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[1] S. I. Pekar, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1022 (1957), Soviet Phys. JETP **6**, 785 (1958); J. Phys. Chem. Solids **5**, 11 (1958).

[2] Ia. I. Frenkel', Phys. Rev. **37**, 17, 1276 (1931).

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THE MOTIONS OF ROTATING MASSES IN THE GENERAL THEORY OF RELATIVITY

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The relativistic equations of translational and rotational motion for spherically symmetrical rotating bodies, developed in a previous paper,[1] have been integrated. Some novel relativistic effects, due to the proper rotations of the bodies, appear and are discussed.

1. INTRODUCTION

In an article by one of the authors[1] the equations of translational and rotational motion for spherically symmetrical rotating bodies were derived from Einstein's gravitational equations. In the present paper we shall study the solutions of these equations. In view of the well-known difficulties in the general problem of celestial mechanics, we shall limit ourselves here to a study of the two-body problem. In their well-known paper,[2] Thirring and Lense studied the relativistic effects of rotation as applied to the very simple (though important) case of a very light, non-rotating body moving in the field of a massive rotating body, by making use of the properties of geodesics. In our problem, however, both bodies are treated on an equal footing — they may be of comparable mass, and each may rotate about its own axis.

Let a^i and b^i, m_a and m_b, M_a^{ik} and M_b^{ik} be the coordinates, masses, and proper rotational moments of the two bodies, and let r be the distance between them and γ the Newtonian gravitational constant. In Ref. 1 the equations

$$\ddot{a}^i - \left(\frac{\gamma m_b}{r}\right)_{,a^i} = F_a^i + D_a^i \tag{1.1}$$

were derived for the translational motion, and

$$M_a^{ik} = L_a^{ik} \tag{1.2}$$

for the proper rotation. (Analogous equations hold for body b). Here F_a^i is the relativistic correction to the Newtonian force when the rotation of the body is neglected; it has been discussed by numerous authors.[3,4,5] D_a^i is the relativistic correction due to the rotation derived in Ref. 1 (cf. Eq. (5.6) of that paper). L_a^{ik} is an abbreviation for the right-hand side of equation (6.2) of Ref. 1. We shall not repeat here the complicated expressions for F_a^i, D_a^i, and L_a^{ik}.

In the Newtonian approximation, $F_a^i = D_a^i = L_a^{ik} = 0$, and we obtain the familiar solution, with $r \equiv |\mathbf{a} - \mathbf{b}|$,

$$\frac{1}{r} = \frac{1}{p}(1 + e\cos\varphi), \tag{1.3}$$

$$M_a^{ik} = \text{const}, \quad M_b^{ik} = \text{const}, \tag{1.4}$$

subject to the conservation laws

$$M_1 = M_2 = 0, \quad M_3 \equiv r^2\dot{\varphi} = (\gamma m p)^{1/2} = \text{const}, \tag{1.5}$$

$$E \equiv \frac{1}{2}v^2 - \frac{\gamma m}{r} = -\frac{\gamma m}{2a} = \text{const}. \tag{1.6}$$

Here e is the eccentricity of the orbit, if we take $e < 1$; p is a parameter, a is the major semiaxis of the relative orbit, v is the relative velocity, and $m \equiv m_a + m_b$.

2. THE ROTATION OF A BODY ABOUT ITS OWN AXIS

Let us consider the rotation of a body about its axis in the first non-Newtonian approximation. As in Ref. 1, we shall use the specific proper moments σ_a and σ_b instead of M_a^{ik} and M_b^{ik} in this approximation: $M_a^{ik} = m_a \delta_{ike} \sigma_a^e$. Substituting the first-approximation values given in (1.3) to (1.5) into the right hand side of (1.2) (cf. Eq. (6.2) in Ref. 1), we obtain equations which are easily integrable. Considering only the secular terms for body a, we have:

$$\tilde{\sigma}_a^1 = \sigma_a^1 - \frac{\gamma m_b}{c^2 mp}(2m_a + m_b)\,\sigma_a^2 \varphi.$$

$$+ \frac{\gamma m_b}{c^2 p\,|\mathbf{M}|}\left(\frac{1}{4}\sigma_a^2\sigma_a^3 - \sigma_a^2\sigma_b^3 + \frac{1}{2}\sigma_a^3\sigma_b^2\right)\varphi,$$

$$\tilde{\sigma}_a^2 = \sigma_a^2 + \frac{\gamma m_b}{c^2 mp}(2m_a + m_b)\,\sigma_a^1 \varphi$$

$$+ \frac{\gamma m_b}{c^2 p\,|\mathbf{M}|}\left(-\frac{1}{4}\sigma_a^1\sigma_a^3 + \sigma_a^1\sigma_b^3 - \frac{1}{2}\sigma_a^3\sigma_b^1\right)\varphi,$$

$$\tilde{\sigma}_a^3 = \sigma_a^3 + \frac{\gamma m_b}{c^2 p\,|\mathbf{M}|}\left(\frac{3}{4}\sigma_a^1\sigma_a^2 + \frac{1}{4}\sigma_a^1\sigma_b^2 + \frac{1}{2}\sigma_a^2\sigma_b^1\right)\varphi, \quad (2.1)$$

where φ is the angle of rotation from (1.3); analogous equations are also obtained for body b. Here and throughout the rest of this paper we use the following notation: corresponding quantities in the Newtonian and non-Newtonian approximations are denoted by the same letters, the higher approximation being distinguished by a tilde over the letter. Thus, in Eq. (2.1) each σ on the right-hand side (and also each m) is Newtonian (i.e., constant), while the $\tilde{\sigma}$ on the left hand side includes the relativistic corrections.

It is easy to see that the ratio of the terms which are bilinear in σ to the linear correction terms is of the same order of magnitude as the ratio of the proper moment to the orbital moment, σ/M; for all astronomical applications this is an extremely small fraction. Therefore we may neglect the bilinear terms and write (2.1) in the form

$$\tilde{\boldsymbol{\sigma}}_a = \boldsymbol{\sigma}_a + \frac{\gamma}{c^2}\frac{m_b(2m_a + m_b)}{mp}[\mathbf{n} \times \boldsymbol{\sigma}_a]\,\varphi, \quad (2.2)$$

where $\mathbf{n} \equiv \mathbf{M}/|\mathbf{M}|$ is a unit pseudovector normal to the plane of the Newtonian orbit; there is an analogous equation for $\tilde{\boldsymbol{\sigma}}_b$. From this it is evident that $\boldsymbol{\sigma}_a$ changes only when $\boldsymbol{\sigma}_a$ is inclined with respect to $\mathbf{n}$.

Forming the scalar product of (2.2) with $\tilde{\boldsymbol{\sigma}}_a$, and the product of the analogous equation for $\tilde{\boldsymbol{\sigma}}_b$ with $\tilde{\boldsymbol{\sigma}}_b$, we obtain, to the corresponding degree of approximation,

$$\tilde{\boldsymbol{\sigma}}_a^2 = \boldsymbol{\sigma}_a^2 = \text{const}, \quad \tilde{\boldsymbol{\sigma}}_b^2 = \boldsymbol{\sigma}_b^2 = \text{const}, \quad (2.3)$$

i.e., the proper-moment vectors of the bodies remain constant in magnitude. Note that if we take into account the linear terms of (2.1), this condition is violated:

$$\tilde{\boldsymbol{\sigma}}_a^2 = \boldsymbol{\sigma}_a^2 + \frac{3\gamma m_b}{4c^2 p\,|\mathbf{M}|}\sigma_a^1\sigma_a^3(\sigma_a^2 + \sigma_b^2)\,\varphi, \quad (2.4)$$

so that in the general case there will be a certain, though slight secular variation in the absolute magnitude of the proper rotation of the body.

Returning now to the approximation (2.2), it is evident from (2.2) and (2.3) that the moment $\tilde{\boldsymbol{\sigma}}_a$ (together with $\tilde{\boldsymbol{\sigma}}_b$) executes a pure precession about the axis $\mathbf{n}$. If we denote by τ the period of motion of the two bodies in their relative orbit (i.e., the time during which φ changes by 2π), we can obtain from (2.2) the value of the precessional period

$$T_a^{\text{prec}} = \left[\frac{\gamma}{c^2}\frac{m_b(2m_a + m_b)}{mp}\right]^{-1}\tau. \quad (2.5)$$

The quadratic terms in (2.1) would introduce a weak perturbation of this simple precession.

3. ORBITAL ANGULAR MOMENTUM

We now turn to a consideration of the effect of proper rotations of the bodies upon the orbital angular momentum as a constant of motion. Writing $\mathbf{r} \equiv \mathbf{a} - \mathbf{b}$, we can obtain the orbital velocity moment $\mathbf{M} \equiv \mathbf{r} \times \dot{\mathbf{r}}$ from (1.1):

$$\dot{\mathbf{M}} = [\mathbf{r} \times (\mathbf{F}_a - \mathbf{F}_b)] + [\mathbf{r} \times (\mathbf{D}_a - \mathbf{D}_b)]. \quad (3.1)$$

To integrate these equations we substitute into the right hand side (which is similar in form to Eq. (5.6) of Ref. 1) the values of the quantities in the first approximation, (1.3) to (1.5), whereupon the integration can be carried out directly. If we retain only the cyclic terms in the equations thus obtained, we have

$$\tilde{\mathbf{M}} = \mathbf{M} - \frac{\gamma}{c^2 mp}\{(2m_a^2 - m_b^2)[\mathbf{n} \times \boldsymbol{\sigma}_a]$$

$$+ (2m_b^2 - m_a^2)[\mathbf{n} \times \boldsymbol{\sigma}_b]\}\,\varphi$$

$$+ \frac{3\gamma m}{4c^2 p\,|\mathbf{M}|}\{(\mathbf{n} \cdot \boldsymbol{\sigma}_a)[\mathbf{n} \times \boldsymbol{\sigma}_a] + (\mathbf{n} \cdot \boldsymbol{\sigma}_b)[\mathbf{n} \times \boldsymbol{\sigma}_b]$$

$$- 2(\mathbf{n} \cdot \boldsymbol{\sigma}_a)[\mathbf{n} \times \boldsymbol{\sigma}_b] - 2(\mathbf{n} \cdot \boldsymbol{\sigma}_b)[\mathbf{n} \times \boldsymbol{\sigma}_a]\}\,\varphi, \quad (3.2)$$

where all the symbols are the same as in (2.1) and (2.2). Here we have omitted the terms arising from the non-rotational relativistic corrections to the forces $\mathbf{F}_a$ and $\mathbf{F}_b$ in (1.1) or (3.1), since they do not lead to secular terms in $\widetilde{\mathbf{M}}$, in agreement with Robertson.[6] It is not difficult to verify that the total moment $(\widetilde{m}_a\widetilde{m}_b/\widetilde{m})\widetilde{\mathbf{M}}$ would differ from the right hand side of (3.2) only by a constant factor, since the relativistic correction to the masses of the bodies contains no secular terms.

Hence if the bodies have proper rotations, the orbital moment undergoes a secular deviation from its Newtonian value.

From (3.2) it is evident that the secular perturbations of the orbital moment lie in the plane of the Newtonian orbit: $\mathbf{n}\cdot(\widetilde{\mathbf{M}}-\mathbf{M})=0$; and to the present degree of approximation, the absolute value of the moment does not change:

$$\widetilde{\mathbf{M}}^2=\mathbf{M}^2=\gamma mp=\text{const.} \tag{3.3}$$

Hence the perturbation consists of a secular wobble of the orbital moment vector, and consequently a secular deviation of the plane of the relativistic orbit from the Newtonian position. As can be seen from (3.2), the angle of inclination is of the order of

$$\Delta\theta\sim\frac{\gamma m}{c^2p}\frac{|\sigma|}{|\mathbf{M}|}\varphi, \tag{3.4}$$

where $|\sigma|$ is understood to represent the larger of the quantities $|\sigma_a|$ and $|\sigma_b|$. The exact value of $\Delta\theta$, and also the direction of the wobble, depend on the orientations of σ_a and σ_b and on the mass ratio of m_a to m_b, as can be see from (3.2). The more accurate values of these quantities are easy to obtain but extremely complicated. Note that, just as in the case of the proper moments, the orbital moment undergoes no secular perturbations if both the proper moments σ_a and σ_b are normal to the orbital plane.

In conclusion, we shall consider the special case where $m_b \ll m_a$, which has been studied previously.[2] From (3.2) and (2.2) without the bilinear terms, we obtain

$$(\widetilde{\sigma}_a\cdot\widetilde{\mathbf{M}})=(\sigma_a\cdot\mathbf{M})-\frac{\gamma m_a}{c^2p}(\mathbf{n}\cdot[\sigma_a\times\sigma_b])\varphi. \tag{3.5}$$

If we assume, following Lense and Thirring,[2] that $\sigma_b=0$, we obtain $\widetilde{\sigma}_a\cdot\widetilde{\mathbf{M}}=\sigma_a\cdot\mathbf{M}$, which together with (2.3) and (3.3) implies that the angle between $\widetilde{\sigma}_a$ and $\widetilde{\mathbf{M}}$ is constant. Lense and Thirring quote this result, in their system of coordinates, as proof that the angle of inclination is constant. Now, however, we can see that the rotation of the second body (the lighter one) leads to secular variations in the angle of inclination. This effect persists, of course, even in the case of bodies with comparable masses.

4. ROTATION OF THE PERIHELION

Let us now consider the effect of the proper rotations of the bodies on their orbital motion. The complete solution of the problem turns out to be very complicated and clumsy. In this paper we shall limit ourselves to the simple case in which the proper moments of both bodies, σ_a and σ_b, are normal to the plane of the Newtonian orbit and where, in consequence, neither σ_a nor σ_b causes any secular perturbation of the orbital plane.

In this case, a direct integration of Eqs. (3.1), using the Newtonian approximations (1.3) to (1.5) on the right-hand side, leads to $\widetilde{M}_1=\widetilde{M}_2=0$ and

$$\begin{aligned}\widetilde{M}_3=M_3\Big[1-\frac{2\gamma}{mc^2r}(2m_a^2+2m_b^2+3m_am_b)\Big]&\\+\frac{\gamma}{mc^2r}\big[(2m_a^2-m_b^2+3m_am_b)\sigma_a^3&\\+(2m_b^2-m_a^2+3m_am_b)\sigma_b^3\big].&\end{aligned} \tag{4.1}$$

In the same way, for the energy (in the Newtonian sense of $\widetilde{E}=\frac{1}{2}\widetilde{v}^2-\gamma m/\widetilde{r}$) we obtain

$$\begin{aligned}\widetilde{E}=E\Big\{1+\frac{\gamma}{mc^2r}\Big[(6m_a^2+6m_b^2+5m_am_b)&\\-\frac{5a}{r}(2m_a^2+2m_b^2+3m_am_b)+\frac{ap}{r^2}m_am_b\Big]\Big\}&\\+\frac{\gamma}{c^2}\frac{3M_3}{me^2}\big[(2m_a^2-m_b^2-m_am_b)\sigma_a^3+(2m_b^2-m_a^2-m_am_b)\sigma_b^3\big]&\\\times\Big(\frac{7a-10p}{60ap^3}+\frac{1}{2ar^2}-\frac{2}{3r^3}+\frac{p}{4r^4}\Big)+\frac{\gamma}{c^2}\frac{2M_3}{mr^3}m_am_b(\sigma_a^3+\sigma_b^3)&\\+\frac{\gamma}{c^2}\frac{4m}{r^3}\Big(-\sigma_a^3\sigma_b^3+\frac{1}{4}\sigma_a^3\sigma_a^3+\frac{1}{4}\sigma_b^3\sigma_b^3\Big).&\end{aligned} \tag{4.2}$$

By eliminating the time variable from the left-hand sides of these equations, expressed in polar coordinates (i.e., from $\frac{1}{2}(\dot{\widetilde{r}}^2+\widetilde{r}^2\dot{\widetilde{\varphi}}^2)-\gamma m/\widetilde{r}$ and $\widetilde{r}^2\dot{\widetilde{\varphi}}$), we obtain an equation for the trajectories. After carrying out the calculations according to the well-known methods,[6] and considering only the secular correction terms, we arrive at the final result

$$\frac{1}{\widetilde{r}}=\frac{1}{p}\{1+e\cos[(1-\alpha-\alpha^*)\widetilde{\varphi}]\}. \tag{4.3}$$

Here $\alpha\equiv 3\gamma m/c^2p$, as usual, describes the rotation of the perihelion of a relativistic orbit when the proper rotations of the bodies are neglected. The quantity

$$\begin{aligned}\alpha^*\equiv\frac{\gamma}{c^2}\frac{1}{2p}\Big\{\frac{1}{mM_3}\Big[\Big(-\frac{19}{2}m_a^2+\frac{19}{4}m_b^2+\frac{3}{4}m_am_b\Big)\sigma_a^3&\\+\Big(-\frac{19}{2}m_b^2+\frac{19}{4}m_a^2+\frac{3}{4}m_am_b\Big)\sigma_b^3\Big]&\\+\frac{24}{\gamma p}\Big(-\sigma_a^3\sigma_b^3+\frac{1}{4}\sigma_a^3\sigma_a^3+\frac{1}{4}\sigma_b^3\sigma_b^3\Big)\Big\}&\end{aligned} \tag{4.4}$$

leads to an additional rotation of the perihelion caused by the proper rotations. It is easy to show that the order of magnitude of the ratio α^*/α is the same as that of σ/M, the ratio of proper rotational moments, σ_a and σ_b, to the orbital moment. Hence the effect α^* will be observable only in exceptional cases. Nonetheless, it is still of great physical interest (see, for instance, Ginzburg[7]). The magnitude and sign of α^* depend on the magnitudes and signs of σ_a and σ_b, and on the ratio of the masses m_a and m_b. Equation (4.4) is the generalization of the corresponding result of Lense and Thirring[2] to the case of comparable masses and $\sigma_a \neq 0$, $\sigma_b \neq 0$ (but $\sigma_a^1 = \sigma_a^2 = \sigma_b^1 = \sigma_b^2 = 0$).

It can be seen that the right hand side of Eq. (4.2) contains no secular terms, and it can be shown that this remains true in the general case where $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ havy any arbitrary orientation. We therefore find that the energy, defined in the Newtonian sense, is not subject to secular perturbations.

5. MOTION OF THE NEWTONIAN CENTER OF INERTIA

It is also of interest to study the motion of the center of inertia, defined in the Newtonian sense:

$$\tilde{c}^i \equiv \frac{1}{m}(m_a \tilde{a}^i + m_b \tilde{b}^i). \qquad (5.1)$$

With the aid of equation (1.1) we have

$$\ddot{\tilde{c}}^i = \frac{1}{m}(m_a F_a^i + m_b F_b^i) + \frac{1}{m}(m_a D_a^i + m_b D_b^i). \qquad (5.2)$$

Into the right hand side of this equation we substitute the Newtonian approximations (1.3) to (1.5) and transform to the mean values for one Newtonian period τ:

$$\ddot{\tilde{c}}_\tau^i \equiv \frac{1}{\tau}\int_0^\tau \ddot{\tilde{c}}^i dt = \frac{1}{2\pi a\sqrt{ap}}\int_0^{2\pi} \ddot{\tilde{c}}^i r^2 d\varphi. \qquad (5.3)$$

Averaging the groups of terms arising from F_a^i and F_b^i, which do not involve the proper rotations of the bodies, leads to a null result, in agreement with the work of Robinson.[6] Consideration of the second term in (5.2) leads to

$$\ddot{\tilde{c}}_\tau^1 = \frac{\gamma}{c^2}\frac{3m_a m_b e}{m^2 a p^2}\sqrt{\frac{\gamma m}{a}}\left[(2m_b + m_a)\sigma_b^3 - (2m_a + m_b)\sigma_a^3\right],$$

$$\ddot{\tilde{c}}_\tau^2 = 0, \qquad (5.4)$$

$$\ddot{\tilde{c}}_\tau^3 = \frac{\gamma}{c^2}\frac{3m_a m_b e}{m^2 a p^2}\sqrt{\frac{\gamma m}{a}}\left[(2m_b + m_a)\sigma_b^1 - (2m_a + m_b)\sigma_a^1\right].$$

This shows that in the general case the proper rotations of the bodies result in a finite residual mean acceleration of the Newtonian center of inertia. It is not difficult to prove that the substitution of $\tilde{m}_a$, $\tilde{m}_b$, and $\tilde{m}$ for m_a, m_b, and m in (5.1) does not alter this conclusion from (5.4). This acceleration is absent only when the relativistic orbit is circular (so that $e = 0$), or when both moments $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ are parallel to the minor axis of the Newtonian ellipse.

The acceleration (5.4) is extremely small, and its ratio to the Newtonian acceleration is of the order

$$|\ddot{\tilde{\mathbf{c}}}|/|\ddot{\mathbf{r}}| \sim e\frac{\gamma m}{c^2 p}\frac{|\sigma|}{|M|} \sim e\alpha\frac{|\sigma|}{|M|} \sim e\alpha^* \qquad (5.5)$$

[cf. Eq. (4.3)]. The existence of a constant acceleration in (5.4), no matter how small, is rather unexpected. However, it must be borne in mind that the point determined by the condition (5.1) is not really the inertial center of the system.[8,9] In addition, it is quite possible that the constant acceleration (5.4) is actually an artefact of the method of successive approximations which has been used in this paper and in Ref. 1. The question as to what is the true motion of the Newtonian center of inertia over long periods of time remains open.

[1] A. P. Riabushko, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1387 (1957), Soviet Phys. JETP **6**, 1067 (1958).
[2] J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918).
[3] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).
[4] N. M. Petrova, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 989 (1949).
[5] A. Papapetrou, Proc. Phys. Soc. **A64**, 57 (1951).
[6] H. P. Robertson, Ann. Math. **39**, 101 (1938).
[7] V. L. Ginzburg, Usp. Fiz. Nauk **59**, 11 (1956).
[8] V. A. Fock, Dokl. Akad. Nauk SSSR **32**, 28 (1941).
[9] M. F. Shirokov, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 251 (1954).

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