For $q^2 \ll M^2$ ($q^2 \lesssim \mu^2$) we neglect all the terms in (17) except for the first one, containing Φ_{B-H}. For $q^2 \gg m^2$ it is possible to neglect Φ_1 entirely, the remaining functions assume the simple form

$$\Phi_{B-H} = (p_+^2 + p_-^2)\, q^2 L \,/\, \sqrt{q^4 + 4q^2m^2} - 2p_+p_-;$$
$$\Phi = -4\omega^2 \ln (p_+\vartheta_{\max} / m). \tag{19}$$

We leave L in its previous form, in view of the large coefficients in the terms with m^2.

Formula (17) can still be integrated over dp_+ for $p_+ + p_- = \text{const} \approx \omega$ ($q^2 = \text{const}$) and $2M dp_- = -dq^2$. Then, for $q^2 \gg m^2$:

$$d\sigma = -\alpha^3 \frac{dq^2}{3q^4}\Big\{a^2(q^2)\Big[\frac{q^2}{\sqrt{q^4+4q^2m^2}}L - \frac{1}{2}\Big] - 3ab\frac{q^2}{M^2}\Big[\ln\frac{\omega\vartheta_{\max}}{2m} - 1\Big] + b^2\frac{q^2}{4M^2}\Big[\frac{q^2L}{\sqrt{q^4+4q^2m^2}} + 6\ln\frac{\omega\vartheta_{\max}}{2m} - \frac{13}{2}\Big]\Big\}. \tag{20}$$

Analogous results can also be obtained for bremsstrahlung. For this purpose it is necessary to replace in the matrix elements (2)

$$\mathbf{p}_+,\ \varepsilon_+ \to -\mathbf{p}_1,\ -\varepsilon_1,\quad \mathbf{p}_-,\ \varepsilon_- \to \mathbf{p}_2,\ \varepsilon_2;\quad \boldsymbol{\omega},\ \omega \to -\boldsymbol{\omega},\ -\omega$$

and in formulas (9) to take $d^3p_2d^3\omega/(2\pi)^6$ instead of the statistical factor $d^3p_+d^3p_-/(2\pi)^6$. The results can be obtained from formulas (10) to (15) by substituting

$$\frac{p_+p_-dp_+}{\omega^3}\, do_+do_- \to \frac{p_2}{p_1}\frac{d\omega}{\omega}\, do_{\mathbf{p}_2}do_\omega;$$

$$\vartheta_+ \to \vartheta;\ \ \vartheta_- \to \vartheta - \vartheta_2,\ \ p_+ \to -p_1$$

(ϑ_2 is the angle between $\mathbf{p}_2$ and $\mathbf{p}_1$).

In conclusion, the author expresses sincere gratitude to I. M. Shmushkevich for suggesting the topic and for valuable advice.

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THE POLARIZATION OF THE INTERNAL-CONVERSION ELECTRONS EMITTED AFTER β-DECAY

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A discussion is given of the correlation of the polarization of internal-conversion electrons with the direction of emission of the electrons in the preceding β-decay. If one neglects the Coulomb field of the nucleus, then in the case of a magnetic multipole the polarization is longitudinal and does not depend on the energy. In the case of an electric multipole both longitudinal and transverse polarizations occur, with dependence on the energy.

1. Owing to the nonconservation of parity in β-decay the daughter nucleus is polarized in the direction of the emitted β-decay electron (the parent nucleus is supposed unpolarized, and the direction of emission of the neutrino is not observed). Therefore if an internal-conversion process occurs after the β-decay, the conversion electrons must possess a preferred polarization.* This effect can be used both in studying β-decay and also in studying the properties of nuclear levels, since (as will be shown below) the character of the polarization

*Our attention was called to the existence of such an effect by A. I. Alikhanov and V. A. Liubimov.

of the conversion electrons depends in an essential way on the order and type (electric or magnetic) of the multipole involved in the nuclear transition.

The general expression for the polarization vector $\langle\sigma\rangle$ of the internal-conversion electron, for the case of an allowed β-decay transition, has the following form:

$$a(\mathbf{v}\cdot\mathbf{n})\mathbf{n} + b(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}), \qquad (1)$$

where a and b are constants depending on the angular momenta of the nuclear states and the energy of the transition, $c\mathbf{v}$ is the velocity of the β particle, and $\mathbf{n}$ is the unit vector of the direction of the conversion transition.

In fact, Eq. (1) is a general expression with the following properties: (1) it is a polar vector, corresponding to the fact that the polarization appears only as a result of the nonconservation of parity in β-decay ($\langle\sigma\rangle$ is an axial vector); (2) it is invariant under replacement of $\mathbf{n}$ by $-\mathbf{n}$, which corresponds to the conservation of parity in the internal-conversion process; (3) it is proportional to the velocity vector of the β particle, which determines the polarization of the daughter nucleus.

2. Let us find the density matrix characterizing the polarization state of the nucleus formed as a result of the β-decay. If this nucleus is in a state with angular momentum j_2, and if the initial nucleus was unpolarized and the direction of the neutrino was not observed, then the general expression for the density matrix that determines the distribution of the angular-momentum component m_2 must have the following form:

$$\rho_{m_2 m_2'} = \frac{1}{2j_2+1}\left\{\delta_{m_2 m_2'} + \left(\frac{j_2+1}{j_2}\right)^{1/2} \zeta C^{j_2 m_2}_{j_2 m_2', 1\mu} v^{\mu}\right\}, \qquad (2)$$

where $v^{\mu} = (-1)^{\mu} v_{-\mu}$ are the components of the vector ($v^0 = v_z$; $v^{\pm 1} = \mp (v_x \mp i v_y)/2^{1/2}$), and $C^{\cdot\cdot}_{\cdot\cdot}$ are the coefficients of vector composition (Clebsch-Gordan coefficients), which differ from the matrix elements of the angular-momentum operator $\mathbf{J}$ only by a normalizing factor:

$$(jm\,|\,J_{\mu}\,|\,jm') = (jm'\,|\,J^{\mu}\,|\,jm) = \sqrt{j(j+1)}\, C^{jm}_{jm', 1\mu}.$$

The constant ζ can be expressed in terms of the constant that appears in the expression W for the angular distribution of the β particles from the decay of polarized nuclei.

For this purpose let us consider the β-decay of a polarized nucleus with angular momentum j_2 into a nucleus with angular momentum j_3. The probability for the decay can be written in the form

$$W = \mathrm{Sp}\, \rho^0 U^+ U,$$

where U is the operator describing the transition $j_2 \rightarrow j_3$ and ρ^0 is the density matrix of the state j_2. If the state of polarization is defined by the average angular momentum vector $\langle\mathbf{J}\rangle$, then

$$\rho^0_{m_2 m_2'} = \frac{1}{2j_2+1}\left\{\delta_{m_2 m_2'} + \frac{3\langle J_{\mu}\rangle}{\sqrt{j_2(j_2+1)}} C^{j_2 m_2}_{j_2 m_2', 1\mu}\right\}.$$

The matrix U^+U is proportional to the expression (2). Indeed, the density matrix (2) is determined by the transition $j_3 \rightarrow j_2$, and since the state j_3 is not polarized,

$$\rho_{m_2 m_2'} = (U^+U)_{m_2 m_2'} / \mathrm{Sp}\, U^+U.$$

Consequently, apart from a common factor

$$W \sim \left(\delta_{m_2 m_2'} + \frac{3\langle J_{\nu}\rangle}{\sqrt{j_2(j_2+1)}} C^{j_2 m_2}_{j_2 m_2', 1\nu}\right)$$

$$\times \left(\delta_{m_2 m_2'} + \left(\frac{j_2+1}{j_2}\right)^{1/2} \zeta C^{j_2 m_2}_{j_2 m_2', 1\mu} v^{\mu}\right).$$

Performing the summation over m_2 and m_2', we get

$$W = 1 + \zeta \langle\mathbf{J}\rangle \mathbf{v} / j_2. \qquad (3)$$

Thus we can omit consideration of the β-decay stage of the process, and determine the constant ζ in Eq. (2) from a comparison of Eq. (3) with the known expression for W.[1,2] In particular, for an allowed β transition in the case of S, T, A, and V interaction variants (with neglect of Coulomb forces) ζ is given by

$$\zeta = 2\,\mathrm{Re}\Big\{(c_S c_T'^* + c_S' c_T^* - c_V c_A'^* - c_V' c_A^*)$$
$$\times \left(\frac{j_2}{j_2+1}\right)^{1/2} \delta_{j_2 j_3} M_F M_{GT}^* + (c_T c_T'^* - c_A c_A'^*)\, \Lambda_{j_2 j_3} |M_{GT}|^2\Big\}$$
$$\times \{(|c_S|^2 + |c_S'|^2 + |c_V|^2 + |c_V'|^2)|M_F|^2$$
$$+ (|c_T|^2 + |c_T'|^2 + |c_A|^2 + |c_A'|^2)|M_{GT}|^2\}^{-1},$$

where

$$\Lambda_{j_2 j_3} = \{j_2(j_2+1) - j_3(j_3+1) + 2\} / \{2j_2(j_2+1)\},$$
$$M_F = \left(\int 1\right), \quad M_{GT} = \left(\int \sigma\right).$$

3. Suppose the nucleus makes a further transition from the state $j_2 m_2$ to the state $j_1 m_1$. The matrix element of the internal-conversion process can be written in the following form[3] (omitting common factors that are of no importance for our purpose):

$$M_{m_2 m_1} = (j_2 m_2\,|\,Q^{(\lambda)}_{LM}\,|\,j_1 m_1)^* \int \psi_2^*(\mathbf{r})\, B^{(\lambda)}_{LM}(\mathbf{r})\, \psi_1(\mathbf{r})\, d\mathbf{r}. \qquad (4)$$

Here $Q^{(\lambda)}_{LM}$ is the operator of the 2^L-pole electric ($\lambda = 1$) or magnetic ($\lambda = 0$) moment of the nucleus, corresponding to the transition in question; ψ_1 and ψ_2 are the wave functions of the electron for the initial and final states; and $B^{(\lambda)}_{LM}$

is the operator of the interaction of the electron with the multipole field. This operator has the following form

$$B_{LM}^{(0)} = \boldsymbol{\alpha}\mathbf{Y}_{LLM}(\mathbf{r}/r)\,G_L(\omega r),$$

$$B_{LM}^{(1)} = Y_{LM}\left(\frac{\mathbf{r}}{r}\right)G_L(\omega r)$$

$$+\sqrt{\frac{2L+1}{L}}\,\boldsymbol{\alpha}\mathbf{Y}_{L,\,L-1,\,M}\left(\frac{\mathbf{r}}{r}\right)G_{L-1}(\omega r).$$

$$G^l(x) = i^l H^{(1)}_{l+1/2}(x)/\sqrt{x},$$

where α is the Dirac matrices, ω is the energy of the transition, $H^{(1)}$ is a Hankel function, Y_{LM} is a spherical harmonic, and $\mathbf{Y}_{LlM}$ is a spherical vector, the components of which are defined in the following way:

$$(Y_{LlM})^{\mu} = C^{LM}_{lm,1\,\mu}Y_{lm}.$$

We confine ourselves here to the free-electron approximation

$$\psi_2 = \begin{pmatrix} u \\ \frac{\mathbf{q}\boldsymbol{\sigma}}{\varepsilon+m}u \end{pmatrix} e^{i\mathbf{qr}};\quad \psi_1 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix};\quad \varepsilon = m+\omega,$$

where ϵ and $\mathbf{q}$ are the energy and momentum of the conversion electron, u and u_0 are two-component spinors, and σ is the Pauli matrices. Then the integral in Eq. (4) reduces to the following:

$$\int e^{i\mathbf{qr}}Y_{lm}\left(\frac{\mathbf{r}}{r}\right)G_l(\omega r)\,d\mathbf{r} \sim Y_{lm}(\mathbf{n})\left(\frac{q}{\omega}\right)^l;\quad \mathbf{n} = \frac{\mathbf{q}}{q}.$$

Omitting unimportant factors, we get the expression for the matrix element

$$M_{m_2m_1} = (j_2m_2\,|\,Q^{(\lambda)}_{LM}\,|\,j_1m_1)\,u^*V^{(\lambda)}_{LM}u_0, \tag{5}$$

where in the case of a magnetic multipole ($\lambda = 0$)

$$V^{(0)}_{LM} = (\boldsymbol{\sigma}\cdot\mathbf{n})\,\boldsymbol{\sigma}\mathbf{Y}_{LLM}(\mathbf{n}), \tag{5a}$$

and in the case of an electric multipole ($\lambda = 1$)

$$V^{(1)}_{LM} = Y_{LM}(\mathbf{n}) + \sqrt{\frac{2L+1}{L}}\,\varkappa(\boldsymbol{\sigma}\cdot\mathbf{n})\,\boldsymbol{\sigma}\mathbf{Y}_{L,\,L-1,\,M}(\mathbf{n}); \tag{5b}$$

$$\varkappa = \frac{\varepsilon-m}{\varepsilon+m}.$$

The probability of internal conversion is given by the quantity

$$W = \rho_{m_2m_2'}M_{m_1m_2}M^*_{m_2'm_1}. \tag{6}$$

If we represent W in the form

$$W = P_{\alpha\beta}u_{\alpha}u^*_{\beta},$$

then the density matrix of the conversion electron is obviously equal to P/Sp P. Furthermore, if P has the form

$$P = A(1+\boldsymbol{\xi}\boldsymbol{\sigma}), \tag{6a}$$

then the polarization vector of the conversion electron is given by

$$\langle\boldsymbol{\sigma}\rangle = \boldsymbol{\xi}. \tag{6b}$$

4. Substituting Eq. (5) into Eq. (6), we use the fact that since the electron in the initial state is not polarized $u^0_{\alpha}u^{0*}_{\beta} = \delta_{\alpha\beta}$. Then we get the following expression for P:

$$P = \rho_{m_2m_2'}\,(j_2m_2\,|\,Q^{(\lambda)}_{LM}\,|\,j_1m_1)^*\,(j_2m_2'\,|\,Q^{(\lambda)}_{LM'}\,|\,j_1m_1)\,V^{(\lambda)}_{LM}V^{(\lambda)*}_{LM'}. \tag{7}$$

The product of the last two factors, which is a matrix with respect to the spin variables, can be represented in the form

$$V^{(\lambda)}_{LM}V^{(\lambda)*}_{LM'} = R^{(\lambda)}_{MM'} + \boldsymbol{\sigma}\mathbf{S}^{(\lambda)}_{MM'}. \tag{8}$$

Using the expressions (5a) and (5b), we get

$$R^{(0)}_{MM'} = \mathbf{Y}_{LLM}\cdot\mathbf{Y}^*_{LLM'},\quad \mathbf{S}^{(0)}_{MM'} = i\,[\mathbf{Y}_{LLM}\times\mathbf{Y}^*_{LLM}];$$

$$R^{(1)}_{MM'} = (1+2\varkappa)\,Y_{LM}Y^*_{LM'} + \frac{2L+1}{L}\varkappa^2\mathbf{Y}_{L,\,L-1,\,M}\mathbf{Y}^*_{L,\,L-1,\,M}; \tag{8a}$$

$$\mathbf{S}^{(1)}_{MM'} = i\frac{L+1}{L}\varkappa^2\,[\mathbf{Y}_{LLM}\times\mathbf{Y}^*_{LLM}]$$

$$-(\varkappa+\varkappa^2)\sqrt{\frac{L+1}{L}}(\mathbf{Y}_{LLM}Y^*_{LM'} + Y_{LM}\mathbf{Y}^*_{LLM'}).$$

The matrix element of the multipole moment can be represented in the following form:

$$(j_2m_2\,|\,Q^{(\lambda)}_{LM}\,|\,j_1m_1) = Q^{(\lambda)}C^{j_2m_2}_{j_1m_1,\,LM}, \tag{9}$$

where $Q^{(\lambda)}$ does not depend on the quantum numbers m_1 and m_2. Substituting Eqs. (2), (8) and (9) into Eq. (7), we get, omitting unimportant common factors:

$$P = \left[\delta_{m_2m_2'} + \left(\frac{j_2+1}{j_2}\right)^{1/2}\zeta C^{j_2m_2}_{j_2m_2',\,1\mu}v^{\mu}\right]C^{j_2m_2}_{j_1m_1,\,LM}C^{j_2m_2'}_{j_1m_1,\,LM'}\,[R^{(\lambda)}_{MM'} + \boldsymbol{\sigma}\mathbf{S}^{(\lambda)}_{MM'}]$$

(summation over all repeated indices is understood), and using the relations

$$C^{j_2m_2}_{j_1m_1,\,LM}C^{j_2m_2}_{j_1m_1,\,LM'} = \frac{2j_2+1}{2L+1}\delta_{MM'};\quad C^{j_2m_2}_{j_2m_2',\,1\mu}C^{j_2m_2}_{j_1m_1,\,LM}C^{j_2m_2'}_{j_1m_1,\,LM'} = \frac{2j_2+1}{2L+1}\,\frac{L(L+1)+j_2(j_2+1)-j_1(j_1+1)}{2\sqrt{L(L+1)\,j_2(j_2+1)}}\,C^{LM}_{LM',\,1\mu},$$

we have

$$P = R^{(\lambda)}_{MM} + \boldsymbol{\sigma}\mathbf{S}^{(\lambda)}_{MM} + \zeta v^{\mu}\frac{L(L+1)+j_2(j_2+1)-j_1(j_1+1)}{2j_2\sqrt{L(L+1)}}\,C^{LM}_{LM',\,1\mu}(R^{(\lambda)}_{MM'} + \boldsymbol{\sigma}\mathbf{S}^{(\lambda)}_{MM'}). \tag{10}$$

In Eq. (10) the first two terms, after they are summed over M, cannot depend on $\mathbf{n}$. Therefore

$$S^{(\lambda)}_{MM} = 0, \quad R^{(\lambda)}_{MM} = \text{const} = \frac{1}{4\pi}\int R^{(\lambda)}_{MM} do.$$

From the expression (8a) it follows that

$$R^{(0)}_{MM} = \frac{2L+1}{4\pi}, \quad R^{(1)}_{MM} = \left(1 + 2\varkappa + \frac{2L+1}{L}\varkappa^2\right)\frac{2L+1}{4\pi}.$$

For the calculation of the last two terms in Eq. (10) we take the z axis in the direction of the vector $\mathbf{v}$. Then instead of the sum over μ there remains just the one term with $\mu = 0$, and $M' = M$. Using the explicit expression of the coefficient

$$C^{LM}_{LM,10} = M / \sqrt{L(L+1)}$$

and the fact that $R^{(\lambda)}_{MM}$ does not depend on the sign of M, we get

$$\sum_M M R^{(\lambda)}_{MM} = 0.$$

We have still to find the last term in Eq. (10), i.e., the quantity

$$v C^{LM}_{LM',10} S^{(\lambda)}_{MM}. \tag{11}$$

According to Eq. (8a), in the case of a magnetic multipole the vector $\mathbf{S}^{(0)}$ is directed along $\mathbf{n}$, since $\mathbf{Y}_{LLM}$ is a transverse vector. This means that for magnetic transitions the coefficient b in Eq. (1) is zero.

Comparing Eq. (11) with the first term in Eq. (1), we see that

$$v \sum_M M S^{(0)}_{MM} = f(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} = f v n_z \mathbf{n}' \tag{12}$$

where f is a constant. To determine this constant we integrate Eq. (12) with respect to the solid angle; this gives

$$\frac{4\pi}{3} f = \sum_M M \int S^{(0)}_{z\,MM} do.$$

Substituting the explicit expression for $S^{(0)}_{MM}$, we find

$$f = \frac{3}{4\pi}\frac{1}{L(L+1)}\sum_M M^2 = \frac{2L+1}{4\pi}.$$

According to Eq. (8a), in the case of an electric multipole $\mathbf{S}^{(1)}$ contains two terms. The first term, $\mathbf{S}^{(1)}_{\parallel}$, is directed along $\mathbf{n}$ and is calculated in the same way as $\mathbf{S}^{(0)}$. The second term $\mathbf{S}^{(1)}_{\perp}$, proportional to $\mathbf{Y}_{LLM}$, is perpendicular to $\mathbf{n}$, and consequently must have the form of the second term in Eq. (1), i.e.,

$$v \sum_M M \mathbf{S}^{(1)}_{\perp\,MM} = g\,(\mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}).$$

Using the explicit form of $\mathbf{S}^{(1)}_{\perp}$ and integrating this equation over the angles, we find

$$g = \frac{2L+1}{4\pi}\sqrt{L(L+1)}.$$

Substituting these results into Eq. (10), we get on the basis of Eqs. (6a) and (6b) the following expressions for the polarization vector of the conversion electron:

(a) in the case of a magnetic multipole

$$\langle \sigma \rangle = (r\zeta / j_2)\,\mathbf{n}(\mathbf{n}\cdot\mathbf{v}); \tag{13}$$

(b) in the case of an electric multipole

$$\langle \sigma \rangle = r\frac{L+1}{1 + 2\varkappa + \varkappa^2(2L+1)/L}\frac{\zeta}{j_2}$$
$$\times \left\{(\varkappa + \varkappa^2)(\mathbf{n}(\mathbf{n}\cdot\mathbf{v}) - \mathbf{v}) + \frac{\varkappa^2}{L}(\mathbf{n}\cdot\mathbf{v})\mathbf{n}\right\}, \tag{14}$$

$$r = [L(L+1) + j_2(j_2+1) - j_1(j_1+1)] / 2L(L+1).$$

We see from Eq. (13) that in the case of a magnetic multipole the polarization is longitudinal and does not depend on the energy of the polarization electron. This feature is, however, closely connected with the free-electron approximation which has been used here. Therefore a treatment of this problem with exact wave functions for the conversion electron would be of interest.

According to Eq. (14), in the case of an electric dipole the polarization is decidedly energy-dependent. When the speed v_k of the conversion electron is small, the transverse polarization is proportional to v_k^2/c^2, and the longitudinal polarization to $(v_k/c)^4$. These results also need to be made more precise, since for small velocities the effect of the Coulomb field of the nucleus can be important.

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