28, 734 (1955), Soviet Phys. JETP 1, 591 (1955); Атомная энергия (Atomic Energy), 2, 501 (1957).

[3] L. D. Landau and E. M. Lifshitz, Квантовая механика (Quantum Mechanics), Moscow, 1948 [English translation, Addison Wesley, 1957].

[4] E. V. Inopin, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 901 (1956), Soviet Phys. JETP 4, 764 (1957).

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KINETIC THEORY OF MAGNETOHYDRODYNAMIC WAVES

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We take account of thermal motion of electrons and ions in considering the propagation of magnetohydrodynamic waves in an ionized gas.

As has been shown by Åström[1,2] and Ginzburg,[3] magnetohydrodynamic waves in an ionized gas are nothing more than low-frequency ordinary and extraordinary electromagnetic waves, familiar from the theory of the propagation of radio waves in the ionosphere. The frequency of these waves is much less than the Larmor frequency of the ions. In the above-cited works the electron and ion motions were described by equations for their mean velocities. The phase velocity V_Φ of a magnetohydrodynamic wave is usually much less than the velocity of light c, and may be comparable with the mean thermal velocity v_T^e and v_T^i of the electrons and ions. One can therefore expect that if $V_\Phi \lesssim v_T^e$, the thermal velocity of the charged particles will strongly influence the propagation of the magnetohydrodynamic waves.

If the frequency ω of the magnetohydrodynamic waves is much less than the frequency ν_c of "short-range" collisions, and if the wavelength λ is large compared with the mean free path, a local Maxwell distribution is established during a time on the order of $2\pi/\omega$. In this case, as is well known, the equations of hydrodynamics can be used, and it follows that in addition to magnetohydrodynamic waves of the Alfven type, two mixed magnetosound waves may propagate in the plasma. If, on the other hand, $\omega \gg \nu_c$, the thermal motion of the charged particles can be taken into account by finding the magnetohydrodynamic wave propagation using the kinetic equation with self-consistent interaction.[4]

The present work is devoted to the kinetic theory of magnetohydrodynamic waves propagating in a plasma at any angle θ with respect to an external magnetic field. "Short-range" collisions leading to damping of the waves are not included. The case $\theta = 0$ has been treated by Gershman[5] (see also Dungey[6]). It is found that if $\theta = 0$, the "short-range" collisions give only a small contribution even if it is not true that $\nu_c \ll \omega$.[3,5,6] In any case, the effect of "short-range" collisions will be small for arbitrary θ if $\nu_c \ll \omega$.

1. DISPERSION EQUATION

Consider electromagnetic waves propagating in a plasma of electrons and singly ionized ions. Let $f_{0\alpha}$ be the equilibrium value of the distribution function for particles of type α ($\alpha = e$ denotes electrons, and $\alpha = i$ denotes ions). We shall write a kinetic equation for $f_\alpha(\mathbf{v}, \mathbf{r}, t)$, the small difference between the actual value of the distribution function and $f_{0\alpha}$, assuming that the frequency of the waves is so high that we may neglect the collision integral in this equation. We then have

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha}\mathbf{E}\frac{\partial f_{0\alpha}}{\partial \mathbf{v}} - \omega_H^\alpha \frac{\partial f_\alpha}{\partial \vartheta} = 0, \qquad (1)$$

$$\omega_H^\alpha = e_\alpha H_0 / m_\alpha c, \quad f_{0\alpha} = n_0 (m_\alpha / 2\pi T_\alpha)^{3/2} \exp(-m_\alpha v^2 / 2T_\alpha).$$

Here e_α and m_α are the charge and mass of the particles of type α (with $e_i = e > 0$), $\mathbf{H}_0$ is the external magnetic field strength, ϑ is the polar angle in velocity space ($\mathbf{v}$ is the velocity of particles of type α, and the z axis is parallel to $\mathbf{H}_0$), T_α is the temperature of the gas of particles of type α, and n_0 is the equilibrium electron density, which is equal to the equilibrium ion density. The electric field strength is given by

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{4\pi e}{c^2}\frac{\partial}{\partial t}\left(\int \mathbf{v} f_i\, d\mathbf{v} - \int \mathbf{v} f_e\, d\mathbf{v}\right). \tag{2}$$

Let the external action perturbing the equilibrium state of the plasma be turned off at time $t = 0$. Then, using a Fourier-Laplace method to solve Eqs. (1) and (2), it can be shown that after a sufficiently long time t the Fourier components of the electric field strength will be proportional to $e^{-i\omega' t}$. The complex frequencies $\omega' = \omega - i\gamma$ are defined as the solutions of the dispersion equation for the lowest γ. To obtain the dispersion equation we insert expressions for f_α and $\mathbf{E}$ proportional to $e^{i(\mathbf{kr}-\omega' t)}$ into (1) and (2), where $\mathbf{k}$ is a given real vector and $\mathrm{Im}\,\omega' < 0$, which means that f_α and $\mathbf{E}$ are in the form of plane waves. The condition on ω' corresponds to solving Eqs. (1) and (2) by a Laplace transform in time. The dispersion equation will then be of the form[7]

$$An'^4 + Bn'^2 + C = 0, \quad n' = kc/\omega', \tag{3}$$

where

$$\begin{aligned} A &= \varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + 2\varepsilon_{13}\cos\theta\sin\theta, \\ B &= 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13})\cos\theta\sin\theta + \varepsilon_{13}^2 - \varepsilon_{11}\varepsilon_{33} \\ &\quad - (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2)\cos^2\theta - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2)\sin^2\theta, \end{aligned} \tag{4}$$

$$C = \mathrm{Det}\,|\varepsilon_{ik}| = \varepsilon_{33}(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) + \varepsilon_{11}\varepsilon_{23}^2 + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{13} - \varepsilon_{22}\varepsilon_{13}^3.$$

Here θ is the angle between the wave vector $\mathbf{k}$ and the magnetic field $\mathbf{H}_0$. If $\mathrm{Im}\,\omega' < 0$, the dielectric constant $\epsilon_{ik}(\omega', k)$ (with $i, k = 1, 2, 3$) is of the form

$$\begin{aligned} \varepsilon_{ik}(\omega', \mathbf{k}) &= \delta_{ik} - i\sum_\alpha \frac{4\pi e e_\alpha}{\omega'\omega_H^\alpha T_\alpha}\int v_i f_{0\alpha}\exp(ia_\alpha\sin\vartheta + ib_\alpha\vartheta) \\ &\quad \times \int^{\vartheta} v_k \exp(-ia_\alpha\sin\psi - ib_\alpha\psi)\, d\psi\, d\mathbf{v}, \\ a_\alpha &= k_x v_\perp/\omega_H^\alpha, \quad b_\alpha = (k_z v_z - \omega')/\omega_H^\alpha. \end{aligned} \tag{5}$$

The z axis is parallel to $\mathbf{H}_0$, and the x axis lies in the plane containing $\mathbf{k}$ and $\mathbf{H}_0$.

Let us write Eq. (5) in a different form. Consider, for example, ϵ_{11}. Bearing in mind the relations

$$e^{-ia\sin\psi} = \sum_{n=-\infty}^{\infty} J_n(a)\, e^{-in\psi}; \quad \int_0^{2\pi} e^{ia\sin\psi - in\psi}\, d\psi = 2\pi J_n(a) \tag{6}$$

and the expression for the second exponential integral of Weber,[8] we find that

$$\varepsilon_{11} = 1 - \sum_\alpha \frac{v_\alpha z_0^\alpha}{\sqrt{\pi}\,\mu_\alpha} e^{-\mu_\alpha} \sum_{n=-\infty}^{\infty} n^2 I_n(\mu_\alpha)\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z_n^\alpha - t}\, dt, \tag{7}$$

where $I_n(\mu_\alpha)$ is the modified Bessel function, and

$$\begin{aligned} v_\alpha &= \Omega_\alpha^2/\omega'^2, \quad \Omega_\alpha = (4\pi n_0 e^2/m_\alpha)^{1/2}, \quad \mu_\alpha = (k_x v_T^\alpha/\omega_H^\alpha)^2; \\ z_n^\alpha &= (\omega' - n|\omega_H^\alpha|)/\sqrt{2}k_z v_T^\alpha; \quad v_T^\alpha = (T_\alpha/m_\alpha)^{1/2}. \end{aligned} \tag{8}$$

The series of Eq. (7) can be summed by noting that

$$\int_0^\infty e^{i\lambda\cos\varphi + i\gamma\varphi}\, d\varphi = i\sum_{n=-\infty}^{\infty}\frac{I_n(\lambda)}{\gamma - n} \quad (\mathrm{Im}\,\gamma > 0). \tag{9}$$

After integrating over t, we obtain an expression for ϵ_{11} in the form of a single integral. The other components of ϵ_{ik} can be expressed similarly. As a result we arrive at

$$\begin{aligned} \varepsilon_{11} &= 1 - i\sum_\alpha \frac{\Omega_\alpha^2}{\omega'|\omega_H^\alpha|}\int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) \\ &\quad + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_\alpha^2}{4}\varphi^2\Big\}\left(i\frac{\omega'}{|\omega_H^\alpha|} - \frac{1}{2}\varkappa_\alpha^2\varphi\right)\sin\varphi\, d\varphi, \\ \varepsilon_{12} &= i\sum_\alpha \frac{\Omega_\alpha^2}{\omega'\omega_H^\alpha}\int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) \\ &\quad + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_\alpha^2}{4}\varphi^2\Big\}(\cos\varphi - 1)\left(\frac{i\omega'}{|\omega_H^\alpha|} - \frac{1}{2}\varkappa_\alpha^2\varphi\right)d\varphi, \\ \varepsilon_{13} &= -i\cot\theta\sum_\alpha \frac{\Omega_\alpha^2}{\omega'|\omega_H^\alpha|}\int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) \\ &\quad + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_\alpha^2}{4}\varphi^2\Big\}\left(1 + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{1}{2}\varkappa_\alpha^2\varphi^2\right)d\varphi, \\ \varepsilon_{22} &= 1 + i\sum_\alpha \frac{\Omega_\alpha^2}{\omega'|\omega_H^\alpha|}\int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) \\ &\quad + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_\alpha^2}{4}\varphi\Big\}\{\mu_\alpha + (1 - 2\mu_\alpha)\cos\varphi + \mu_\alpha\cos^2\varphi\}\, d\varphi, \\ \varepsilon_{23} &= i\sum_\alpha \frac{\Omega_\alpha^2\varkappa_\alpha\sqrt{\mu_\alpha}}{\sqrt{2}\omega'\omega_H^\alpha} \\ &\quad \times \int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_2^\alpha}{4}\varphi\Big\}(1 - \cos\varphi)\, d\varphi, \\ \varepsilon_{33} &= 1 + i\sum_\alpha \frac{\Omega_\alpha^2}{\omega'|\omega_H^\alpha|} \\ &\quad \times \int_0^\infty \exp\Big\{\mu_\alpha(\cos\varphi - 1) + \frac{i\omega'}{|\omega_H^\alpha|}\varphi - \frac{\varkappa_\alpha^2}{4}\varphi\Big\}\left(1 - \frac{1}{2}\varkappa_\alpha^2\varphi^2\right)d\varphi, \\ \varepsilon_{21} &= -\varepsilon_{12}; \quad \varepsilon_{31} = \varepsilon_{13}; \quad \varepsilon_{32} = -\varepsilon_{23}, \\ \varkappa_\alpha &= \sqrt{2}k_z v_T^\alpha/|\omega_H^\alpha|. \end{aligned} \tag{10}$$

Equations (10) for the ϵ_{ik} are analytic functions of ω' over the whole ω' plane. We shall use them in solving the dispersion equation (3). We note that if H_0 is replaced by $-H_0$, the components ϵ_{11}, ϵ_{22}, ϵ_{33}, and ϵ_{13} remain invariant, whereas ϵ_{12} and ϵ_{23} change sign.

Let us rewrite Eq. (10). Expanding $\exp(\mu_\alpha \cos\varphi)$ in Eq. (10) in a series of the $I_n(\mu_\alpha)$ functions and making use of the relation[9]

$$I(z) = \frac{z}{\sqrt{\pi}} \int\limits_C \frac{e^{-t^2}}{z-t} dt = -i\sqrt{\pi}\, z w(z),$$

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int\limits_0^z e^{t^2} dt\right), \tag{11}$$

where the integration over t is taken along a contour C going from $-\infty$ to $+\infty$ and circling the point $t = z$ from below, we obtain

$$\varepsilon_{11} = 1 - \sum_\alpha \frac{v_\alpha z_0^\alpha}{\sqrt{\pi \mu_\alpha}} e^{-\mu_\alpha} \sum_{n=-\infty}^{\infty} n^2 I_n(\mu_\alpha) \int\limits_C \frac{e^{-t^2}}{z_n^\alpha - t} dt,$$

$$\varepsilon_{12} = -i \sum_\alpha \frac{e_\alpha v_\alpha z_0^\alpha}{\sqrt{\pi}} e^{-\mu_\alpha} \sum_{n=-\infty}^{\infty} n\left[I'_n(\mu_\alpha) - I_n(\mu_\alpha)\right] \int\limits_C \frac{e^{-t^2}}{z_n^\alpha - t} dt,$$

$$\varepsilon_{13} = -\sum_\alpha \frac{\sqrt{2} v_\alpha z_0^\alpha}{\sqrt{\pi \mu_\alpha}} e^{-\mu_\alpha} \sum_{n=-\infty}^{\infty} n I_n(\mu_\alpha) \int\limits_C \frac{t e^{-t^2}}{z_n^\alpha - t} dt,$$

$$\varepsilon_{22} = 1 - \sum_\alpha \frac{v_\alpha z_0^\alpha}{\sqrt{\pi}} e^{-\mu_\alpha}$$

$$\times \sum_{n=-\infty}^{\infty} \left[\left(\frac{n^2}{\mu_\alpha} + 2\mu_\alpha\right) I_n(\mu_\alpha) - 2\mu_\alpha I'_n(\mu_\alpha)\right] \int\limits_C \frac{e^{-t^2}}{z_n^\alpha - t} dt,$$

$$\varepsilon_{23} = i \sum_\alpha \frac{\sqrt{2} e_\alpha v_\alpha z_0^\alpha \sqrt{\mu_\alpha}}{\sqrt{\pi}} e^{-\mu_\alpha}$$

$$\times \sum_{n=-\infty}^{\infty} \left[I'_n(\mu_\alpha) - I_n(\mu_\alpha)\right] \int\limits_C \frac{t e^{-t^2}}{z_n^\alpha - t} dt, \tag{12}$$

$$\varepsilon_{33} = 1 - \sum_\alpha \frac{2 v_\alpha z_0^\alpha}{\sqrt{\pi}} e^{-\mu_\alpha} \sum_{n=-\infty}^{\infty} I_n(\mu_\alpha) \int\limits_C \frac{t^2 e^{-t^2}}{z_n^\alpha - t} dt,$$

where $e_i = 1$ and $e_e = -1$. If $\cos\theta > 0$, the integral over t in (12) is taken along a contour C that circles $t = z_n^\alpha$ from below, whereas if $\cos\theta < 0$, the contour C passes above $t = z_n^\alpha$. To be specific, we shall assume below that $\cos\theta > 0$. We note that expression (12) for ϵ_{ik} can be obtained directly from an equation of the form of (7) by replacing the integral over t along the real axis by an integral along C.

In attempting to determine the excitation of electromagnetic vibrations in a plasma by external currents, one must also consider Eq. (3), which defines the wave number $k' = k + i\kappa$ as a function of the frequency ω [in Eqs. (3), (4), and (10) to (12) one must set $\omega' = \omega - i\delta$, where ω is a given real number and $\delta \to +0$].

Equation (3) can be solved in several limiting cases. If $V_\Phi \gg v_T^\alpha$ (and $T_\alpha \to 0$), Eq. (12) leads to known expressions for the ϵ_{ik} (quasi-hydrodynamic approximation), and Eq. (3) gives the index of refraction for the ordinary and extraordinary waves. If the separate terms entering into A are much greater than $|B/n^2|$ and $|C/n^4|$, we obtain by setting $A = 0$ the dispersion equation for longitudinal vibrations of the plasma in the magnetic field. In the present work the dispersion equation (3) is treated for low-frequency waves, when $\omega \ll \omega_H^i$.

2. ANALYSIS OF THE DISPERSION EQUATION

We shall now consider the dispersion equation (3) for a strong magnetic field, when $kv_T^\alpha \ll \omega_H^\alpha$. In view of these inequalities, we may consider that $\mu_\alpha \ll 1$ and $|z_n^\alpha| \gg 1$ for $n = \pm1, \pm2, \ldots$. The functions $I_n(\mu_\alpha)$ and $e^{-\mu_\alpha}$ in (12) can therefore be expanded in powers of μ_α, and in the resulting sums we need retain only the first few terms. In addition, the integrals taken along C which contain z_n^α (with $n = \pm1, \pm2, \ldots$) will be expanded in the asymptotic series

$$I(z) = \frac{z}{\sqrt{\pi}} \int\limits_C \frac{e^{-t^2}}{z-t} dt \sim 1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots - i\sqrt{\pi}\, z e^{-z^2}$$

$$(|\mathrm{Re}\, z| \gg 1, \quad \mathrm{Im}\, z \ll 1). \tag{13}$$

Considering further that $|\omega'| \ll \omega_H^\alpha$, we obtain the following expressions for the ϵ_{ik}:

$$\varepsilon_{11} = n_A^2 \left[1 + \frac{u_i}{v_i} + \frac{1}{u_i} + \frac{3\beta_i^2 n'^2}{u_i}\left(\cos^2\theta - \frac{1}{4}\sin^2\theta\right) + \frac{m_e}{m_i}\right],$$

$$\varepsilon_{12} = \frac{i n_A^2}{\sqrt{u_i}} \left[1 + \beta_i^2 n'^2 \left(\cos^2\theta - \frac{3}{2}\sin^2\theta\right)\right],$$

$$\varepsilon_{13} = -\frac{2\beta_i^2 n'^2 n_A^2}{u_i} \cos\theta \sin\theta; \tag{14}$$

$$\varepsilon_{22} = \varepsilon_{11} - 2\beta_i^2 n'^2 n_A^2 \sin^2\theta \left[I(z_0^i) + \frac{T_e}{T_i} I(z_0^e)\right],$$

$$\varepsilon_{23} = -\frac{i v_i \sin\theta}{\sqrt{u_i} \cos\theta} \left[I(z_0^i) - I(z_0^e)\right],$$

$$\varepsilon_{33} = \frac{v_i}{\beta_i^2 n'^2 \cos^2\theta} \left[1 + \frac{T_i}{T_e} - I(z_0^i) - \frac{T_i}{T_e} I(z_0^e)\right], \tag{15}$$

where

$$n_A = (v_i/u_i)^{1/2} = (4\pi n_0 m_i c^2 / H_0^2)^{1/2}, \quad \beta_\alpha = v_T^\alpha / c, \quad u_\alpha = \omega^2 / (\omega_H^\alpha)^2.$$

If $y_\alpha = -\mathrm{Im}\, z_0^\alpha \ll 1$, expressions (15) for ϵ_{22}, ϵ_{23}, and ϵ_{33}, which contain the integral $I(z_0^\alpha)$, can be simplified. In this case we expand $w(z)$ in Eq. (11) in powers of $y = -\mathrm{Im}\, z$. Dropping terms of order y, we obtain (noting that $z = x - iy$)

$$I(z) = I(x) + O(y); \quad I(x) = 2xF(x) - i\sqrt{\pi}\, x e^{-x^2},$$

$$F(x) = e^{-x^2} \int\limits_0^x e^{t^2} dt. \tag{16}$$

Taking (16) into account, (15) can be written

$$\varepsilon_{11}-\varepsilon_{22}=2\beta_i^2 n'^2 n_A^2 \sin^2\theta\,[2x_iF(x_i)+2x_eF(x_e)\,T_e/T_i - i\sqrt{\pi}\,x_i e^{-x_i^2} - i\sqrt{\pi}\,x_e e^{-x_e^2}\,T_e/T_i],$$
$$\varepsilon_{23}=-\frac{iv_i\sin\theta}{\sqrt{v_i}\cos\theta}[2x_iF(x_i)-2x_eF(x_e) - i\sqrt{\pi}\,x_i e^{-x_i^2}+i\sqrt{\pi}\,x_e e^{-x_e^2}],$$
$$\varepsilon_{33}=\frac{v_i}{\beta_i^2 n^2\cos^2\theta}[1+T_i/T_e-2x_iF(x_i)-2x_eF(x_e)\,T_i/T_e + i\sqrt{\pi}\,x_i e^{-x_i^2} - i\sqrt{\pi}\,x_e e^{-x_e^2}T_i/T_e], \tag{17}$$

where

$$x_\alpha=(\sqrt{2}\,\beta_\alpha n\cos\theta)^{-1},\quad n=kc/\omega.$$

In the most interesting cases, in which it is possible to speak of wave propagation at all, the index of refraction n for magnetohydrodynamic waves is on the order of n_A. We shall assume that $n_A \gg 1$, or that $V_\Phi \ll c$, since only then will the inclusion of thermal motion give significant corrections to n'. Equation (3) then becomes

$$(\cos^2\theta\, n'^2-\varepsilon_{11})(n'^2-\varepsilon_{22}-\varepsilon_{23}^2/\varepsilon_{33}) = -\varepsilon_{12}^2-2n'^2\cos\theta\sin\theta\,\varepsilon_{12}\varepsilon_{23}/\varepsilon_{33}+\dots. \tag{18}$$

In view of the fundamental inequalities

$$v_e \gg v_i \gg u_i \gg 1 \gg \beta_i^2 n^2/u_i \gg \beta_e^2 n^2/u_e. \tag{19}$$

the terms discarded in (18) are small compared with those which remain.

Let us now go on to a consideration of (18) for various special cases.

(a) Consider first propagation of magnetohydrodynamic waves along the magnetic field. Setting $\theta = 0$, we obtain $\epsilon_{11} = \epsilon_{22}$ and $\epsilon_{13} = \epsilon_{23} = 0$. The left side of (3) then breaks up into the product of three factors. We equate each of these to zero, obtaining

$$\varepsilon_{33}=0;\quad n'^2-\varepsilon_{11}\mp\sqrt{-\varepsilon_{12}^2}=0. \tag{20}$$

The first relation in (20) gives the dispersion equation for the longitudinal plasma vibrations investigated by Vlasov[4] and Landau.[10] The second is the dispersion equation for ordinary and extraordinary electromagnetic waves which, for $\theta = 0$, are purely transverse (div **E** = 0). This equation agrees with the dispersion equation obtained by Gershman.[5] If we take account of (19), Eq. (20) gives the indices of refraction for the ordinary and extraordinary waves in the form

$$n_{1,2}^2=n_A^2/(1\mp\sigma);\quad \sigma=\beta_i^2 n_A^2/\sqrt{u_i}. \tag{21}$$

According to this equation, the thermal motion of the ions gives corrections to the indices of refraction which are significant only if $V_\Phi \ll v_T^i$, when $\beta_i^2 n_A^2/\sqrt{u_i} \sim 1$. If we include the exponentially small terms in (13), we obtain the damping constant

$$\left(\frac{\gamma}{\omega}\right)_{1,2}=\sqrt{\frac{\pi}{8}}\frac{u_i}{\beta_i n_{1,2}}\frac{1\mp\sigma}{2\mp\sigma}\exp\{-(z_{\pm1}^i)^2\};\quad (z_{\pm1}^i)^2=\frac{(1\mp\omega_H^i/\omega)^2}{2\beta_i^2 n_{1,2}^2} \tag{22}$$

The imaginary part of the wave vector k′ will be equal to (the frequency ω is given)

$$(\varkappa/k)_{1,2}=\sqrt{\frac{\pi}{8}}\frac{u_i}{\beta_i n_{1,2}}\exp\{-(z_{\pm1}^i)^2\}. \tag{23}$$

Both $(\gamma/\omega)_{1,2}$ and $(\kappa/k)_{1,2}$ are extremely small because $\beta_i n/\sqrt{u_i} \ll 1$ even if $\beta_i^2 n_A^2 \sim \sqrt{u_i}$.

(b) Let us now consider the propagation of magnetohydrodynamic waves at a small angle $\theta \ll 1$ to the magnetic field. We find from (18) that if $\beta_i^2 n_A^2/\sqrt{u_i} \ll 1$, then $n_1 \approx n_2 \approx n_A$. Writing

$$n'_{1,2}=n_A(1+q'_{1,2});\quad q'_{1,2}=q_{1,2}+i\gamma_{1,2}/\omega;\quad \gamma_{1,2}=\varkappa_{1,2}c/n_A\quad (|q'_{1,2}|\ll 1), \tag{24}$$

we find that the quantity

$$q''_{1,2}=q'_{1,2}-\frac{1}{2}\left(\frac{1}{n_A^2}+\frac{1}{u_i}+\frac{3\beta_i^2 n_A^2}{u_i}+\frac{m_e}{m_i}+\cdots\right), \tag{25}$$

is given by

$$q''^2_{1,2}+\frac{1}{2}\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{n_A^2}-\frac{\varepsilon_{23}^2}{n_A^2\varepsilon_{33}}-\theta^2\right)q''_{1,2} +\frac{1}{4}\left(\frac{\varepsilon_{12}^2}{n_A^4}+\theta^2\frac{\varepsilon_{22}-\varepsilon_{11}}{n_A^2}+\theta^2\frac{\varepsilon_{23}^2}{n_A^2\varepsilon_{33}}+2\frac{\varepsilon_{12}\varepsilon_{23}\theta}{n_A^2\varepsilon_{33}}\right)=0. \tag{26}$$

We now make use of expressions (17) for $\epsilon_{11} - \epsilon_{22}$, ϵ_{23}, and ϵ_{33}, obtaining

$$q''_{1,2}=\frac{\theta^2}{4}(1-\beta_i^2 n_A^2 D)\pm\left\{\frac{\theta^4}{16}(1+\beta_i^2 n_A^2 D)^2 +\frac{(1+\beta_i^2 n_A^2)}{4u_i}[1+\beta_i^2 n_A^2-2\theta^2\beta_i^2 n_A^2(I_i-I_e)\,G]\right\}^{1/2}, \tag{27}$$

$$D=2I_i+2I_eT_e/T_i+(I_i-I_e)^2/(1+T_i/T_e-I_i-I_eT_i/T_e),$$
$$G=[1+T_i/T_e-I_i-I_eT_i/T_e]^{-1},$$
$$I_\alpha=I(x_\alpha),\quad x_\alpha=(\sqrt{2}\,\beta_\alpha n_A)^{-1}.$$

Let us now consider (27) in some special cases. Let the phase velocity $V_\Phi = c/n_A$ be much greater than the mean thermal velocity of the ions, or $\beta_i n_A \ll 1$. Equation (27) then leads to

$$\mathrm{Re}\,q''_{1,2}=\frac{\theta^2}{4}\pm\frac{1}{2}\sqrt{\frac{\theta^4}{4}+\frac{1}{u_i}}; \tag{28}$$

$$\left(\frac{\gamma}{\omega}\right)_{1,2}=\sqrt{\frac{\pi}{32}}\,\theta^2\frac{m_e}{m_i}\beta_e n_A\left(1\mp\frac{\theta^2}{\sqrt{\theta^4+4/u_i}}\right)\exp\left\{-\frac{1}{2\beta_e^2 n_A^2}\right\}. \tag{29}$$

It follows from (29) that as the phase velocity decreases, the damping increases. If $\beta_e n_A \ll 1$, the ratio γ/ω is exponentially small. The damping given by (29), however, will be much greater than that given by (22) for all except the smallest values of θ, since the quantity $u_i/2\beta_i^2 n_A^2$, whose exponential enters into (22), is much greater than $1/2\beta_e^2 n_A^2$. If $V_\Phi \lesssim v_T^e$, then $(\gamma/\omega)_{1,2} \sim \theta^2 \beta_e n_A m_e/m_i$. As is seen from (29), the extraordinary wave is more highly damped than the ordinary one, or $\gamma_2 > \gamma_1$.

Equation (29) is valid if $\beta_i n_A \ll 1$. If, however, $\beta_i n_A \sim 1$, which means that $V_\Phi \sim v_T^i$, then as follows from (27) we have

$$\operatorname{Re} q''_{1,2} \sim 1/\sqrt{u_i}, \quad (\gamma/\omega)_{1,2} \sim a_1\theta^2 + a_2\theta^4/\sqrt{u_i},$$

where $a_{1,2} \sim 1$ and $\theta^2 \lesssim 1/\sqrt{u_i}$.

For $V_\Phi \ll v_T^i$, Eq. (27) gives

$$q''_{1,2} = i\sqrt{\frac{\pi}{8}}\beta_i n_A\theta^2 \pm \frac{1}{2}\left[\frac{\beta_i^4 n_A^4}{u_i} - \frac{\pi}{2}\beta_i^2 n_A^2\theta^4\right]^{1/2}, \tag{30}$$

or

$$(\gamma/\omega)_{1,2} = \sqrt{\pi/8}\,\beta_i n_A\theta^2 \text{ for } 1/u_i > \pi\theta^4/2\beta_i^2 n_A^2, \tag{31}$$

$$(\gamma/\omega)_{1,2} = \sqrt{\frac{\pi}{8}}\beta_i n_A\theta^2 \mp \frac{1}{2}\left[\frac{\pi}{2}\beta_i^2 n_A^2\theta^4 - \frac{\beta_i^4 n_A^4}{u_i}\right]^{1/2} \text{ for } \frac{1}{u_i} < \frac{\pi}{2}\frac{1}{\beta_i^2 n_A^2}\theta^4. \tag{32}$$

Equations (30) to (32) are valid only if $\beta_i n_A \gg 1$, $\beta_i^2 n_A^2/\sqrt{u_i} \ll 1$, and $\beta_i n_A\theta^2 \ll 1$. If however, $\beta_i n_A \gg 1$, but the inequality $\beta_i^2 n_A^2/\sqrt{u_i} \ll 1$ is not fulfilled, the initial approximation $n_1 \approx n_2 \approx n_A$ becomes invalid. Let $\beta_i^2 n_A^2 \sim \sqrt{u_i}$. Then assuming that in the zeroth approximation the indices of refraction of the ordinary and extraordinary waves are given by (21), we obtain

$$\left(\frac{\gamma}{\omega}\right)_{1,2} = \sqrt{\frac{\pi}{2}\frac{1\mp\sigma}{2\mp\sigma}}\beta_i n_{1,2}\theta^2, \quad \left(\frac{\varkappa}{k}\right)_{1,2} = \sqrt{\frac{\pi}{8}}\beta_i n_{1,2}\theta^2,$$

$$(\beta_i n_A\theta^2 \ll 1). \tag{33}$$

If $\sigma = \beta_i^2 n_A^2/\sqrt{u_i} \ll 1$, then (33) leads to Eq. (31) for $\gamma_{1,2}$. Thus Eqs. (27) to (33) for small θ will give $\gamma_{1,2}$ for all σ.

(c) Let $\theta \sim 1$. The right side of (18) contains quantities small compared with the individual terms on the left side. Therefore we can obtain an approximate solution of (18) by equating each of the factors on the left side to zero. The index of refraction of the ordinary wave is then given by

$$n_1 = n_A/\cos\theta. \tag{34}$$

We note that Åström calls the wave whose index of refraction is that given by (34) the extraordinary wave.

Let us find the corrections to (34). Writing

$$n'_1 = n_A(1 + q'_1)/\cos\theta, \quad q'_1 = q_1 + i(\gamma/\omega)_1, \quad \gamma_1 = \varkappa_1 c\cos\theta/n_A, \quad |q'_1| \ll 1, \tag{35}$$

we find from (18) that

$$q'_1 = q''_1 + \frac{1}{2}\left(\frac{1}{n_A^2} + \frac{1}{u_i} + \frac{3\beta_i^2 n_A^2}{u_i}\left(1 - \frac{1}{4}\tan^2\theta\right) + \frac{m_e}{m_i}\right), \tag{36}$$

$$q''_1 = \frac{[1 + \beta_i^2 n_A^2(1 - {}^3/_2\tan^2\theta)]^2\,[\cot^2\theta + 2\beta_i^2 n_A^2(I_e - I_i)\,a]}{2u_i(1 + \beta_i^2 n_A^2 b)}, \tag{37}$$

$$a^{-1} = [1 + \beta_i^2 n_A^2(1 - {}^3/_2\tan^2\theta)]\,[1 + T_i/T_e - I_i - I_e T_i/T_e],$$

$$b = (I_i - I_e)^2/(1 + T_i/T_e - I_i - I_e T_i/T_e) + 2I_i + 2I_e T_e/T_i,$$

$$I_\alpha = I(x_\alpha), \quad x_\alpha = (\sqrt{2}\,\beta_i n_A)^{-1}.$$

If $\beta_i n_A \ll 1$, we obtain

$$q_1 = \frac{1}{2}\left(\frac{1}{n_A^2} + \frac{1 + \cot^2\theta}{u_i} + \frac{m_e}{m_i}\right), \tag{38}$$

$$\left(\frac{\gamma}{\omega}\right)_1 = \sqrt{\frac{\pi}{8}}\frac{m_e}{m_i}\frac{\beta_e n_A}{u_i}\cot^2\theta\exp\{-x_e^2\}. \tag{39}$$

If $\beta_e n_A \ll 1$, the quantity $(\gamma/\omega)_1$ is exponentially small, while if $\beta_e n_A \sim 1$, we have $(\gamma/\omega)_1 \sim m_e\beta_e n_A/m_i u_i$.

If $\beta_i n_A \sim 1$, it is easily seen from (37) that

$$\operatorname{Re} q''_1 \sim \operatorname{Im} q''_1 \sim 1\,u_i^{-1}.$$

Finally, if $\beta_i n_A \gg 1$, Eq. (37) leads to

$$\left(\frac{\gamma}{\omega}\right)_1 = \frac{\cot^2\theta\,(1 - {}^3/_2\tan^2\theta)^2}{\sqrt{8\pi}}\frac{\beta_i^3 n_A^3}{u_i}, \tag{40}$$

and $\operatorname{Im} q''_1 \gg \operatorname{Re} q''_1$. Equation (40) is valid if $(\gamma/\omega)_1 \ll 1$, or if $\beta_i^3 n_A^3/u_i \ll 1$. If $\beta_i^3 n_A^3/u_i \sim 1$, it follows from (18) that $\operatorname{Re} n' \sim \operatorname{Im} n' \sim n_A$. Thus the ordinary wave is weakly damped $(\gamma_1 \ll \omega_1)$ only if $\beta_i^3 n_A^3/u_i \ll 1$.

(d) Let us now consider the propagation of the extraordinary wave for $\theta \sim 1$. We equate the second factor on the left side of (18) to zero, writing

$$n'^2 - \varepsilon_{11} + (\varepsilon_{11} - \varepsilon_{22}) - \varepsilon_{23}^2/\varepsilon_{33} = 0. \tag{41}$$

Assuming that $\epsilon_{11} \approx n_A^2$ is much greater than either $|\epsilon_{11} - \epsilon_{22}|$ or $|\epsilon_{23}^2/\epsilon_{33}|$, we can use (41) to find the index of refraction of the extraordinary wave (which Åström calls the ordinary wave). This is

$$n_2 = n_A. \tag{42}$$

Let us now find the corrections to (42). We set

$$n'_2 = n_A(1 + q'_2); \quad q'_2 = q_2 + i(\gamma/\omega)_2; \quad \gamma_2 = \varkappa_2 c/n_A; \quad |q'_2| \ll 1. \tag{43}$$

Then it follows from (18) that

$$q_2 = \frac{1}{2}\left(\frac{1}{n_A^2} + \frac{m_e}{m_i} - \frac{\cot^2\theta}{u_i}\right) - \frac{1}{2}\beta_i^2 n_A^2\sin^2\theta\left(2 + \frac{T_e}{T_i} + 2x_e F(x_e)\frac{T_e}{T_i}\right). \tag{44}$$

$$\left(\frac{\gamma}{\omega}\right)_2 = \sqrt{\frac{\pi}{2}}\frac{m_e}{m_i}\frac{\sin^2\theta}{\cos\theta}\beta_e n_A e^{-x_e^2}, \quad x_e = (\sqrt{2}\beta_e n_A \cos\theta)^{-1}. \quad (45)$$

Comparison of (29) and (45) shows that if $\beta_e n_A \lesssim 1$, the extraordinary wave is damped much more strongly than the ordinary one, or $\gamma_2/\gamma_1 \sim u_i \gg 1$.

If $\beta_i n_A \ll 1$, then $|q_2'| \ll 1$. If, on the other hand, $\beta_i n_A \sim 1$, then n' must be found from (41), which then becomes

$$n'^2 - n_A^2 + \beta_i^2 n'^2 n_A^2 \sin^2\theta \left[\frac{I^2(z_0^i)}{1 + T_e/T_i - I(z_0^i)} + 2I(z_0^i)\right] = 0, \quad (46)$$

where $I(z_0^i)$ is the integral defined by (11). Equation (46) is obtained on the assumption that $|\beta_e n' \cos\theta| \gg 1$. It follows from (46) that $\mathrm{Re}\, n_2' \sim \mathrm{Im}\, n_2' \sim n_A$, which means that if $\beta_i n_A \sim 1$, the extraordinary wave is strongly damped. Exact solutions of (46) can be obtained numerically, using the tables of Faddeeva and Terent' ev.[9]

We now make one remark regarding the propagation of electromagnetic waves perpendicular to the magnetic field. As $\theta \to \pi/2$, we find that $|z_n^\alpha| \to \infty$, and the imaginary parts of ϵ_{ik} in (12) vanish. Therefore the damping of the electromagnetic waves for $\theta = \pi/2$ is determined entirely by "short-range" collisions.

3. CONCLUSIONS

The kinetic equation was used to investigate the propagation of magnetohydrodynamic waves whose frequency is much greater than the frequency of "short-range" collisions of charged particles both with each other and with neutral particles. It is shown that magnetohydrodynamic waves propagating at an angle $\theta \neq \pi/2$ are damped (damping is similar to that found by Landau[10] for longitudinal plasma waves). The damping constant increases as the phase velocity $V_\Phi \sim c/n_A$ decreases, and is no longer exponentially small when $V_\Phi \sim v_T^e$. If $\theta \ll 1$ and $V_\Phi \ll v_T^i$, the damping of magnetohydrodynamic waves is small only in a very narrow angle interval $\theta^2 \ll 1/\beta_i n_A \ll 1$. If $\theta \sim 1$ and $V_\Phi \sim v_T^e$, the damping constant γ_2 for the extraordinary wave is much greater than the damping constant γ_1 for the ordinary wave, and we may write $\gamma_2/\gamma_1 \sim u_i \gg 1$. The ordinary wave is strongly damped ($\mathrm{Re}\, n_1' \sim \mathrm{Im}\, n_1'$) for $V_\Phi \ll v_T^i$, when $\beta_i^3 n_A^3 \sim u_i$. Strong damping ($\mathrm{Re}\, n_2' \sim \mathrm{Im}\, n_2' \sim n_A$) does not allow the extraordinary wave to propagate when the phase velocity becomes of the order of v_T^i.

[1] E. Åström, Ark. f. Fysik **2**, 443 (1951).

[2] E. Åström, Nature **165**, 1019 (1950).

[3] V. L. Ginzburg, J. Exptl. Theoret. Phys. (U.S.S.R.) **21**, 788 (1951).

[4] A. A. Vlasov, J. Exptl. Theoret. Phys. (U.S.S.R.) **8**, 291 (1938).

[5] B. N. Gershman, J. Exptl. Theoret. Phys. (U.S.S.R.) **24**, 453 (1953).

[6] J. W. Dungey, J. Geophys. Research 59, **323** (1954).

[7] A. G. Sitenko and K. N. Stepanov, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 642 (1956), Soviet Phys. JETP **4**, 512 (1957).

[8] G. N. Watson, A Treatise on the Theory of Bessel Functions (Russ. Transl.), IIL, M. 1948, part 1, p. 432.

[9] V. N. Faddeeva and M. N. Terent' ev, Таблицы значении интеграла вероятностеи от комплексного аргумента (Tables of the Probability Integral of a Complex Argument), edited by Academician V. A. Fock, GITTL, M. 1954.

[10] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **16**, 574 (1946).

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ERRATA TO VOLUME 7

Page	Reads	Should Read
533, title	Nuclear magnetic moments of Sr^{87} and Mg^{95}	Nuclear magnetic moments of Sr^{87}
645 Eq. (1)	$\ldots + \varkappa \sqrt{j_0 (j_0 + 1)}$	$\ldots - \varkappa \sqrt{j_0 (j_0 + 1)}$
647 Eq. (11)	$(L+1) \lvert B_L^- \rvert^2 - L \lvert B_L^+ \rvert^2$	$L(L+1) [\lvert B_L^- \rvert^2 - \lvert B_L^+ \rvert^2]$
894 Eq. (12)	$\varepsilon_{11} = 1 - \sum \frac{\cdots}{\sqrt{\pi/\mu}}$	$\varepsilon_{11} = 1 - \sum \frac{\cdots}{\sqrt{\pi}\,\mu}$
897 Eq. (45)	$\sqrt{\pi/2}$	$\sqrt{\pi/8}$
979 Table II, heading	$\lvert E_\gamma > 50$ **Mev** $\lvert E_\gamma > 50$ **Mev**	$\lvert E_\gamma < 50$ **Mev** $\lvert E_\gamma > 50$ **Mev**
1023 Figure caption		a) $\omega < \omega_H$, b) $\omega > \omega_H$
1123 Eq. (2)	$\Gamma = \mu_2/\mu_1$	$\Gamma = \mu_2/\mu_1$, $\mu_\perp = (\mu_1^2 - \mu_2^2)/\mu_1$

ERRATA TO VOLUME 8

Page	Reads	Should Read
375 Figure caption	a) positrons of energy up to 0.4 ϵ, b) positrons of energy up to 0.3 ϵ.	a) positrons of energy up to 0.3 ϵ, b) positrons of energy up to 0.4 ϵ.
816 Beginning of Eq. (8)	$I_2^5 = (4\pi)^2 \ldots$	$I_2^2 = (4\pi)^5 \ldots$