the opening of which a Dewar was inserted bearing the specimen.

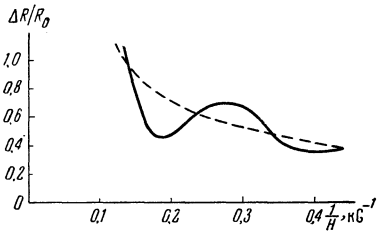
The germanium samples were of different purity and at room temperature (300°K) possessed the following specific resistances: $\rho = 54\,\Omega$ cm (sample No. 1), $\rho = 20\,\Omega$ cm (sample No. 2), and $\rho = 7\,\Omega$ cm (sample No. 3).

In the range of the magnetic field of 25 — 120 kG and at T = 300°K, the dependence of $\Delta R/R_0$ on the field was linear for all three specimens, with a slope of $34 \times 10^{-2}\text{kG}^{-1}$ (No. 1), $1.65 \times 10^{-2}\text{kG}^{-1}$ (sample No. 2) and $1.07 \times 10^{-2}\text{kG}^{-1}$ (sample No. 3). At 77°K, in this same range of fields, the linear dependence of $\Delta R/R_0$ was maintained only for samples 2 and 3; however, the angle of slope of the lines increased in this case. So far as sample No. 1 is concerned, at 77°K, the dependence of $\Delta R/R_0$ on the field, beginning at 25 kG, bore a curvilinear character with a tendency toward saturation; this is seen from the fact that at fields with intensities of 25, 50, 75 and 100 kG, $\Delta R/R_0$ is equal to 2.4; 3.3; 4.6; 5.9, respectively.

The change of the resistance in a magnetic fields $\Delta R/R_0$ of sample No. 1 ($\rho = 54\,\Omega$ cm) was also studied at 20°K in fields up to 110 kG. These measurements gave very interesting results.

When sample No. 1 was at the temperature of liquid hydrogen and the magnetic field was turned on (H = 110 kG), then the resistance of the sample decreased, instead of increasing as it does ordinarily. The magnitude of the decrease in the specific resistance ρ (20°K, H = 0) $-\rho$ (20°K, H = 110) = 670 − 390 = 280 Ω cm. However, the resistance of the sample was reduced in proportion to the decrease in the amplitude of the magnetid field from 110 kG to zero, up to its initial value of ρ (20°K, H = 0).

Moreover, for this sample of n-type germanium ($\rho = 54\,\Omega$ cm), we observed the phenomenon of the oscillation of the electric resistance in the field range from 25 to 110 kG. The period of these oscillations amounted to $0.18\ \text{kG}^{-1}$, while its maximum amplitude was found at a field of H = 55 kG. In the Figure, the dependence of $\Delta R/R_0$ on the reciprocal of the magnetic field intensity is shown for 20°K.



L. Shubnikov and de Haas[1] first discovered the oscillation of the electrical resistance in a transverse static magnetic field, while studying bismuth at low temperatures,[1] and then Frederikse and Hosler,[2] Kanai and Sasaki,[3] and also Busch, Kern and Lüthi,[4] observed this same effect on a specimen of InSb. The latter authors carried out their measurements (as did we) with pulsed magnetic fields.

The theory of the oscillations of galvanometric effects is set forth in the researches of G. E. Zil'berman.[5]

So far as we know, the Shubnikov-de Haas effect has never been observed for germanium up to the present time.

Information on the experimental details and the arrangements for strong magnetic fields will be published in a subsequent paper.

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Translated by R. T. Beyer
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ON A FUNCTIONAL RELATION IN QUANTUM MECHANICS

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WE consider a system of non-interacting particles in a stationary state in some external field with potential $V(\mathbf{r})$.* The density of the number of particles in such a distribution will, generally speaking, be a very complicated function of V and all its derivatives

$$\rho(\mathbf{r}) = \rho(V, \nabla_i V, \nabla_i \nabla_k V \ldots), \tag{1}$$

where all arguments on the right hand side are taken at the point $\mathbf{r}$.

We shall show below that, independent of the character of the occupation of the levels of the field V, i.e., independent of the statistics and the temperature,† the quantity (1) satisfies the relation

$$D\rho / DV = \partial\rho / \partial V, \tag{2}$$

where the symbol D/DV denotes the Euler derivative,

$$D\rho / DV \equiv \sum_{n=0}^{\infty} (-1)^n \nabla_{i_1} \ldots \nabla_{i_n} \{\partial\rho / \partial(\nabla_{i_1} \ldots \nabla_{i_n} V)\},$$

and $\partial/\partial V$ the derivative with respect to the first argument in (1).

To prove this, we consider first the case of a degenerate Fermi system. The Hamiltonian of a particle has the form, $\hat{H} = \hat{T} - V$, where $\hat{T}$ is the kinetic-energy operator. The density of the number of particles and the kinetic-energy density can be put in operator form,[1,2,3] and the appearance of the gradients of V in (1), is due to the commutators of the operators $\hat{T}$ and V:

$$\rho = \langle \theta(\hat{H} - E_0)\rangle, \quad \mathcal{E} = \langle \theta(\hat{H} - E_0)\hat{T}\rangle, \tag{3}$$

where $\theta(x) = \frac{1}{2}(1 - x/|x|)$, E_0 is the upper bound of the occupied spectrum, and

$$\langle \hat{a} \rangle \equiv (2\pi\hbar)^{-3} \, \mathrm{Sp} \int d\mathbf{p} \exp(-i\mathbf{pr}/\hbar)\, \hat{a} \exp(i\mathbf{pr}/\hbar).$$

The trace is taken here over the spins and similar variables, and $\hat{a}$ is an arbitrary operator.

Using Eq. (3) and the equation $\langle \hat{a}\delta(\hat{a})\rangle = 0$ we have

$$\partial[\mathcal{E} - \rho(V + E_0)] / \partial E_0 = \partial[\mathcal{E} - \rho(V + E_0)] / \partial V = -\rho. \tag{4}$$

On the other hand, if we transform the equation of the variational principle[4]

$$\delta \int \mathcal{E}\, d\mathbf{r} - \int (V + E_0)\, \delta\rho\, d\mathbf{r} = 0$$

to the functional argument V and take it into account that $\delta \int F d\mathbf{r} = \int (DF/DV)\delta V d\mathbf{r}$ for any arbitrary function $F(V)$, we get

$$D[\mathcal{E} - \rho(V + E_0)] / DV = -\rho. \tag{5}$$

If we compare (4) and (5) and take into account the fact that $\partial/\partial V$ and D/DV commute, we are led to the relation (2) which we are trying to prove.

The transition to the case of an arbitrary occupation of the levels, which corresponds to replacing the function θ in Eq. (3) by some other function f, can be realized without difficulty by means of the identity

$$f(\hat{a}) = -\int_{-\infty}^{\infty} f'(\lambda)\, \theta(\hat{a} - \lambda)\, d\lambda.$$

By multiplying both sides of relation (2) by $f'(E_0)$ and integrating over E_0, we can satisfy ourselves that this relation is universally valid.

We shall give as an example the form of the dependence (1), which follows from the Thomas-Fermi model with quantum correction up to the fourth order in $\hbar$ in the degenerate, non-relativistic case.[1,2]‡

$$\begin{aligned}\rho = {} & \frac{1}{3\pi^2\hbar^3}(2MV)^{3/2} - \frac{M^2}{24\pi^2\hbar\,(2MV)^{1/2}}[(\nabla V)^2 - 4V\Delta V] \\ & + \frac{M^4\hbar}{1920\pi^2(2MV)^{7/2}}\{-64V^3\Delta^2 V + 80V^2(\Delta V)^2 + 192V^2\nabla V\cdot\nabla\Delta V \\ & + 64(\nabla_i\nabla_k V)^2 V^2 - 200V\Delta V(\nabla V)^2 \\ & - 240V\nabla_i V\nabla_k V\nabla_i\nabla_k V + 175(\nabla V)^4\}.\end{aligned} \tag{6}$$

The corresponding expression for the case of a finite temperature is of the form (up to terms of the second order in $\hbar$)[2,3]

$$\rho = \frac{\sqrt{2}M^{3/2}}{\pi^2\hbar^3}\Big\{\theta^{3/2} I_{1/2}(x) + \frac{\hbar^2}{24M}[\theta^{-3/2} I'''_{1/2}(x)(\nabla V)^2 + 2\theta^{-1/2} I''_{1/2}(x)\Delta V]\Big\}; \tag{7}$$

$$I_{1/2}(x) = \int_0^{\infty} \frac{y^{1/2}\,dy}{\exp(y - x) + 1}, \quad x = \theta^{-1}(V + \mu), \quad \theta = kT,$$

where μ is the chemical potential.

One can easily verify that Eqs. (6) and (7) satisfy, indeed, relation (2). It is important to note that the latter connects only terms of the same order in $\hbar$. In particular, terms of the zeroth order in $\hbar$ satisfy it in a trivial manner.

The relation we have obtained is convenient to use to derive and check the correctness of approximate expressions for the density. The problem whether one can generalize this equation to the case where the interaction V depends on the momentum or on the Dirac matrices, and also whether there exists a similar relation for other quantities (for instance, for the Green function of a particle in an external field) requires a special investigation.

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*The relation which we obtain below is completely valid also in the case where the interaction between the particles is considered by the Hartree method; we must in that case take for V the sum of the external and the self-consistent field. However, if we take exchange into account, Eq. (2) is violated.

†We restrict ourselves to the case where the number of occupied levels depends on only one energy.

‡Equation (24) of Ref. 1 for the fourth-order corrections contains a mistake, which is corrected in Eq. (6). Corresponding equations for the second-order corrections were also obtained by Kompaneets and Pavlovskii[5] and Golden.[3]

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REMARKS ON A NOTE BY F. S. LOS' "PHASE OF A SCATTERED WAVE"[1]

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Los' has considered the equation

$$\frac{d^2 G}{d\rho^2} + \left[1 - \frac{l(l+1)}{\rho^2} - U(\rho)\right] G = 0 \qquad (1)$$

and asserts that "it is necessary to find a solution of Eq. (1) which assumes the following form as $\rho \to 0$

$$G = A_0 \rho^{l+1} \qquad (2)$$

and the following asymptotic form at large values of ρ

$$G = \text{const} \cdot \sin(\rho - \pi l/2 + \delta_l); \quad \delta_l = \text{const} \ldots » \qquad (3)$$

It should be noted that under the condition $\int^{\infty} U(\rho)\,d\rho < C$, given in the author's note, when $\rho \to \infty$, $G(\rho)$ cannot assume the asymptotic form given in (3). The case $\int^{\infty} |U(\rho)|\,d\rho < C$ is considered in Courant and Hilbert[2] where the same system of equations is obtained, using the method employed by Los'. Equations (6) and (7) given by Los' correspond to Eq. (56) and (57) in Courant and Hilbert.

In Ref. 2, a proof is given for the asymptotic formula (3) which is more detailed than that given in the note by Los'; in this proof it is not shown that $A(\rho) \to$ const. when $\rho \to \infty$ in Eq. (4).

For $\rho \to 0$, Los' has obtained expressions for $\delta(\rho)$ and $A(\rho)$ to which the following remarks apply:

1. It is not meaningful to express $\delta(\rho)$ in terms of $\int_0^{\rho} \rho\gamma(\rho)\,d\rho$ since in the approximation which is used only the first terms of the series $\gamma(\rho) = \gamma_0 + \gamma_1\rho + \ldots$ should be retained
2. The constant coefficients in (2) and in the expression for $A(\rho)$ are denoted by the same symbol — A_0. These coefficients differ by a factor of $l + 1$.
3. Even if these errors are ignored, it should be noted that the title of the note does not reflect its contents.

The use of the supplementary condition (5) is valid for $\rho \to \infty$, when $A(\rho) \to$ const and $\delta(\rho) \to$ const. However, when $\rho \to 0$, in place of (5) any other supplementary condition can be used. The different supplementary conditions correspond to different phases $\delta(\rho)$. The problem becomes indeterminant. By the definition of the phase of a scattered wave it follows that the phase $\delta(\rho)$ found by means of such an additional condition is not the phase of the scattered wave.

Thus, in the form in which it has been published the note given by F. S. Los' is not useful for an analysis of scattering and can only introduce confusion on the part of the reader.

[1] F. S. Los', J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 273 (1957).
[2] Courant and Hilbert, Methoden der Mathematischen Physik, (Russ. Transl.) 1933, Vol. I, p. 314.

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SECOND MOMENT OF THE PARAMAGNETIC ABSORPTION CURVE WHEN THE SPIN MAGNETISM IS NOT PURE

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In Refs. 1 — 5, equations were given for the second moment $\langle \nu^2 \rangle$ of the curve $f(\nu)$ of paramagnetic absorption in the absence of a static magnetic H_0, and for the second moment $\langle (\Delta\nu)^2 \rangle$ of the