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ON THE THEORY OF HIGH-SPIN PARTICLES

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An algebraic method is suggested for treating the relativistically invariant equations of high-spin particles. The direct product of generalized Dirac algebras underlies the analysis. This method can be used to obtain explicit expressions for the infinitesimal rotation matrix, the spin operator, the metric and reflection operator, as well as to limit the number of representations taken into account. The equations can be treated directly either in tensor or in spin-tensor form. The commutation relations are automatically obtained in parametric form.

1. INTRODUCTION

WE consider relativistically invariant equations of the form

$$\alpha_k \partial_k \psi + \varkappa \psi = 0, \tag{1}$$

where ψ is a particle wave function with a finite number of components which transforms according to some finite-dimensional representation of the Lorentz group. Equation (1) shall be called relativistically invariant if under the Lorentz transformation $x'_i = l_{ik}x_k$ together with the transformation $\psi' = S\psi$ it remains formally invariant, in other words if

$$\alpha_i = l_{ij} S \alpha_j S^{-1}. \tag{2}$$

Now the conditions of (2) are fulfilled if

$$[I_{ik}, I_{jl}] = -\delta_{ij} I_{kl} + \delta_{il} I_{kj} + \delta_{kj} I_{il} - \delta_{kl} I_{ij}, \tag{3}$$

$$[\alpha_i, I_{jk}] = \delta_{ij}\alpha_k - \delta_{ik}\alpha_j, \tag{4}$$

where the I_{ij} are the infinitesimal-rotation matrices, defined by

$$\psi' = \psi + {}^1/{}_2 \varepsilon_{ik} I_{ki} \psi.$$

We shall consider here the problem of finding all relativistically-invariant equations (1) that satisfy the additional requirements that they be invariant under reflection, that there exist a nondegenerate real Lagrangian, that the energy density be positive definite for particles with integral spin or the charge be positive definite for particles with half-integral spin, and that the equations be irreducible.

This problem has been treated in general form by Gel'fand and Iaglom.[1] Their method can be used, in principle, to obtain all possible equations for high-spin particles. In actually obtaining the equations, however, certain difficulties arise. Among these are, in particular, the analysis of irreducibility, the transition to spin-tensor (or tensor) form, and the determination of the algebra of the α_k matrices. Further, there is no way to tell whether all of the equations belonging to a given maximum spin have been found.

While Gel'fand and Iaglom base their considerations on explicit expressions for the infinitesimal operators, Harish-Chandra[2,3] develops an algebraic method based on the study of an algebra designated $U(\alpha)$. This is the α-matrix algebra of the forms

$$G = c^k\alpha_k + c^{kl}\alpha_k\alpha_l + c^{klm}\alpha_k\alpha_l\alpha_m + \cdots,$$

where the $c^{k\cdots}$ are numerical coefficients. Harish-Chandra has shown that the I_{ij} matrices, the reflection matrix T, the metric matrix η (used in the definition of the bilinear invariant Hermitian form $\psi^+\eta\psi$), and the matrix S corresponding to the Lorentz transformation all belong to $U(\alpha)$. He has also shown that the α_k matrices must satisfy commutation relations which uniquely define $U(\alpha)$. He did not, however, give any concrete method for finding the matrices I_{ij}, T, η, and S, or the commutation relations of $U(\alpha)$. The present article (Sec. 2) gives such a method. It is based on a consideration of the direct product of generalized Dirac algebras $U(\Gamma)$

$$U(\alpha) = U(\Gamma)\times U(\Gamma')\times U(\Gamma'')\times\ldots$$

whose matrices Γ_i satisfy the commutation relations of Eq. (9). This method can be used to obtain and analyze equations both invariant and not invariant under reflection (these latter being of interest in connection with the nonconservation of parity in weak interactions). We shall obtain general expressions for I_{ij}, T, η, and α_k in terms of algebraic parameters. The method can be used to arrive at conclusions concerning the irreducibility of the equations, as well as to restrict the number of linked representations. In particular, we shall demonstrate (in Sec. 2) the reducibility* of the many-mass equations for spin-$\frac{1}{2}$, treated by Ulehla.[7] In our method the commutation relations for the α_k matrices are automatically obtained in parametric form. In Sec. 3 we obtain the commutation relation which Harish-Chandra[3] and Petras[5,6] found by selection. This method can be used to establish beforehand all of the equations belonging to a given maximum spin, without separate investigations of all possible methods of composition, among which many are equivalent. All of the analysis (the determination of the mass and spin states, the positive definite conditions) may be performed in spin-tensor (or tensor) form using a single standard method.

Section 2 is a general discussion, while the advantages of the method are shown in Sec. 3 by a concrete analysis of the equations for spin 1 and $\frac{3}{2}$.

2. GENERAL METHOD

We first make some general remarks. Let the $u_p(\alpha)$ (with $p = 1, \ldots, n$) form a linearly independent basis for the algebra $U(\alpha)$. Now $u_p(l\alpha)$ is also an element of $U(\alpha)$, and can therefore be expressed linearly in terms of the basis in the form

$$u_p(l\alpha) = \sum_q L_p^q u_q(\alpha). \qquad (5)$$

Thus to each transformation l in α-space there corresponds a transformation L in U-space. The set of all L is a representation of the Lorentz group.[2] Using Eqs. (2) and (5), we immediately obtain the relation between this representation L and the representation S which transforms the ψ function. This relation is

$$\sum_q L_p^q u_q(\alpha) = u_p(S\alpha S^{-1}) = S u_p(\alpha) S^{-1}. \qquad (6)$$

It is easily shown that this implies that if S can be written as the direct product of the representations which transform the wave functions ψ' and ψ'' that satisfy the equations

$$\alpha'_k\partial_k\psi' + \varkappa\psi' = 0,\quad \alpha''_k\partial_k\psi'' + \varkappa\psi'' = 0,$$

then the algebra $U(\alpha)$ can be written in the form

$$U(\alpha) = U(\alpha')\times U(\alpha''). \qquad (7)$$

The converse is also true. We recall that the direct product of an algebra with a decomposable algebra is also a decomposable algebra, from which we obtain the criterion for the reducibility of the equations.

Note that the equations given by Ulehla[7] for a spin-$\frac{1}{2}$ particle with several different mass states are reducible. Indeed, Ulehla's $U(\alpha)$ algebra corresponds to the representation (using Cartan's notation)

$$S = D(1/2, 3/2) + D(3/2, 1/2) + mD(0, 1/2) + mD(1/2, 0),\quad m \geqslant 2.$$

But this can be written as the direct product of the representations

$$[D(1/2, 1/2) + (m-1)D(0,0)] \text{ or } [D(0, 1/2) + D(1/2, 0)].$$

It is easily shown that any α_k matrix belonging to the first of these two representations is reducible unless m = 2. Thus all of Ulehla's equations are decomposable, except that for a single mass. This will be seen to follow automatically, furthermore, from the theory developed below.

All the finite-dimensional representations of the Lorentz group, which transform some wave function ψ, can be written as the direct sums of quantities D(m, n), where m and n are positive integers or half integers, including zero. Every such sum can be obtained by forming products of the form $\prod_i [k_i D(0, \frac{1}{2}) + l_i D(\frac{1}{2}, 0)]$. Using the crite-

*Ulehla's single-mass equation for spin-1/2 is irreducible.

rion for reducibility, we see immediately that the most general case involves direct products of representations of the form

$$D(0, 1/2) + D(1/2, 0). \tag{8}$$

All irreducible equations can be obtained by treating n algebras $U(\Gamma)$ corresponding to Eq. (8). An algebra $U(\Gamma)$ is a generalized Dirac algebra, for which the commutation relations hold:

$$\Gamma_k\Gamma_i + \Gamma_i\Gamma_k = 2p\delta_{ik} \tag{9}$$

(where p is any complex number). The matrix Γ_k can be written

$$\Gamma_k = \begin{pmatrix} 0 & a\sigma_k \\ b\sigma_k & 0 \end{pmatrix}, \; p = ab,$$

while the ordinary Dirac matrices are

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}.$$

Let us thus consider the direct product of n of these algebras, or

$$U(\alpha) = U(\Gamma)\times U(\Gamma')\times U(\Gamma'')\times \dots . \tag{10}$$

We first find the matrices I_{kj} and α_k of the algebra $U(\alpha)$. To do this, we note that α_k and $I_{kj} = -I_{jk}$ must be covariant in the indices k and kj with respect to Lorentz transformations. For the direct product of two $U(\Gamma)$ algebras we obtain in the most general case, using Eq. (9),

$$\begin{aligned} \alpha_k = {} & \Gamma_k (b_0 + b_1\Gamma_l\Gamma'_l + b_2\Gamma_l\Gamma_m\Gamma'_l\Gamma'_m + b_3\Gamma_l\Gamma_m\Gamma_n\Gamma'_l\Gamma'_m\Gamma'_n + b_4\Gamma_l\Gamma_m\Gamma_n\Gamma_r\Gamma'_l\Gamma'_m\Gamma'_n\Gamma'_r) \\ & + \Gamma'_k (c_0 + c_1\Gamma_l\Gamma'_l + c_2\Gamma_l\Gamma_m\Gamma'_l\Gamma'_m + c_3\Gamma_l\Gamma_m\Gamma_n\Gamma'_l\Gamma'_m\Gamma'_n + c_4\Gamma_l\Gamma_m\Gamma_n\Gamma_r\Gamma'_l\Gamma'_m\Gamma'_n\Gamma'_r), \\ I_{kj} = {} & (\Gamma_k\Gamma_j + \Gamma_j\Gamma_k)(g_0 + g_1\Gamma_l\Gamma'_l + g_2\Gamma_l\Gamma_m\Gamma'_l\Gamma'_m + g_3\Gamma_l\Gamma_m\Gamma_n\Gamma'_l\Gamma'_m\Gamma'_n \\ & + g_4\Gamma_l\Gamma_m\Gamma_n\Gamma_r\Gamma'_l\Gamma'_m\Gamma'_n\Gamma'_r) + (\Gamma'_k\Gamma'_j - \Gamma'_j\Gamma'_k)(g'_0 + g'_1\Gamma_l\Gamma'_l + \cdots) \\ & + (\Gamma_k\Gamma'_j - \Gamma'_j\Gamma_k)(g''_0 + g''_1\Gamma_l\Gamma'_l + \cdots). \end{aligned} \tag{11}$$

For the direct product of a large number of algebras we obtain much more complicated expressions. However, relations (3) and (4) must be satisfied if the equations are to be relativistically invariant and the I_{kj} are to be infinitesimal rotation operators. This places some restrictions on the coefficients b_i, c_i, and g_i. We now insert equations such as (11) into (3) and (4), and after very cumbersome, though algebraically simple calculations, we find that for the most general case of products of 1, 2, ..., or n algebras $U(\Gamma)$, the I_{ij} are given by

$$\begin{aligned} I_{ij} &= g_1(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i), \\ I_{ij} &= g_1(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i) + g_2(\Gamma'_i\Gamma'_j - \Gamma'_j\Gamma'_i), \\ I_{ij} &= \sum_{k=1}^{n} g_k(\Gamma_i^{(k)}\Gamma_j^{(k)} - \Gamma_j^{(k)}\Gamma_i^{(k)}). \end{aligned} \tag{12}$$

Here the coefficients g_k satisfy the conditions $g_k(4p_kg_k - 1) = 0$ (no summation over k).

From these expressions we automatically obtain the minimal equations

$$I_{ij}^2 + \frac{1}{4} = 0; \; I_{ij}(I_{ij}^2 + 1) = 0;$$

$$\left(I_{ij}^2 + \frac{1}{4}\right)\left(I_{ij}^2 + \frac{9}{4}\right) = 0 \text{ etc.} \tag{13}$$

for the squares of the infinitesimal-rotation operators. The I_{ij} corresponding to space rotations are spin-component matrices. Equations (3) and (4) place only one restriction on the α_k. This is that the number of algebras in the direct product of (10) is uniquely determined by the maximum spin for the particle. Thus all relativistically-invariant irreducible equations with maximum spin 1 can be obtained by considering the direct product of two $U(\Gamma)$ algebras. Those with maximum spin 3/2 are obtained by considering the direct product of three $U(\Gamma)$ algebras, etc.

For the equations to be invariant under reflection in space, the reflection matrix must satisfy the relations[1]

$$T^2 = E, \text{ where E is the unit matrix} \tag{14}$$

$$[T, I_{jk}]_- = 0, \; j \neq 0, \; k \neq 0;$$

$$[T, I_{jk}]_+ = 0, \; j = 0, \text{ or } k = 0, \tag{15}$$

$$[T, \alpha_k]_+ = 0, \; k = 1, 2, 3; \; [T, \alpha_0]_- = 0. \tag{16}$$

According to (15), T must be of the form

$$T = \Gamma_0 \times \Gamma'_0 \times \Gamma''_0 \times \dots .$$

If (14) is true, then in the most general case for a single algebra we have

$$\alpha_k = \cos\varphi\,\gamma_k + i\sin\varphi\,\gamma_5\gamma_k.$$

All the values of the parameter φ are equivalent, and we can thus always choose $\alpha_k = \gamma_k$ and $T = \gamma_0$. In the general case, however, we must treat two kinds of direct products, namely

$$U(\gamma_k)\times U(\gamma'_k)\times \dots \text{and } U(i\gamma_5\gamma_k)\times U(i\gamma'_5\gamma'_k)\times \dots .$$

For the direct product of two algebras, T has the two nonequivalent definitions

$$T = \gamma_0\gamma_0', \; T = -\gamma_5\gamma_5'\gamma_0\gamma_0', \qquad (17)$$

corresponding to the tensor and pseudotensor theories. For three algebras there are also two nonequivalent reflection laws, namely

$$T = \gamma_0\gamma_0'\gamma_0'', \; T = -\gamma_0\gamma_5'\gamma_0'\gamma_5''\gamma_0'', \qquad (18)$$

corresponding to the spinor and the pseudospinor theories, etc. In general when treating equations invariant under reflection, it is necessary and sufficient to consider the direct product of ordinary Dirac algebras together with the two different possible reflection laws indicated in (17) and (18).

The existence of an invariant real Lagrangian for Eq. (1) requires the existence of a bilinear invariant Hermitian form $\psi_1^+\eta\psi_2$, where η is the metric matrix. If this is to be a Lorentz invariant, reflection invariant, and Hermitian form, and if the Lagrangian for the equation is to be real, then we must have

$$\left.\begin{array}{l}[\eta, I_{jk}]_- = 0, \; j \neq 0, \; k \neq 0; \; [\eta, I_{jk}]_+ = 0, \; j = 0 \text{ or } k = 0; \\ T\eta T = \eta; \qquad \eta = \eta^+; \end{array}\right\} \qquad (19)$$

$$(\eta\alpha_k)^+ = -\eta\alpha_k, \; k = 1, 2, 3; \; (\eta\alpha_0)^+ = \eta\alpha_0. \qquad (20)$$

It is easily shown that conditions (19) imply that in our representation T and η are represented by matrices which are equivalent up to a constant factor. On going to another representation, however, T and η in general transform differently. The allowable transformations R under which the theory is invariant are of the form[8]

$$\begin{array}{l}\psi \to \psi' = R^{-1}\psi, \quad \eta \to \eta' = R^+\eta R; \\ \alpha_k \to \alpha_k' = R^{-1}\alpha_k R, \quad T \to T' = R^{-1}TR.\end{array} \qquad (21)$$

In any representation, therefore, we may write

$$\eta = \Lambda T, \qquad (22)$$

where Λ is a scalar matrix (see also Harish-Chandra[3]). Relations (20) restrict the coefficients in the expression for the α_k.

These coefficients are restricted also by the requirements that there exist mass and spin eigenstates and that either the charge or energy be positive definite. We proceed with the first of these requirements. Consider the spin operator Z. As is known, $Z^2 = Z_x^2 + Z_y^2 + Z_z^2$. But the infinitesimal space rotation operators are just the spin-component operators. Therefore

$$Z^2 = I_{12}^2 + I_{23}^2 + I_{31}^2. \qquad (23)$$

We find the eigenfunction belonging to a given value of the total spin by solving

$$Z^2\psi_\xi = \xi(\xi + 1)\psi_\xi. \qquad (24)$$

We now apply the mass operator α_0 to each of the eigenfunctions ψ_ξ, obtaining

$$\alpha_0\psi_\xi = \lambda\psi_\xi. \qquad (25)$$

In this way we find the eigenvalues λ and eigenfunctions $\psi_{\xi\lambda}$ of α_0, and hence the mass states.

At this point the analysis can be brought directly into algebraic form. To find states with positive definite energy density, we then must require that

$$\eta\alpha_0^{f(n)+2} \geqslant 0, \qquad (26)$$

whereas if the charge is to be definite, we require

$$\eta\alpha_0^{f(n)+1} \geqslant 0, \qquad (27)$$

where the inequality means that the eigenvalues of the matrix are greater than or equal to zero.[3] Here $f(n) = n$ when n is even, $f(n) = n - 1$ when n is odd, and n is the lowest power in the minimal polynomial $\alpha_0^n(\alpha_0^2 - 1)\ldots(\alpha_0^2 - m^2)$.

The analysis can be cast in spin-tensor (or tensor) form, and the transition to this form is elementary. Then for each simultaneous eigenfunction $\psi_{\xi\lambda}$ of the spin and mass, we must require that

$$\psi_{\xi\lambda}^+\eta\alpha_0\psi_{\xi\lambda} \geqslant 0, \qquad (28)$$

or

$$\psi_{\xi\lambda}^+\eta\psi_{\xi\lambda} \geqslant 0 \qquad (29)$$

for half integral or integral spin, respectively.

As an example, we apply this method in the next section to the equations with maximum spin 1 in algebraic form, and to those with maximum spin $3/2$ in spin-tensor form. The commutation relations are then automatically obtained in parametric form.

3. SPECIFIC EXAMPLES

The case of maximum spin $1/2$ is trivial. There is only one irreducible equation which satisfies the necessary physical requirements, namely the Dirac equation, for which

$$\begin{array}{c}\alpha_k = \gamma_k; \; T = \eta = \gamma_0; \; \eta\alpha_0 = 1 > 0, \; I_{ij} = 1/4(\gamma_i\gamma_j - \gamma_j\gamma_i), \\ \gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.\end{array} \qquad (30)$$

As mentioned in the previous section, we shall treat maximum spin 1 algebraically. The most general form of the α_k for an equation which is relativistically invariant and invariant under reflection is

$$\begin{array}{c}\alpha_k = \gamma_k(b_0 + b_1J + b_2J^2 + b_3Js + b_4s) \\ + \gamma_k'(c_0 + c_1J + c_2J^2 + c_3Js + c_4s), \\ J = \gamma_l\gamma_l', \; s = \gamma_5\gamma_5'.\end{array} \qquad (31)$$

Inserting α_0 and $\eta = \gamma_0\gamma_0'$ into (20), we find that the conditions for the existence of an invariant Lagrangian are

$$\begin{aligned} &\operatorname{Im} c_0 = \operatorname{Im}(b_1 - 2c_2),\ \operatorname{Re} c_1 = 2b_2,\ \operatorname{Re} c_4 = b_3,\\ &\operatorname{Im} b_0 = \operatorname{Im}(c_1 - 2b_2),\ \operatorname{Re} b_1 = 2c_2,\ \operatorname{Re} b_4 = c_3,\\ &\operatorname{Im} c_1 = \operatorname{Im} c_3 = \operatorname{Im} b_2 = \operatorname{Im} b_3 = 0. \end{aligned} \tag{32}$$

Let us perform an equivalence transformation (21) of the form

$$R = a_0 + a_1 J + a_2 J^2 + a_3 Js + a_4 s. \tag{33}$$

We have at our disposal four arbitrary constants. In the transformed matrix we can choose $b_1 = b_4 = 0$. Then, according to (32), $a_3 = b_2 = c_3 = c_2 = 0$, and b_0 and c_0 are real. Thus, if an invariant Lagrangian exists, we may restrict our considerations to $\alpha_k = b_0\gamma_k + c_0\gamma'_k$.

If the energy density is to be positive definite, then

$$\eta\alpha_0^2 \geqslant 0. \tag{34}$$

Since $\eta\alpha_0^2 = (b_0^2 + c_0^2)\gamma_0\gamma'_0 + 2b_0c_0$ and $\gamma_0\gamma'_0$ have eigenvalues ± 1, we may write (34) in the form

$$b_0^2 + c_0^2 \leqslant 2b_0c_0,$$

which can happen only if $b_0 = c_0$. If $b = 1/2$, the minimal equation for α_0 is

$$\alpha_0(\alpha_0^2 - 1) = 0. \tag{35}$$

Thus the α_k are given by

$$\alpha_k = {}^1\!/_2(\gamma_k + \gamma'_k). \tag{36}$$

This leads simply to the Duffin-Kemmer commutation relations

$$\alpha_k\alpha_i\alpha_l + \alpha_l\alpha_i\alpha_k = \delta_{li}\alpha_k + \delta_{ki}\alpha_l. \tag{37}$$

On going to tensor form, the direct product of two bispinors decomposes into a scalar C, a pseudoscalar $\widetilde{C}$, a vector A^m, a pseudovector $\widetilde{A}^m$, and a second-rank tensor H^{nm}. Here the symbol ~ denotes the pseudo-quantity. In tensor form Eq. (1) is

$$\begin{aligned} &\partial_k A^k + \varkappa C = 0,\ (\partial_n \widetilde{A}^m - \partial_m \widetilde{A}^n) + \varkappa \widetilde{H}^{mn} = 0,\\ &\partial_m C + \varkappa A^m = 0,\ \partial_n \widetilde{H}^{mn} + \varkappa \widetilde{A}^m = 0. \end{aligned} \tag{38}$$

In the other theory, which corresponds to $T = \eta = -\gamma_5\gamma'_5\gamma_0\gamma'_0$, we have

$$\begin{aligned} &\partial_k \widetilde{A}^k + \varkappa \widetilde{C} = 0, \qquad (\partial_n A^m - \partial_m A^n) + \varkappa H^{mn} = 0,\\ &\partial_m \widetilde{C} + \varkappa \widetilde{A}^m = 0, \qquad \partial_n H^{mn} + \varkappa A^m = 0. \end{aligned} \tag{39}$$

As is seen, the algebra of the α_k matrices is decomposable. The α_k matrices can be written in the form

$$\alpha_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_k & 0 \\ 0 & 0 & \hat{\beta}_k \end{pmatrix}.$$

The algebra formed by these matrices contains 126 independent matrices. It decomposes into a homogeneous algebra containing only the known matrix, a five-dimensional algebra $U(\beta)$, and a ten-dimensional algebra $U(\hat{\beta})$. From the explicit form

$$(\beta_k)_{nm} = \begin{pmatrix} 0 & \delta_{km} \\ \delta_{nk} & 0 \end{pmatrix},\ (\hat{\beta}_k)_{\substack{rsm\\ r's'm'}} = \begin{pmatrix} 0 & \delta_{rk}\delta_{sm'} - \delta_{sk}\delta_{rm'} \\ \delta_{kr'}\delta_{ms'} - \delta_{ns}\delta_{mr'} & 0 \end{pmatrix} \tag{40}$$

of the matrices, we obtain the commutation relations for the algebras $U(\beta)$ and $U(\hat{\beta})$, namely

$$\begin{aligned} &\beta_i\beta_k\beta_l + \beta_l\beta_k\beta_i = \delta_{ik}\beta_l + \delta_{kl}\beta_i\\ &\beta_i\beta_k\beta_l = 0, \quad \text{if} \quad i \neq k, l \neq k,\\ &\beta_\alpha^2\beta_l = \beta_{\alpha'}^2\beta_l,\ \alpha \neq \alpha' \neq l,\\ &B^2 = 5B - 4E, \text{where } B = \sum_\alpha \beta_\alpha^2. \end{aligned} \tag{41}$$

These four relations uniquely define a $U(\beta)$ algebra with 25 independent matrices.

The algebra $U(\hat{\beta})$ is uniquely defined by the commutation relations

$$\begin{aligned} &\hat{\beta}_i\hat{\beta}_k\hat{\beta}_l + \hat{\beta}_l\hat{\beta}_k\hat{\beta}_i = \delta_{lk}\hat{\beta}_i + \delta_{ik}\hat{\beta}_l.\\ &B^2 = 5B - 6E. \end{aligned} \tag{42}$$

This contains 100 independent matrices, corresponding to the ten-dimensional representation.

We note that (41) and (42) are very important results. For instance, Petras[6] obtained parametric commutation relations in which β was a parameter, but which did not include all of Eq. (41) (the third relation was missing). His expressions cannot be used alone to obtain the complete commutation relations explicitly. A similar remark can be made about the work of Harish-Chandra.[3]

By finding the eigenvalues of the spin operator, we can show that the five-dimensional representation corresponds to a particle with spin 0, and that the ten-dimensional representation corresponds to one with spin 1. Thus there exist only four irreducible equations satisfying the physical requirements. Their matrices satisfy the commutation relations (41) and (42), namely

$$\begin{aligned} &\text{in the } U(\beta) \text{ algebra } I_{ij} = \beta_i\beta_j - \beta_j\beta_i,\ T = \eta = 2\beta_0^2 - 1;\\ &\text{in the } U(\hat{\beta}) \text{ algebra } I_{ij} = \hat{\beta}_i\hat{\beta}_j - \hat{\beta}_j\hat{\beta}_i,\ T = \eta = 2\hat{\beta}_0^2 - 1. \end{aligned} \tag{43}$$

Here we have $T\psi = +\psi$ or $T\psi = -\psi$ depending on whether we are using the tensor or pseudotensor theory.

For the case of maximum spin $3/2$, all allowable α_k matrices are contained in the direct product of three Dirac algebras. We are interested, however, only in those equations for which there exists an invariant Lagrangian. It is known[1] that α_k cannot correspond to an equation for which there exists an invariant Lagrangian if D(m, n) and D(n, m) are not both contained in the representation according to which ψ transforms, or in other words if this representation is not symmetric. Multiplication again by $D(0, \frac{1}{2}) + D(\frac{1}{2}, 0)$ would give an asymmetric expression. In that case the resulting direct product would be one for which no invariant Lagrangian exists. We need therefore consider the direct product only of those algebras for which there is an invariant Lagrangian. Then the α_k matrices of a direct product of two Dirac algebras can always be written in the form $(\gamma_i + \gamma_i')/2$. But the algebra $U[(\gamma_i + \gamma_i')/2]$ is decomposable. Therefore all of the desired irreducible equations with maximum spin $\frac{3}{2}$ can be obtained by considering the direct products

$$U(\gamma_i) \times U(\beta_i), \tag{44}$$

$$U(\gamma_i) \times U(\hat{\beta}_i) \tag{45}$$

(including both the spinor and pseudospinor theories).

In order to exhibit the possibilities of our method, let us consider the algebra indicated in (44) in spinor form. The most general form of the α_k for this direct product is

$$\alpha_k = b_1\gamma_k + b_2\gamma_k\gamma_l\beta_l + b_3\gamma_k\beta_l\beta_l + b_4\gamma_l\beta_k\beta_l + b_5\gamma_l\beta_l\beta_k + b_6\beta_k + b_7\beta_k\beta_l\beta_l + b_8\gamma_k\gamma_l\gamma_m\beta_l\beta_m + b_9\gamma_k\gamma_l\beta_m\beta_m\beta_l. \tag{46}$$

Equation (1) with the α_k given by (46) can be written in the spin-tensor form

$$\begin{aligned} &a_{11}\gamma_k\partial_k\psi_1 + a_{12}\gamma_k\partial_k\psi_2 + a_{13}\partial_k F_k + \varkappa\psi_1 = 0,\\ &a_{21}\gamma_k\partial_k\psi_1 + a_{22}\gamma_k\partial_k\psi_2 + a_{23}\partial_k F_k + \varkappa\psi_2 = 0,\\ &a_{31}\hat{R}_n\psi_1 + a_{32}\hat{R}_n\psi_2 + a_{33}Q_{ns}F_s + \varkappa F_n = 0. \end{aligned} \tag{47}$$

Here $\begin{matrix}\hat{R}_n = \partial_n - \frac{1}{4}\gamma_n\gamma_k\partial_k,\\ Q_{ns} = \delta_{ns}\gamma_k\partial_k - \frac{1}{2}\gamma_n\partial_s,\end{matrix}$ $\begin{pmatrix}\psi_1\\ \psi_2\\ F_n\end{pmatrix}$ is the wave function of the system, ψ_1 and ψ_2 are bispinors, and the spin tensor F_n satisfies the equation $\gamma_n F_n = 0$. The a_{ij} are related to the b_i by

$$\begin{aligned} &a_{11} = b_1 + 4b_3 + b_4 + b_5 - 2b_8, \quad a_{12} = 2b_2 + b_6 + b_7 + 16b_9,\\ &a_{22} = -\frac{1}{2}b_1 - \frac{1}{2}b_3 + b_4 + b_5 + 4b_8,\\ &a_{21} = -\frac{1}{2}b_2 + \frac{1}{4}b_6 + b_7 - \frac{1}{2}b_9,\\ &a_{33} = b_1 + b_3, \qquad a_{13} = b_6 + b_7,\\ &a_{32} = 2b_1 + 2b_3 + 4b_4, \quad a_{31} = 2b_2 + b_6 + 4b_7 + 2b_9,\\ &a_{23} = \frac{1}{2}b_1 + \frac{1}{2}b_3 + b_5. \end{aligned} \tag{48}$$

To find the spin and mass states, we note that according to (12) and (43) the infinitesimal rotation operator can be written

$$I_{ij} = \frac{1}{4}(\gamma_i\gamma_j - \gamma_j\gamma_i) + \beta_i\beta_j - \beta_j\beta_i. \tag{49}$$

The squares of the I_{ij} satisfy

$$(I_{ij}^2 + \frac{1}{4})(I_{ij}^2 + \frac{9}{4}) = 0.$$

To separate the states with spin $\frac{1}{2}$ from those with spin $\frac{3}{2}$, we shall write out the spin operator Z^2. According to (23) and (49), the result of operating on ψ with Z^2 is

$$\begin{matrix} \frac{3}{4}\psi_1\\ \frac{3}{4}\psi_2\\ 2(\delta_{n1}F_1 + \delta_{n2}F_2 + \delta_{n3}F_3) - [(\delta_{n2}\gamma_2 + \delta_{n3}\gamma_3)\gamma_1 F_1 + (\delta_{n3}\gamma_3 + \delta_{n1}\gamma_1)\gamma_2 F_2 + (\delta_{n1}\gamma_1 + \delta_{n2}\gamma_2)\gamma_3 F_3] + \frac{3}{4}F_n. \end{matrix} \tag{50}$$

Setting $\xi = \frac{1}{2}$ in (24), we obtain the spin-$\frac{1}{2}$ eigenfunction. This is the wave function in which ψ_1, ψ_2, and F_0 are arbitrary, but $-\gamma_1F_1 = -\gamma_2F_2 = -\gamma_3F_3 = (\frac{1}{3})\gamma_0F_0$ (twelve independent quantities). We now set $\xi = \frac{3}{2}$, obtaining $F_0 = \psi_1 = \psi_2 = 0$, and $\gamma_1F_1 + \gamma_2F_2 + \gamma_3F_3 = 0$ (only eight independent quantities).

The eigenvalues and eigenfunctions of the mass operator α_0 are determined by the equations

spin 3/2: $$a_{33}\gamma_0 F_i = \lambda F_i; \tag{51}$$

$$a_{11}\gamma_0\psi_1 + a_{12}\gamma_0\psi_2 + a_{13}F_0 = \lambda\psi_1,$$

spin 1/2: $$a_{21}\gamma_0\psi_1 + a_{22}\gamma_0\psi_2 + a_{23}F_0 = \lambda\psi_2,$$

$$\frac{3}{4}a_{31}\gamma_0\psi_1 + \frac{3}{4}a_{32}\gamma_0\psi_2 + \frac{1}{2}a_{33}F_0 = \lambda\gamma_0 F_0. \tag{52}$$

Using the explicit expression for γ_0, it can be shown that the eigenfunctions of γ_0 belonging to the eigenvalues ± 1 are not linked. Thus to every positive root $+\lambda$ there corresponds a negative root $-\lambda$ (see also Gel'fand and Iaglom[4]). For the state with spin $\frac{3}{2}$ we have $\lambda = a_{33}$.

It is easily seen from (52) that if at least one state with spin $\frac{1}{2}$ is to exist, we must have

$$\begin{aligned} \frac{3}{4}a_{13}a_{31} &= -\frac{a_{11}^2(a_{22} + a_{33}/2)}{a_{22} - a_{11}};\\ \frac{3}{4}a_{23}a_{32} &= \frac{a_{22}^2(a_{11} + a_{33}/2)}{a_{22} - a_{11}}. \end{aligned} \tag{53}$$

In this case

$$\lambda_{1,2} = \pm(a_{11} + a_{22} + \frac{1}{2}a_{33}), \quad \lambda_{3,4} = \lambda_{5,6} = 0.$$

Since we have restricted our considerations to (44), there are three inequivalent possibilities for the matrix η. These are

$$\eta = \gamma_0(\gamma_0'\gamma_0''), \quad \eta = \gamma_0(i\gamma_5'\gamma_0' \cdot i\gamma_5''\gamma_0''),$$
$$\eta = i\gamma_5\gamma_0 \cdot (i\gamma_5'\gamma_0' \cdot \gamma_0'').$$

In the algebra given by (44) these correspond to

$$T = \eta = \gamma_0(2\beta_0^2 - 1), \tag{54}$$

$$T = \eta = \gamma_0(2\beta_0^2 - 1) - \frac{1}{2}\gamma_l\gamma_0\gamma_m\beta_l\beta_m, \tag{55}$$

$$T = \eta = \gamma_0(2\beta_0^2 + \frac{2}{3} - \frac{5}{3}\beta_l^2) - \frac{1}{2}\gamma_l\gamma_0\gamma_m\beta_l\beta_m. \tag{56}$$

Inserting (54) and α_0 into the quadratic form $\psi^+\eta\alpha_0\psi$, we have

$$\psi^+\eta\alpha_0\psi = a_{11}\psi_1^+\psi_1 + a_{13}\psi_1^+\gamma_0 F_0 + a_{22}\psi_2^+\psi_2 + a_{23}\psi_2^+\gamma_0 F_0 + a_{31}F_0^+\gamma_0\psi_1 + a_{32}F_0^+\gamma_0\psi_2 + a_{33}F_0^+F_0 - a_{33}F_n^+F_n.$$

It is seen from this that a necessary condition for the existence of an invariant real Lagrangian is that

$$\bar{a}_{11} = a_{11},\ \bar{a}_{22} = a_{22},\ \bar{a}_{33} = a_{33},\ a_{13} = \bar{a}_{31},\ a_{23} = \bar{a}_{32}. \quad (57)$$

We now investigate the conditions under which the charge density is positive definite, using (28). It can be shown that if two or three mass states with spin $1/2$ exist, the conditions on the a_{ik} are necessarily contradictory. If there is only one mass state with spin $1/2$, we have

$$-a_{33}F_n^+F_n > 0,\ \frac{(a_{11} + a_{22} + a_{33}/2)^2}{(a_{11} + a_{33}/2)(a_{22} + a_{33}/2)} F_0^+F_0 \geqslant 0. \quad (58)$$

Conditions (53), (57), and (58) are contradictory, i.e., it is impossible to introduce a positive definite charge density in the case given by (54).

Similar considerations hold for equations (55) and (56). If there exists only one state with spin $1/2$, a definite charge density can be introduced. If

$$T = \eta = \gamma_0(2\beta_0^2 - 1) - {}^1/_2\,\gamma_l\gamma_0\gamma_m\beta_l\beta_m$$

(pseudospinor theory) the conditions for the existence of a Lagrangian are

$$\bar{a}_{11} = a_{11},\ \bar{a}_{22} = a_{22},\ \bar{a}_{33} = a_{33},\quad 2\bar{a}_{13} = a_{31},\ -\bar{a}_{23} = a_{32}, \quad (59)$$

and those for a definite charge are

$$a_{33} > 0,\ a_{11} > a_{22},\ a_{11} < -a_{33}/2. \quad (60)$$

If

$$T = \eta = \gamma_0(2\beta_0^2 + {}^2/_3 - {}^5/_3\beta_l^2) - {}^1/_2\,\gamma_l\gamma_0\gamma_m\beta_l\beta_m$$

(spinor theory) the conditions for the existence of a Lagrangian are

$$\bar{a}_{11} = a_{11},\ \bar{a}_{22} = a_{22},\ \bar{a}_{33} = a_{33},\quad -3\bar{a}_{13} = a_{31},\ -\bar{a}_{23} = a_{32}, \quad (61)$$

and those for a definite charge are

$$a_{33} > 0,\ a_{11} > -(a_{22} + a_{33}/2),\ a_{22} < -a_{33}/2. \quad (62)$$

The spin-$1/2$ equations with a single mass state must satisfy the subsidiary conditions

$$(a_{31}/a_{11}^2)\psi_1 + (a_{32}/a_{22}^2)\psi_2 = 0,$$
$$({}^3/_4 M + \gamma_\nu\partial_\nu)(a_{31}\psi_1 + a_{32}\psi_2) + {}^3/_8\, a_{33}M\left(\frac{a_{31}}{a_{11}}\psi_1 + \frac{a_{32}}{a_{12}}\psi_2\right) + a_{33}({}^1/_2\,\partial_\nu F_\nu - \gamma_0\gamma_\nu\partial_\nu F_0) - \varkappa\gamma_0 F_0 = 0;\ \nu = 1, 2, 3;\ M = \varkappa(a_{11} + a_{22} + {}^1/_2\,a_{33})^{-1}. \quad (63)$$

With the aid of (48), we obtain commutation relations for the α_k in the parametric form

$$\alpha_k = (a_{33} - b_3)\gamma_k + b_3\gamma_k\beta_l\beta_l + ({}^1/_4\,a_{32} - {}^1/_2\,a_{33})\gamma_l\beta_k\beta_l + (a_{23} - {}^1/_2\,a_{33})\gamma_l\beta_l\beta_k + b_8\gamma_k\gamma_l\gamma_m\beta_l\beta_m + {}^1/_{12}\{(4a_{31} + a_{13})\gamma_k\gamma_l\beta_l + 2(-a_{31} + 8a_{13})\beta_k + 2(a_{31} - 2a_{13})\beta_k\beta_l\beta_l - (a_{31} + a_{13})\gamma_k\gamma_l\beta_m\beta_m\beta_l\};$$
$$b_8 = {}^1/_4(a_{22} + {}^3/_2\,a_{33} - a_{23} - {}^1/_4\,a_{32});\ b_3 = {}^1/_3(a_{11} + {}^1/_2\,a_{22} + {}^3/_4\,a_{33} - {}^3/_2\,a_{23} - {}^3/_8\,a_{32}). \quad (64)$$

Here the commutation relations between the γ matrices are given by (30), while those between the β matrices are given by (41). The values of the constants a_{13}, a_{31}, a_{32}, and a_{23} depend on the choice of theory.

Thus we see that physical requirements lead to definite restrictions on the coefficients a_{ij}. If we specify the mass states we can write the equations for special cases. Then the values of a_{11}, a_{22}, and a_{33} will be fixed. Inserting these into the appropriate formulas, we obtain the positive definite conditions, the subsidiary conditions, and the algebra of the α_k matrices without performing any further calculations. There exist only four types of equations (with spinor and pseudospinor versions). These are the following.

1. The Ginzburg equation, with $a_{33} = 1$. This equation was obtained by Ginzburg[9] in 1943 in spin-tensor form. In 1952 Bhabha[8] dealt with it in spinor form, and it has been treated in detail by Fainberg,[10] It describes a particle which can be found in two states with spin $3/2$ and $1/2$. If $a_{22} = -a_{33}/2$ or $a_{11} = -a_{33}/2$, it breaks up into the Dirac equation and the Pauli-Fierz equation. We remark that the positive-energy equation given algebraically by Harish-Chandra[3] is a special case of the Ginzburg equation.

2. The Fradkin equation with $a_{33} = 1$, $a_{22} = -(a_{11} + 1/2)$. The existence of this equation was pointed out by Fradkin in 1950.[11] It contains no mass state with spin $1/2$, and involves only the single constant a_{11}. When $a_{11} = 0$, it becomes the Pauli-Fierz equation. In the absence of an external field, the equation together with its subsidiary conditions is identical with the Pauli-Fierz equation. When, however, one introduces a gauge-invariant interaction with the electromagnetic field, these equations differ. The Fradkin equation describes a particle with spin $3/2$ and an anomalous magnetic moment.

3. The Ulehla-Petras equation with $a_{33} = 0$ and

$a_{22} = 1$. This equation was obtained algebraically by Petras and investigated by Ulehla.[7] It contains no state with spin $\frac{3}{2}$ and involves the single arbitrary constant a_{11}. In the free state it coincides with the Dirac equation. When the interaction with an electromagnetic field is included, it describes a particle with spin $\frac{1}{2}$ and anomalous magnetic moment.

4. The Pauli-Fierz equation. This equation was obtained by Fierz[12] and has been treated in spin-tensor form by Ginzburg.[13] If $a_{22} = -a_{33}/2$, terms with an even number of β matrices remain in Eq. (64). Using the explicit form of β given by (40), we find that the product $\beta_k\beta_l$ can be written in the form $\begin{pmatrix} \delta_{kl} & 0 \\ 0 & B^{kl} \end{pmatrix}$, where the B^{kl} satisfy the relations

$$B^{kl}B^{mn} = \delta_{lm}B^{kn}. \tag{65}$$

This equation can therefore be written in the form (setting $a_{11} = 0$)

$$\alpha_k = \gamma_k + \left(\frac{1}{\sqrt{3}} - {}^1\!/_2\right)\gamma_l B^{lk} + \left(-\frac{1}{4\sqrt{3}} - {}^1\!/_2\right)\gamma_l B^{kl} + {}^1\!/_4\left(1 - \frac{\sqrt{3}}{4}\right)\gamma_l\gamma_k\gamma_m B^{lm}. \tag{66}$$

Equations (66), (65), and (30) determine the commutation relations of the algebra, and the minimal polynomial of α_0 is $\alpha_0^2(\alpha_0^2 - 1)$. The reflection matrix is $T = \eta = \gamma_0(2B_{00} - 1) - (\frac{1}{2})\gamma_l\gamma_0\gamma_m B^{lm}$, and the infinitesimal rotation matrices are given by $I_{ij} = (\frac{1}{4})(\gamma_i\gamma_j - \gamma_j\gamma_i) + B^{ij} - B^{ji}$. Petras[5] obtained the Pauli-Fierz equation by selection in the somewhat different form in which

$$\alpha_k = \gamma_k + \gamma_l(B^{lk} - B^{kl})/\sqrt{3}. \tag{67}$$

It is easily shown that the equivalence transformation

$$R = E + (\sqrt{3}/8)\gamma^l\gamma^m B^{lm} \tag{68}$$

will bring (67) into the form given by (66). This exhausts all of the irreducible equations in the direct product of (44). By performing a similar analysis for (45), we can write down all of the desired irreducible equations for particles with maximum spin $\frac{3}{2}$. We are at present in the process of analyzing (45) and obtaining an explicit expression for the algebra given by (44). The method of the present article can be standardized for particles of higher spins $(2, \frac{5}{2}, \ldots)$ and has certain definite advantages over other existing methods.

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