PERIODIC SOLUTIONS OF THE NONLINEAR GENERALIZED DIRAC EQUATION

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Submitted to JETP editor January 15, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1587-1592 (June, 1958)

The nonlinear Dirac equation is considered, and a method for its solution is given. Periodic solutions are obtained in general form. A relation is established between the nonlinear Dirac equation and the Klein-Gordon equation.

As has been pointed out repeatedly in the literature, it is possible that an important part in the solution of problems of the theory of elementary particles will be played by nonlinear generalized equations, and in particular by the nonlinear generalized Dirac equation.

In the present paper we do not consider the question of the derivation of nonlinear generalized field equations. Taking the equations as given, we examine the possibility of solving them analytically. Furthermore, on the basis of physical considerations we seek solutions that are periodic functions; under certain restrictions we are able to obtain these solutions in a closed analytic form.

1. THE GENERAL FORM OF THE NONLINEAR GENERALIZED DIRAC EQUATION

Let us consider the nonlinear Dirac equation in its general form,* which has as special cases the generalization of the basic Dirac equation by the addition of terms $\lambda(\bar{\psi}\psi)\psi$ or $\lambda(\bar{\psi}\gamma_5\psi)\gamma_5\psi$, first suggested by Ivanenko[1,2] and by Ivanenko and Mirianashvili,[3] or in the case of vanishing rest mass by Heisenberg:[4]

$$\left\{-\frac{\partial}{i\partial t}-\frac{1}{i}\alpha\nabla-\rho_3 A(\psi^*,\psi)\right\}\psi=0 \quad (1.1)$$

and its complex conjugate equation:

$$\psi^*\left\{\frac{\partial}{i\partial t}+\frac{1}{i}\alpha\nabla+\overline{A^*(\psi^*,\psi)}\rho_3\right\}=0, \quad (1.2)$$

where

$$\psi^*\overline{A^*(\psi^*,\psi)}\rho_3=(\rho_3 A(\psi^*,\psi)\psi)^*. \quad (1.3)$$

We introduce the usual notation[5]

$$\gamma_n=-i\rho_3\alpha_n,\ \gamma_4=\rho_3,\ \psi^+=-i\psi^*\gamma_4,$$
$$x_4=it,\ x_\mu(x_n,x_4). \quad (1.4)$$

Then Eqs. (1.1) and (1.2) are written in the forms

$$\{\gamma_\mu\partial/\partial x_\mu+A(\psi^*,\psi)\}\psi=0, \quad (1.5)$$
$$\psi^+\{\gamma_\mu\partial/\partial x_\mu+\gamma_4\overline{A^*(\psi^*,\psi)}\gamma_4\}=0. \quad (1.6)$$

The nonlinearity of Eq. (1.1) is contained in $A(\psi^*,\psi)$, which does not depend on derivatives and is a function of ψ^*, ψ only. By considerations of invariance, $A(\psi^*,\psi)$ must have the form[6]

$$A(\psi^*,\psi)=A((\psi^*\Gamma_{l_1}^{(1)}\psi)T_{\nu_1}^{(1)}(\psi^*\Gamma_{l_2}^{(2)}\psi)T_{\nu_2}^{(2)}\ldots(\psi^*\Gamma_{l_n}^{(n)}\psi)T_{\nu_n}^{(n)}),$$

where

$$\Gamma_{l_1}^{(1)},\Gamma_{l_2}^{(2)}\ldots\Gamma_{l_n}^{(n)};\ T_{\nu_1}^{(1)}T_{\nu_2}^{(2)}\ldots T_{\nu_n}^{(n)}, \quad (1.7)$$

are, generally speaking, matrices. In addition, the invariance of the expression requires that the indices $l_i\nu_i$ not include any free indices.

The limitation that we impose here on $A(\psi^*,\psi)$ is that we consider only the case

$$A(\psi^*,\psi)=A((\psi^*\Gamma_{l_1}^{(1)}\psi)(\psi^*\Gamma_{l_2}^{(2)}\psi)\ldots(\psi^*\Gamma_{l_n}^{(n)}\psi)). \quad (1.8)$$

As is well known, the solution of Eqs. (1.5), (1.6) depends not only on x_μ, but also on the spin coordinate s.

We assume that the solutions of the equation allow the separation of the spin and (four-dimensional) space coordinates. We shall try to find these solutions in the form

$$\psi=\chi(s)\varphi(x_\mu),\ \psi^*=\chi^*(s)\varphi^*(x_\mu) \quad (1.9)$$

with the condition $\chi^*(s)\chi(s)=1$.

We now introduce the notation

$$\rho(x_\mu)=\varphi(x_\mu)\varphi^*(x_\mu)=\rho^*(x_\mu). \quad (1.10)$$

Then Eq. (1.8) takes the form

$$A(\psi^*,\psi)=A((\chi^*\Gamma_{l_1}^{(1)}\chi)\ldots(\chi^*\Gamma_{l_n}^{(n)}\chi)\rho^n(x_\mu))=A(\rho), \quad (1.11)$$

where we have used the fact that, $(\chi^*\Gamma_l^{(1)}\chi)\ldots(\chi^*\Gamma_{l_n}^{(n)}\chi)$, is a numerical quantity which can be included in the constant coefficient of the function $\rho^n(x_\mu)$. Thus we get

*The notations are the same as in reference 1 ($\hbar=c=1$).

$$\gamma_4\overline{A^*(\psi^*,\psi)}\gamma_4 = \gamma_4\overline{A^*(\rho)}\gamma_4 = \gamma_4 A^*(\rho)\gamma_4 = A^*(\rho). \quad (1.12)$$

Then Eqs. (1.5) and (1.6) take the forms

$$\{\gamma_\mu\partial/\partial x_\mu + A(\rho)\}\chi(s)\varphi(x_\mu) = 0; \quad (1.13)$$

$$\chi^+(s)\{\gamma_\mu\partial/\partial x_\mu - A^*(\rho)\}\varphi^*(x_\mu) = 0. \quad (1.14)$$

2. SOLUTION OF THE NONLINEAR DIRAC EQUATION

On the basis of physical considerations we shall look for solutions of Eqs. (1.13), (1.14) in the class of periodic functions:

$$\varphi(x_\mu) = \varphi(\sigma),\ \sigma = k_\mu x_\mu,\ k_\mu(k_n, k_4),\ k_4 = i\omega. \quad (2.1)$$

Then we have

$$\{\gamma_\mu k_\mu d/d\sigma + A(\rho)\}\chi(s)\varphi(\sigma) = 0, \quad (2.2)$$

$$\chi^+(s)\{\gamma_\mu k_\mu d/d\sigma - A^*(\rho)\}\varphi^*(\sigma) = 0. \quad (2.3)$$

Let us consider the system of equations

$$d\varphi/d\sigma = -A(\rho)\varphi/i\lambda,\ d\varphi^*/d\sigma = A^*(\rho)\varphi^*/i\lambda. \quad (2.4)$$

As we see, if φ and φ^* are solutions of (2.4), then Eqs. (2.2) and (2.3) reduce to a system of equations for the spin factor of the solution

$$(i\gamma_\mu k_\mu + \lambda)\chi(s) = 0,\ \chi^+(s)(i\gamma_\mu k_\mu + \lambda) = 0. \quad (2.5)$$

We supplement Eq. (2.5) with the following two equations for the spin*

$$(ks - \sigma_s \mathbf{k})\chi(s) = 0,\ \chi^+(s)(ks - \sigma_s\mathbf{k}) = 0. \quad (2.6)$$

The solutions of the system of equations (2.5) and (2.6) are well known.[1]

Let us now return to the solution of the system of equations (2.4). We write the functions $A(\rho)$ and $A^*(\rho)$ in the form†

$$A(\rho) = a(\rho) + ib(\rho),\ A^*(\rho) = a(\rho) - ib(\rho). \quad (2.7)$$

Then Eq. (2.4) gives

$$\frac{d\rho}{d\sigma} = -\frac{2}{\lambda}b(\rho)\rho;\quad \int\frac{d\rho}{\rho b(\rho)} = -\frac{2\sigma}{\lambda} + c_1. \quad (2.8)$$

We denote the solution of Eq. (2.8) by $\rho = f(\sigma)$, and substituting it into Eq. (2.4) we get

$$d\varphi/\varphi = -A(\sigma)d\sigma/i\lambda,\quad d\varphi^*/\varphi^* = A^*(\sigma)d\sigma/i\lambda. \quad (2.9)$$

The solutions of these equations have the form

$$\frac{\varphi}{\varphi_0} = \exp\left\{\frac{i}{\lambda}\int A(\sigma)d\sigma\right\} \quad (2.10)$$

$$= \exp\left\{-\frac{1}{\lambda}\int b(\sigma)d\sigma + \frac{i}{\lambda}\int a(\sigma)d\sigma\right\},$$

$$\frac{\varphi^*}{\varphi_0^*} = \exp\left\{-\frac{i}{\lambda}\int A^*(\sigma)d\sigma\right\} \quad (2.11)$$

$$= \exp\left\{-\frac{1}{\lambda}\int b(\sigma)d(\sigma) - \frac{i}{\lambda}\int a(\sigma)d\sigma\right\},$$

$$\frac{\rho}{\rho_0} = \exp\left\{-\frac{2}{\lambda}\int b(\sigma)d\sigma\right\}. \quad (2.12)$$

As we see, the density is determined by $b(\rho)$ only. From this it follows in particular that for $b(\rho) = 0$ the nonlinear Dirac equation has no other complex solution besides

$$\rho = \rho_0 = \text{const},\ \varphi = \varphi_0 e^{ik_\mu x_\mu},\ \varphi^* = \varphi_0^* e^{-ik_\mu x_\mu},$$
$$k_\mu^2 = -\lambda^2 = -a^2(\rho_0), \quad (2.13)$$

and in the case $a(\rho_0) = k_0$ we arrive at an ordinary solution of the linear Dirac equation.

The real solutions of the equations (2.9) can be obtained directly from the solutions (2.10) and (2.11). In fact, for $a(\rho) = 0$ we have

$$\frac{\varphi}{\varphi_0} = \frac{\varphi^*}{\varphi_0^*} = \exp\left\{-\frac{1}{\lambda}\int b(\sigma)d\sigma\right\}. \quad (2.14)$$

For $b(\rho) = 0$, if we replace $(i\lambda)$ by $-\lambda'$ with $\lambda'^2 > 0$, Eqs. (2.9) and (2.5) become[7]

$$d\varphi/d\sigma = d\varphi^*/d\sigma = a(\rho)\varphi/\lambda' = a(\rho)\varphi^*/\lambda', \quad (2.15)$$

$$(i\gamma_\mu k_\mu + (i\lambda'))\chi(s) = 0,\ \chi^+(s)(i\gamma_\mu k_\mu + (i\lambda')^*) = 0, \quad (2.16)$$

and the solutions for the spin factors are given by[1]

$$\chi(s) = \Omega B(s, \varepsilon, (i\lambda')),\ \chi^* = B^*(s, \varepsilon, (i\lambda')^*)\Omega^*,$$
$$B^*(s, \varepsilon, (i\lambda)^*) = \frac{1}{2c_0}(\sqrt{1+s}f^*(\varepsilon), \quad (2.17)$$
$$\sqrt{1-s}f^*(\varepsilon), \varepsilon\sqrt{1+s}f^*(\varepsilon), \quad -\varepsilon\sqrt{1-s}f^*(-\varepsilon)),$$
$$f^*(\varepsilon) = \sqrt{1+(i\lambda')^{*2}/K^2},\quad k_\mu^2 = k^2 - K^2 = \lambda'^2,\quad K = \mp\omega,$$
$$c_0 = \sqrt{1+\lambda'^2/K^2} = |k|/K.$$

It must be noted that, just as in the linear theory, one can choose in the nonlinear theory also two of the four amplitudes arbitrarily and determine the other two from (2.5). The expressions for the amplitudes will be just the same as in the linear theory, except that the mass k_0 is replaced by λ.

In the Newtonian approximation $K \gg \lambda$, just as

*As is well known, in the linear theory one introduces for the spin the operator $\sigma\nabla/ik$. The necessity of this is basically due to the existence of a superposition principle for the linear equations. In a nonlinear theory the introduction of a differential operator leads to serious difficulties, but because of the absence of a superposition principle it is also not necessary. We here use the single term $\sigma_s\mathbf{k}$.

†In $A(\psi^*, \psi)$ the function of $i = (-1)^{1/2}$, which changes the sign of the term with $b(\rho)$ when one goes from the equation to its complex conjugate, can be performed by γ_5. In such a case, however, one must remove the restriction (1.8) and replace λ by $\lambda_1 + \gamma_5\lambda_2$ in Eq. (2.5).

in the linear theory, two of the amplitudes can be neglected in comparison with the other two.[8]

3. THE RELATION BETWEEN THE NONLINEAR DIRAC AND KLEIN-GORDON EQUATIONS

Let us consider the Dirac equation (1.5). Applying to it the operator $\gamma_\nu \partial/\partial x_\nu$, we get

$$\frac{\partial^2\psi}{\partial x_\nu^2} + \left(\gamma_\nu \frac{\partial}{\partial x_\nu} A(\psi^*, \psi)\right)\psi + \gamma_\nu A(\psi^*, \psi)\frac{\partial\psi}{\partial x_\nu} = 0. \quad (3.1)$$

Under the restrictions imposed on our present problem we have

$$\gamma_\nu A(\psi^*, \psi)\frac{\partial\psi}{\partial x_\nu} = A(\rho)\gamma_\nu\frac{\partial\psi}{\partial x_\nu} = -A^2(\rho)\psi, \quad (3.2)$$

$$\left(\gamma_\nu\frac{\partial}{\partial x_\nu}A(\psi^*, \psi)\right)\psi = \frac{dA(\rho)}{d\rho}\frac{d\rho}{d\sigma}\gamma_\nu k_\nu\psi \quad (3.3)$$

$$= -2i\rho b(\rho)\frac{dA(\rho)}{d\sigma},$$

where we have used Eqs. (2.9) and (2.5). Substituting Eqs. (3.2) and (3.3) into Eq. (3.1), we arrive at the equation

$$\partial^2\psi/\partial x_\mu^2 - B(\rho)\psi = 0, \quad (3.4)$$

where

$$B(\rho) = A^2(\rho) + i2\rho b(\rho)\frac{dA}{d\rho}$$
$$= a^2 - \frac{d}{d\rho}(\rho b^2(\rho)) + i2b(\rho)\frac{d}{d\rho}(\rho a(\rho)). \quad (3.5)$$

We also get an analogous relation for the complex conjugate function.

We arrive at the same result if we apply the operator $d/d\sigma$ to Eq. (2.4) and use Eqs. (2.8) and (2.1).

We now suppose that we are given a nonlinear Klein-Gordon equation of the form (3.4) and its complex conjugate equation. It is required to find the corresponding Dirac equation. The problem reduces to the determination of $A(\rho)$ and $A^*(\rho)$ in terms of $B(\rho)$ and $B^*(\rho)$. Equation (3.5) and its complex conjugate give a system of nonlinear differential equations to determine $a(\rho)$ and $b(\rho)$ in terms of the given values of $B(\rho)$ and $B^*(\rho)$.

If we introduce the notations

$$\tau = -1/\rho,\ \xi = \rho a(\rho) = -\frac{1}{\tau}a(\tau),$$

$$\eta = \sqrt{\rho}\, b(\rho) = b(\tau)/\sqrt{-\tau}, \quad (3.6)$$

$$f_1(\tau) = -\tau^{-2}B_1(\tau), \quad f_2(\tau) = \frac{1}{2}(-\tau)^{-5/2}B_2(\tau),$$

where

$$B_1(\tau) = (B(\tau) + B^*(\tau))/2, \quad (3.7)$$
$$B_2(\tau) = (B(\tau) - B^*(\tau))/2i,$$

then Eq. (3.5) and its complex conjugate can be written in the form

$$\eta\, d\xi/d\tau = f_2(\tau), \quad (3.8)$$

$$d\eta^2/d\tau - \xi^2 = f_1(\tau). \quad (3.9)$$

In the particular case in which $f_2(\tau) = 0$ ($B_2(\tau) = 0$), the system of equations (3.8), (3.9) is easily solved, and we find:

a) $\eta = 0,\quad \xi = \sqrt{-f_1(\tau)};\quad b(\rho) = 0 \quad a(\rho) = \sqrt{B_1(\rho)},$

b) $\eta \neq 0,\quad \xi = \xi_0,\quad \eta = \{\int (f_1(\tau) + \xi_0^2)\, d\tau\}^{1/2},$ (3.10)

that is,

$$a(\rho) = \frac{C_1}{\rho},$$
$$b(\rho) = \left\{\left(C_2 - \frac{C_1^2}{\rho} - \int B_1(\rho)\, d\rho\right)\Big/\rho\right\}^{1/2}. \quad (3.11)$$

If, on the other hand, $f_2(\tau) \neq 0$, then the system of equations (3.8) and (3.9) can be put in the form

$$\xi'' - (f_2'/f_2)\xi' + (1/2f_2^2)(\xi^2 - f_1)\xi'^3 = 0, \quad (3.12)$$

$$\eta = \left(\int \{f_1(\tau) + \xi^2\}\, d\tau\right)^{1/2}. \quad (3.13)$$

In view of the complexity of these equations, however, it is difficult to find their solution in general form.

As we see, in the nonlinear theory A and B are related through (ordinary) differential equations, unlike the linear theory, in which this connection is purely algebraic.

On this account, for given B and B^* the functions A and A^* are determined only to within two arbitrary constants if $f_2(\tau) = 0$ and three arbitrary constants if $f_2(\tau) \neq 0$. A unique connection between B and A can be secured in the usual way, by adjoining supplementary conditions to the differential equation.[7]

In cases in which one is able to reduce the nonlinear Klein-Gordon equation (3.4), (3.5) to a nonlinear Dirac equation, the solution can be found as indicated above.

I regard it as my obligation to express my deep gratitude to Professor D. D. Ivanenko for his constant interest in this work.

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Translated by W. H. Furry
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ELECTROMAGNETIC WAVES IN ISOTROPIC AND CRYSTALLINE MEDIA CHARACTERIZED BY DIELECTRIC PERMITTIVITY WITH SPATIAL DISPERSION

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Submitted to JETP editor January 6, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 1593-1604 (June, 1958)

The dielectric permittivity tensor ϵ_{ik} is usually taken to be a function of frequency alone, i.e. one neglects spatial dispersion — the dependence of ϵ_{ik} on the wavelength. However, even in non-gyrotropic media spatial dispersion must be considered in cases of weak absorption, when the refractive index increases rapidly and becomes infinite if dispersion and absorption are not taken into account. Spatial dispersion is also important in the analysis of longitudinal (plasma) waves which propagate in an isotropic medium or along the principal dielectric axes in crystals. It is also shown that spatial dispersion leads to a weak optical anisotropy in cubic crystals. In addition to the above, an analysis is made of the collective (discrete) energy losses in solids.

1. INTRODUCTION

In analyzing the propagation of light and electromagnetic waves of longer wavelengths in a medium, one usually uses the local relation

$$D_i = \varepsilon_{ik}(\omega)E_k, \qquad (1.1)$$

where **D** and **E** are taken at ω, the frequency of the Fourier components of the electric induction and field intensity at the point **r**. If there is absorption, the tensor ϵ_{ik} becomes complex and **D** must be replaced by $\mathbf{D} - i(4\pi/\omega)\mathbf{j}$ where **j** is the density of the conduction current. In order to simplify the analysis this substitution is implied below, but not carried out explicitly.

The relation in (1.1) does not reflect the nature of the field variation in space, that is to say, it applies only if we neglect spatial dispersion — the dependence of the tensor ϵ_{ik} on the wavelength. The spatial dispersion can be characterized by the parameter $a/\lambda = an/\lambda_0$, where a is a characteristic length for a given medium (molecular dimensions, lattice constants, Debye radius, etc.), $\lambda_0 = 2\pi c/\omega$ is the wavelength in vacuum, $\lambda = \lambda_0/n$ is the wavelength in the medium and n is the index of refraction. In condensed media in the optical region usually $a/\lambda_0 \sim 1$ to 3×10^{-3} so that spatial dispersion is negligibly small in most cases.* This, however, is not the case if we are interested in effects associated with spatial inhomogeneities of the field. A well-known example of this type is natural optical activity — an effect which is of order a/λ. It will be shown below that taking terms of order $(a/\lambda)^2$ into account leads to an additional effect — weak optical anisotropy in cubic crystals.

*The time dispersion, which leads to a dependence of ε_{ik} on ω may be large under these same conditions because it is characterized by the parameter ω/ω_j, where ω_j is a characteristic frequency of the medium.