heat flux vanishes at the center and at the wave front:

$$r^2 \frac{\partial T^k}{\partial r}\bigg|_0 = r^2 \frac{\partial T^k}{\partial r}\bigg|_{r_f} = T(r_f, t) = 0, \tag{4}$$

we obtain an infinite number of relations which, taken together, are equivalent to Eq. (2). The first of these relations ($m = 2$) gives us the law of conservation of energy [Eq. (3)], the second ($m = 3$) leads to the additional condition

$$\frac{d}{dt}\int_0^{r_f} (aT_*^\lambda + bT_*^4) r^3 dr = 2c_0 \int_0^{r_f} rT_*^k dr. \tag{5}$$

In spite of the fact that the remaining equations remain unsatisfied, we can hope that there will be reasonably good agreement between the approximate and the exact solutions, since the temperature behind the wave front varies smoothly and monotonically.

Our problem is characterized by the following parameters, whose dimensions are expressed in terms of length [L], time [t], temperature [T] and quantity of heat [Q] as follows:

$$[a] = [Q]/[L]^3[T]^\lambda, \quad [b] = [Q]/[L]^3[T]^4, \quad [c_0] = [Q]/[L][T]^k[t], \quad [Q_0] = [Q]. \tag{6}$$

It is not possible to form a dimensionless constant from these quantities. The solution has therefore the important property of similarity. It can be recalculated independently for arbitrary values of the parameters a, b, c_0, and Q_0. If we eliminate T from (3) and (5) and introduce dimensionless variables, we obtain the law of motion of the wave front:

$$x^3\left[\left(\frac{1}{x^2}\frac{dx}{d\tau}\right)^{\lambda/k} + \left(\frac{1}{x^2}\frac{dx}{d\tau}\right)^{4/k}\right] = 1,$$
$$x = \left[\frac{4\pi}{3}\frac{a}{Q_0}\left(\frac{a}{b}\right)^{\lambda/(4-\lambda)}\right]^{1/3} r, \quad \tau = \left[\frac{16\pi^2}{9aQ_0^2}\left(\frac{a}{b}\right)^{(3k-\lambda)/(4-\lambda)}\right]^{1/3} 4c_0 t. \tag{7}$$

By denoting $x^{-2}dx/d\tau = p$ we can easily write down the solutions of Eq. (7) in parametric form:

$$\tau = \int_p^\infty \frac{x^2(\lambda p^{\lambda/k} + 4p^{4/k})\,dp}{3kp^2}, \quad x = (p^{\lambda/k} + p^{4/k})^{-1/3}. \tag{8}$$

In those cases when any one of the combinations

$$(\lambda - 3k)/\beta, \quad (3\lambda - k - 8)/\beta,$$

(where $\beta = 12 - 3\lambda$) and one of the two ratios $(12 - 3k - 2\lambda)/\beta$ and $(4 - 3k)/\beta$ are integers simultaneously, the integral (8) can be expressed in terms of elementary functions. Figure 1 shows the dependence of the dimensionless coordinate of the wave front x on the time τ obtained from the solution (8) for $\lambda = 1$ and $k = 4$ and 6. From an analysis of (7) it follows that the solution of our problem $x(\tau)$ approaches, for small values of τ, the limiting self-similar solution $x_1 = \left(\frac{(3k-4)}{4}\tau\right)^{4/(3k-4)}$ while for large τ it tends to $x_2 = \left(\frac{3k-\lambda}{\lambda}\tau\right)^{\lambda/(3k-\lambda)}$

For all values of τ, the curve $x(\tau)$ lies lower than in both self-similar cases (dotted curve in Fig. 2). However, the solution $x(\tau)$ practically turns out to be close to $x_1(\tau)$ as long as the value of the quantity $x^{(12-3\lambda)/4} = \mu$ is small compared to unity. If, for example, we consider $\mu = 0.03$ as small, then our solution coincides with the curve $x_1(\tau)$, so long as $x < x_{10}$, $x_{10} = (0.03)^{4/(12-3\lambda)}$ (if $\lambda = 1$, $x_{10} = 0.210$). In the case $x > x_{20}$, when we can neglect the quantity $x_2^{(3\lambda-12)/\lambda} = \nu$ compared to unity, our solution approaches the other self-similar solution $x_2(\tau)$ (if we set $\nu = 0.03$ then $x_{20} = 1.475$ for $\lambda = 1$). In spite of the fact that in terms of dimensionless coordinates the zone of the non-self-similar solution is important only within the narrow interval $x_{10} < x < x_{20}$, it turns out in practice to be the most interesting one, since outside this interval the solution (8) loses its meaning in a number of cases, owing to

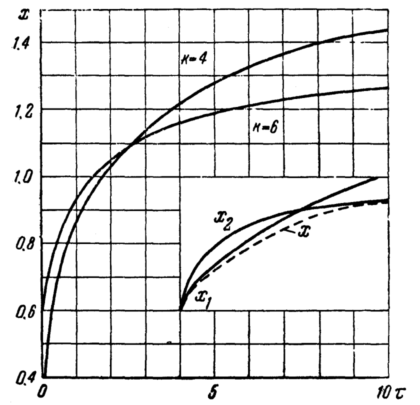

FIG. 1

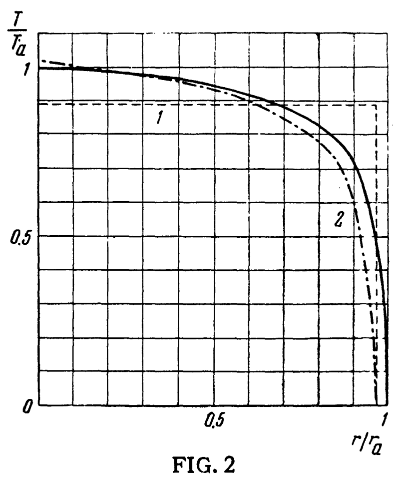

FIG. 2

the fact that the gas begins moving. The large pressure gradients at the wave front, correspond-to the temperature gradients, lead to the formation of a shock wave. After this has occurred the energy is transferred by a gas-dynamic mechanism, whereby the law of motion of a strong discontinuity differs from the law of propagation of a thermal wave, and our solution becomes unjustified.

A strong discontinuity can exist only if small perturbations propagated with the speed of sound $c_*(T)$ can catch up to the wave front. Since the thermal wave is slowed down very sharply, the limits of validity of the solution may be found, following a remark of A. S. Kompaneets, from the physical requirement $c_*(T_*) = \dot{r}_f$. In this connection it turns out that, for example, for air of normal density, $x_* < x_{20}$.

It is interesting to compare the solution of a self-similar problem obtained by the method just described with the exact solution, which in the case $\lambda = b = 0$ is given by the following formulas:

$$r_a = \left(t \frac{c_0}{a}\right)^{1/(3k-1)} \left[\frac{Q_0}{2\pi a \gamma(k)}\right]^{(k-1)/(3k-1)};$$
$$T = \left[\frac{a}{c_0} \frac{k-1}{k} r_f \dot{r}_f\right]^{1/(k-1)} [\eta - 0.5\eta^2]^{1/(k-1)}, \quad \eta = 1 - r/r_f; \tag{9}$$
$$\gamma(k) = \left[\frac{k-1}{2k(3k-1)}\right]^{1/(k-1)} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{k}{k-1}\right) \left[\Gamma\left(\frac{3}{2} + \frac{k}{k-1}\right)\right]^{-1};$$

where Γ is the gamma-function.

For the approximate solution we obtain:

$$r_0 = \left[t \frac{c_0}{a} (12k-4)\right]^{1/(3k-1)} \left[\frac{3Q_0}{4\pi a}\right]^{(k-1)/(3k-1)},$$
$$T_*(t) = \frac{3Q_0}{4\pi a r_0^3}. \tag{10}$$

Figure 2 gives the temperature distribution for $\lambda = b = 0$ and $k = 6$, obtained by means of the exact solution (9) (solid line) and by the approximate method, on the assumption that there is a temperature "plateau" behind the wave front (curve 1). The temperature distribution is given in dimensionless form in terms of its value $T_a = T(0, t)$, at the center of the self-similar wave, and is shown as a function of the ratio of the coordinate r to the radius of the front of the self-similar wave $r_a(t)$. If we compare (9) and (10) we find that the approximate solution agrees well with the exact one. For $k = 6$ we have

$$r_0 / r_a = 0.962, \quad T_* / T(0, t) = 0.889.$$

If we know the law of wave propagation (10) we can also find the temperature distribution near the front by expanding the solution in powers of η:

$$T = T_1(\eta, t) + T_2(\eta, t) + \cdots \tag{11}$$

We write Eq. (2) in terms of the variables t and $\eta(r, t)$:

$$\frac{\partial(aT^\lambda + bT^4)}{\partial t} + \frac{\dot{r}_f}{r_f}(1-\eta)\frac{\partial(aT^\lambda + bT^4)}{\partial \eta}$$
$$= \frac{2c_0}{r_f^2(\eta-1)} \frac{\partial T^k}{\partial \eta} + \frac{c_0}{r_f^2} \frac{\partial^2 T^k}{\partial \eta^2}. \tag{12}$$

Near the front of the wave $(\eta \to 0)$ the temperature tends to zero, and therefore the main terms of Eq. (12) must cancel one another:

$$\dot{r}_f r_f \partial T_1^\lambda / \partial \eta = c_0 \partial^2 T_1^k / \partial \eta^2.$$

This leads to the well known law for the variation of temperature near the wave front:

$$T_1 = \left[\frac{a}{c_0} \eta \frac{k-\lambda}{k} r_f \dot{r}_f\right]^{1/(k-\lambda)} = A_1 \eta^{1/(k-\lambda)}. \tag{13}$$

For the second term we obtain:

$$T_2 = A_2 \eta^{(5-\lambda)/(k-\lambda)}, \quad A_2 = bA_1^{5-\lambda} / a(4 + k - 2\lambda). \tag{14}$$

The series expansion gives the solution as a function of one arbitrary function $r_f(t)$, which can be approximately determined by the method described earlier. We note that the method of solving the problem for small values of η is analogous to the method of "short" waves proposed by Khristianovich[5] for treating a number of gas-dynamic problems that are physically characterized by the fact that an appreciable variation in the parameters of the gas occurs only in a narrow zone close to the front.

In the case of a self-similar problem, the first two terms of the expansion (11) give good agreement with the exact solution over the whole range $0 \leq \eta \leq 1$. The solution obtained by means of such an expansion in the case $\lambda = b = 0$, $k = 6$, is given by curve 2 of Fig. 2. Therefore the function $r_f(\dot{t})$ may be determined for such problems directly from the condition of heat balance:

$$Q_0 = 4\pi a A_1(t) r_f^3 \int_0^1 (1-\eta)^2 \left[\eta^{1/(k-1)} + \eta^{k/(k-1)} \frac{A_2}{A_1} + \cdots \right] d\eta. \tag{15}$$

It follows hence that γ_0 is given by formulas (9), in which one should set $\gamma \approx \gamma_0$:

$$\gamma_0 = 2\left[\frac{k-1}{k(3k-1)}\right]^{1/(k-1)} \left[\frac{4k^2-5k+2}{k(3k-2)} - \frac{8k^2-11k+4}{(2k-1)(4k-3)}\right]. \tag{16}$$

This value of γ_0 agrees well with the exact one. Thus, for example, for $k = 6$ we get $\gamma = 0.251$ and $\gamma_0 = 0.252$.

The approximate method just described may be made more precise if, for example, we replace T_* of Eqs. (3) and (5) with an expression for T which takes into account the variation of temperature in the zone near the wave front.

By the method described above we can also investigate the problem arising when both the internal energy and the range of radiation are expressed by formulas that contain several terms.

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