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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 4 OCTOBER, 1958

ON THE RELATIONS BETWEEN THE CROSS SECTIONS FOR MULTIPLE PRODUCTION OF PIONS

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Submitted to JETP editor November 8, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 916-921 (April, 1958)

Isotopic relations have been obtained between the cross sections of transformation of a pion and a γ-quantum into three pions during collisions between the pion or γ-quantum and nucleons or deuterons. It is shown that near the threshold of the reaction, and also for production of pions with identical momenta, some auxiliary relations hold.

1. Relations have been obtained previously by the author between the cross sections of transformation of a single pion into two during collisions of pions with nuclei and deuterons.[1] Close to the reaction threshold, and also in the production of two pions with identical momenta, some auxiliary relations hold. In the present note it is shown that some auxiliary relations also hold between the cross sections of the various reactions of production of three pions during collisions of pions with nucleons and deuterons close to the threshold, or in the formation of pions with identical momenta.

Let us consider the collision of pions with nucleons (n) as the result of which the pion is transformed into three pions, i.e., the reaction

$$\pi + n \rightarrow n' + \pi' + \pi'' + \pi'''. \tag{1}$$

The initial state is the superposition of states with isotopic spin T equal to $\frac{1}{2}$ and $\frac{3}{2}$. The wave function of the system of three pions is a superposition of states with isotopic spin t, equal to 0, 1, 2, 3. The wave function of the final state with total isotopic spin $T = \frac{1}{2}$ and $\frac{3}{2}$ can be constructed from the wave function of the nucleon (isotopic spin $\frac{1}{2}$) and from the wave function of the system of three pions with isotopic spin $t = 0$, 1, 2. The state with isotopic spin $t = 3$ is forbidden.

We denote by A_t^T, B_t^T, f_t^T, g_t^T and h_t^T the amplitudes of transitions into states with total isotopic spin T and isotopic spin of the system of three pions t and with a definite (see below) type of symmetry of the coordinate wave function of the pions.

Since the pions obey Bose statistics, then the type of symmetry of the coordinate part of the wave function and isotopic spin for the system of pions is determined by one and the same Young scheme.[2] The symmetry of the amplitude A_t^T is determined by a Young scheme consisting entirely of one row. The symmetric state of the system of three pions relative to an arbitrary pair of permutations of the charge variables is a state with total isotopic spin t equal to unity. Therefore, we have in all two different amplitudes, $A_1^{1/2}$ and $A_1^{3/2}$.

The symmetry of the amplitude B_t^T is determined by a Young scheme consisting of a single column. Since the antisymmetric state of the system of three pions relative to an arbitrary pair of permutations of the charge variables is the state with $t = 0$, then we have just the single amplitude $B_0^{1/2}$

The type of symmetry of the amplitudes f_t^T, g_t^T and h_t^T is determined by a Young scheme consisting of two rows. In his case t can take on the values 1 and 2.

For the differential cross sections σ of the different processes, we have

$$\sigma_1(\pi^+ + p \to p + \pi^+ + \pi^0 + \pi^0) = \left| A_1^{3/2} + f_1^{3/2} + \frac{1}{\sqrt{5}} f_2^{3/2} \right|^2,$$

$$\sigma_2(\pi^+ + p \to p + \pi^+ + \pi^+ + \pi^-) = \left| 2A_1^{3/2} - f_1^{3/2} + \frac{1}{\sqrt{5}} f_2^{3/2} \right|^2,$$

$$\sigma_3(\pi^+ + p \to n + \pi^+ + \pi^+ + \pi^0) = \left| \sqrt{\frac{8}{5}} f_2^{3/2} \right|^2,$$

$$\sigma_4(\pi^- + p \to p + \pi^- + \pi^0 + \pi^0) = \left| -\frac{1}{3}(2A_1^{1/2} + A_1^{3/2}) - \frac{1}{3}(2f_1^{1/2} + f_1^{3/2}) + \frac{1}{\sqrt{5}} f_2^{3/2} \right|^2,$$

$$\sigma_5(\pi^- + p \to p + \pi^- + \pi^+ + \pi^-) = \left| \frac{2}{3}(2A_1^{1/2} + A_1^{3/2}) - \frac{1}{3}(2f_1^{1/2} + f_1^{3/2}) - \frac{1}{\sqrt{5}} f_2^{3/2} \right|^2,$$

$$\sigma_6(\pi^- + p \to n + \pi^0 + \pi^0 + \pi^0) = \left| \sqrt{2}(A_1^{1/2} - A_1^{3/2}) \right|^2,$$

$$\sigma_7(\pi^- + p \to n + \pi^0 + \pi^+ + \pi^-) = \left| \frac{\sqrt{2}}{3}(-A_1^{1/2} + A_1^{3/2}) - \frac{\sqrt{2}}{3} B_0^{1/2} - \frac{\sqrt{2}}{3}(f_1^{1/2} - f_1^{3/2}) - \frac{1}{3}\sqrt{\frac{2}{5}}(f_2^{3/2} + 2g_2^{3/2}) \right|^2,$$

$$\sigma_8(\pi^0 + p \to p + \pi^0 + \pi^0 + \pi^0) = \left| A_1^{1/2} + 2A_1^{3/2} \right|^2,$$

$$\sigma_9(\pi^0 + p \to p + \pi^0 + \pi^+ + \pi^-) = \left| \frac{1}{3}(A_1^{1/2} + 2A_1^{3/2}) - \frac{1}{\sqrt{3}} B_0^{1/2} + \frac{1}{3}(f_1^{1/2} + 2f_1^{3/2}) - \frac{1}{3}\sqrt{\frac{2}{5}}(f_2^{3/2} + 2h_2^{3/2}) \right|^2, \qquad (2)$$

$$\sigma_{10}(\pi^0 + p \to n + \pi^+ + \pi^0 + \pi^0) = \left| \frac{\sqrt{2}}{3}(A_1^{1/2} - A_1^{3/2}) + \frac{\sqrt{2}}{3}(f_1^{1/2} - f_1^{3/2}) + \sqrt{\frac{2}{5}} f_2^{3/2} \right|^2,$$

$$\sigma_{11}(\pi^0 + p \to n + \pi^+ + \pi^+ + \pi^-) = \left| \frac{2\sqrt{2}}{3}(-A_1^{1/2} + A_1^{3/2}) + \frac{\sqrt{2}}{3}(f_1^{1/2} - f_1^{3/2}) - \sqrt{\frac{2}{5}} f_2^{3/2} \right|^2.$$

From these we obtain two relations between the cross sections σ^t which are not difficult to obtain by the method laid down in Refs. 3, 4:

$$\sigma_1^t + \sigma_2^t + \sigma_3^t + \sigma_4^t + \sigma_5^t + \sigma_6^t + \sigma_7^t = 2(\sigma_8^t + \sigma_9^t + \sigma_{10}^t + \sigma_{11}^t); \qquad (3)$$

$$\sigma_2^t + \sigma_5^t + \sigma_{11}^t = \sigma_1^t + \sigma_4^t + 2(\sigma_6^t + \sigma_8^t) + \sigma_{10}^t. \qquad (4)$$

Close to the reaction threshold of (1), the pions are formed in an S state. Since the coordinate part of the wave function of the system of three pions in the final state must be symmetric relative to permutations of the pions, then, close to the reaction threshold, only the amplitudes A_t^T are different from zero.

Substituting $B_t^T = f_t^T = g_t^T = h_t^T = 0$ in (2), we obtain the following relations between the differential cross sections close to the reaction threshold:

$$4\sigma_1 = \sigma_2, \quad 4\sigma_4 = \sigma_5, \quad 9\sigma_9 = \sigma_8,$$
$$\sigma_7 = \frac{1}{9}\sigma_6 = \frac{1}{4}\sigma_{11} = \sigma_{10}, \quad \sigma_1 + \sigma_4 = 2\sigma_9 + \sigma_7, \qquad (5)$$

while for the total cross sections,

$$4\sigma_1^t = \sigma_2^t, \quad 4\sigma_4^t = \sigma_5^t, \quad 3\sigma_9^t = 2\sigma_8^t,$$
$$3\sigma_7^t = 2\sigma_6^t = \frac{3}{2}\sigma_{11}^t = 6\sigma_{10}^t, \quad 2\sigma_1^t + 2\sigma_4^t = 2\sigma_9^t + \sigma_7^t. \qquad (6)$$

As is seen from (2), the $\pi^+ + p \to n + \pi^+ + \pi^+ + \pi^0$ is forbidden close to the threshold.

The same relations hold away from the reaction threshold if the three pions in reaction (1) are formed with identical momenta $(\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3)$, since in this case also the wave function is symmetric relative to the coordinates of the pions.

Now let us consider the transformation of a pion into three pions during collisions of pions with deuterons, i.e., the reaction

$$\pi + d \to n' + n'' + \pi' + \pi'' + \pi'''. \qquad (7)$$

The isotopic spin of the entire system T in the initial and final states of the system is unity. We denote by $A_{t_1t_2}$, $B_{t_1t_2}$, $f_{t_1t_2}$, $g_{t_1t_2}$, $h_{t_1t_2}$ the amplitudes of the transition to a state with isotopic spins of the system of the two nucleons t_1 and the system of three pions t_2, and with a definite type of symmetry of the coordinate part of the wave function of the pions.

For the differential cross sections σ of the different processes, we have

$$\sigma_1(\pi^+ + d \to p + p + \pi^0 + \pi^0 + \pi^0) = \left| \frac{3}{\sqrt{2}} A_{11} \right|^2,$$

$$\sigma_2(\pi^+ + d \to p + p + \pi^0 + \pi^+ + \pi^-) = \left| \frac{1}{\sqrt{2}} A_{11} + \frac{1}{\sqrt{2}} f_{11} - B_{10} + \frac{1}{\sqrt{30}}(f_{12} + 2g_{12}) \right|^2,$$

$$\sigma_3(\pi^+ + d \to p + n + \pi^+ + \pi^0 + \pi^0) = \left| \frac{1}{2} A_{11} + \frac{1}{2} f_{11} - \frac{1}{\sqrt{2}} A_{01} - \frac{1}{\sqrt{2}} f_{01} + \sqrt{\frac{3}{20}} f_{12} \right|^2,$$

$$\sigma_4(\pi^+ + d \to p + n + \pi^+ + \pi^+ + \pi^-) = \left| A_{11} - \sqrt{2} A_{01} - \frac{1}{2} f_{11} + \frac{1}{\sqrt{2}} f_{01} + \sqrt{\frac{3}{20}} f_{12} \right|^2,$$

$$\sigma_5(\pi^+ + d \to n + n + \pi^+ + \pi^+ + \pi^0) = \left| \sqrt{\frac{6}{5}} f_{12} \right|^2,$$

$$\sigma_6(\pi^0 + d \to p + p + \pi^- + \pi^0 + \pi^0) = \left| \frac{1}{\sqrt{2}} A_{11} + \frac{1}{\sqrt{2}} f_{11} - \sqrt{\frac{3}{10}} f_{12} \right|^2, \qquad (8)$$

$$\sigma_7(\pi^0 + d \to p + p + \pi^- + \pi^+ + \pi^-) = \left| \sqrt{2} A_{11} - \frac{1}{\sqrt{2}} f_{11} - \sqrt{\frac{3}{10}} f_{12} \right|^2,$$

$$\sigma_8(\pi^0 + d \to p + n + \pi^0 + \pi^0 + \pi^0) = \left| \frac{3}{\sqrt{2}} A_{01} \right|^2,$$

$$\sigma_9(\pi^0 + d \to p + n + \pi^0 + \pi^+ + \pi^-) = \left| \frac{1}{\sqrt{2}} A_{01} + \frac{1}{\sqrt{2}} f_{01} + \frac{1}{\sqrt{2}} B_{10} - \frac{1}{\sqrt{15}} (f_{12} + 2h_{12}) \right|^2.$$

We then obtain two relations between the total cross sections which are also easily obtained with the method outlined in Refs. 3, 4:

$$\sigma_1^t + \sigma_2^t + \sigma_3^t + \sigma_4^t + \sigma_5^t = 2(\sigma_6^t + \sigma_7^t) + \sigma_8^t + \sigma_9^t; \tag{9}$$

$$\sigma_4^t + \sigma_7^t = 2\sigma_1^t + \sigma_3^t + \sigma_6^t + \sigma_8^t. \tag{10}$$

Close to the threshold of the reaction (7), and also in the formation in this reaction of pions with identical momenta, we have

$$B_{t_1t_2} = f_{t_1t_2} = g_{t_1t_2} = h_{t_1t_2} = 0.$$

In these cases the following relations are valid between the differential cross sections:

$$4\sigma_3 = \sigma_4, \quad 9\sigma_9 = \sigma_8, \quad \frac{1}{9}\sigma_1 = \sigma_2 = \sigma_6 = \frac{1}{4}\sigma_7, \tag{11}$$

and between the total cross sections:

$$4\sigma_3^t = \sigma_4^t, \quad \frac{1}{2}\sigma_9^t = \frac{1}{3}\sigma_8^t, \quad 2\sigma_3^t = \sigma_2^t + \sigma_9^t, \quad \frac{1}{3}\sigma_1^t = \frac{1}{2}\sigma_2^t = \sigma_6^t = \frac{1}{2}\sigma_7^t. \tag{12}$$

The reaction $\pi^+ + d \to n + n + \pi^+ + \pi^+ + \pi^0$ is forbidden in the cases under consideration.

For processes of transformation of a single pion into three, for which splitting of the deuteron does not occur, we have

$$\sigma_1(\pi^+ + d \to d + \pi^+ + \pi^0 + \pi^0) = \sigma_2(\pi^0 + d \to d + \pi^0 + \pi^+ + \pi^-) = |A_{01} + f_{01}|^2,$$

$$\sigma_3(\pi^+ + d \to d + \pi^+ + \pi^+ + \pi^-) = |2A_{01} + g_{01}|^2,$$

$$\sigma_4(\pi^0 + d \to d + \pi^0 + \pi^0 + \pi^0) = |3A_{01}|^2. \tag{13}$$

Close to the threshold of the reaction under consideration, and also for $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3$, we obtain the following relations between the differential cross sections:

$$\sigma_1 = \sigma_2 = \frac{1}{4}\sigma_3 = \frac{1}{9}\sigma_4, \tag{14}$$

and between the total cross sections:

$$\sigma_1^t = \frac{1}{2}\sigma_2^t = \frac{1}{4}\sigma_3^t = \frac{1}{3}\sigma_4^t. \tag{15}$$

2. We consider the reaction of photoproduction of three pions in hydrogen and deuterium. The operator of interaction of the electromagnetic field with the meson-nucleon system consists of two parts $H = S + V_3$,[5] where S is a scalar and V_3 the three components of a vector in isotopic space.

The isotopic spin of the initial state in the reaction $\gamma + p \to n' + \pi' + \pi'' + \pi'''$ is $T = 1/2$, and in the final state, $T = 1/2$ and $3/2$.

The operator S gives the transitions to a state of the meson-nucleon system with isotopic spin $T = 1/2$, the operator V_3, those to the state with $T = 1/2, 3/2$.

We denote by $A^T_{tS(V)}$, $B^T_{tS(V)}$, $f^T_{tS(V)}$, $g^T_{tS(V)}$ and $h^T_{tS(V)}$ the amplitudes of transition (the operator S or V_3 is acting) to a state with total isotopic spin T and isotopic spin of the system of three pions t and with a definite type of symmetry of the coordinate wave function of the pions.

We have for the differential cross sections σ of the different processes:

$$\sigma_1(\gamma + p \to p + \pi^0 + \pi^0 + \pi^0) = 9\left| \frac{1}{\sqrt{3}} A^{1/2}_{1S} - \frac{1}{3} A^{1/2}_{1V} + \frac{\sqrt{2}}{3} A^{3/2}_{1V} \right|^2,$$

$$\sigma_2(\gamma + p \to p + \pi^0 + \pi^+ + \pi^-) = \frac{1}{9}\left| -\sqrt{3}(A^{1/2}_{1S} + f^{1/2}_{1S}) + (A^{1/2}_{1V} + f^{1/2}_{1V}) - \sqrt{2}(A^{3/2}_{1V} + f^{3/2}_{1V}) - 3B^{1/2}_{0S} + \sqrt{3}B^{1/2}_{0V} + (g^{3/2}_{2V} - h^{3/2}_{2V}) \right|^2, \tag{16}$$

$$\sigma_3(\gamma + p \to n + \pi^+ + \pi^0 + \pi^0) = \frac{1}{9}\left| \sqrt{6}(A^{1/2}_{1S} + f^{1/2}_{1S}) - \sqrt{2}(A^{1/2}_{1V} + f^{1/2}_{1V}) - (A^{3/2}_{1V} + f^{3/2}_{1V}) + \sqrt{3}f^{3/2}_{2V} \right|^2,$$

$$\sigma_4(\gamma + p \to n + \pi^+ + \pi^+ + \pi^-) = \frac{1}{9}\left| \sqrt{6}(2A^{1/2}_{1S} - f^{1/2}_{1S}) - \sqrt{2}(2A^{1/2}_{1V} - f^{1/2}_{1V}) - (2A^{3/2}_{1V} - f^{3/2}_{1V}) + \sqrt{3}f^{3/2}_{2V} \right|^2.$$

We then obtain the following inequality between the total cross sections σ^t:

$$3\sigma_2^t \geqslant 2\sigma_1^t. \tag{17}$$

Substituting in (16)

$$B^T_{tS(V)} = f^T_{tS(V)} = g^T_{tS(V)} = h^T_{tS(V)} = 0,$$

we get the following relation between the differential cross sections σ close to the threshold of the reaction:

$$9\sigma_2 = \sigma_1, \quad 4\sigma_3 = \sigma_4, \tag{18}$$

and between the total cross sections:

$$3\sigma_2^t = 2\sigma_1^t, \quad 4\sigma_3^t = \sigma_4^t. \tag{19}$$

We now determine the cross section of photoproduction of three pions in deuterium. The isotopic spin of the initial state in the reaction $\gamma + d \rightarrow n_1 + n_2 + \pi' + \pi'' + \pi'''$ is equal to zero. The isotopic spin of the final state $T = 0$ or 1. The corresponding operators of six transition are S and V_3.

We denote by $A^{t_1}_{tS(V)}$, $B^{t_1}_{tS(V)}$, $f^{t_1}_{tS(V)}$, $g^{t_1}_{tS(V)}$ and $h^{t_1}_{tS(V)}$ the amplitudes of transition to a state with isotopic spin of the system of two nucleons t_1 and of the system of three pions t, and with a definite type of symmetry of the coordinate wave function of the pions.

For the differential cross sections σ of the various processes, we have:

$$\sigma_1 (\gamma + d \rightarrow p + n + \pi^0 + \pi^0 + \pi^0) = \frac{3}{2} \left| A^1_{1S} - A^0_{1V} \right|^2,$$

$$\sigma_2 (\gamma + d \rightarrow p + n + \pi^0 + \pi^+ + \pi^-) = \frac{1}{6} \Big| - (A^1_{1S} + f^1_{1S}) + (A^1_{1V} + f^1_{1V}) + B^1_{0V} + \sqrt{3} B^0_{0S} + 2 (g^1_{2V} - h^1_{2V}) \Big|^2,$$

$$\sigma_3 (\gamma + d \rightarrow n + n + \pi^+ + \pi^0 + \pi^0) = \Big| \frac{1}{\sqrt{3}} (A^1_{1S} + f^1_{1S}) + \frac{1}{3} (A^1_{1V} + f^1_{1V}) - f^1_{2V} \Big|^2, \quad (20)$$

$$\sigma_4 (\gamma + d \rightarrow n + n + \pi^+ + \pi^+ + \pi^-) = \Big| \frac{1}{\sqrt{3}} (2A^1_{1S} - f^1_{1S}) + \frac{1}{3} (2A^1_{1V} - f^1_{1V}) - f^1_{2V} \Big|^2.$$

We then obtain the following inequality:

$$3\sigma^t_2 \geqslant 2\sigma^t_1. \quad (21)$$

Close to the reaction threshold under consideration,

$$B^{t_1}_{tS(V)} = f^{t_1}_{tS(V)} = g^{t_1}_{tS(V)} = h^{t_1}_{tS(V)} = 0.$$

Therefore, close to the threshold of the reaction, the following relations hold between the differential cross sections σ and between the total cross sections σ^t:

$$9\sigma_2 = \sigma_1, \quad 4\sigma_3 = \sigma_4, \quad (22)$$

$$3\sigma^t_2 = 2\sigma^t_1, \quad 4\sigma^t_3 = \sigma^t_4. \quad (23)$$

For reactions which take place without splitting of the deuteron, $t = 0$ or 1; therefore, $\sigma(\gamma + d \rightarrow d + \pi^0 + \pi^0 + \pi^0) = \frac{3}{2} |A^0_{1V}|^2$,

$$\sigma (\gamma + d \rightarrow d + \pi^+ + \pi^- + \pi^0) = \left| \frac{1}{\sqrt{6}} (A^0_{1V} + f^0_{1V}) + \frac{1}{\sqrt{2}} B^0_{0S} \right|^2. \quad (24)$$

It then follows that

$$3\sigma^t (\gamma + d \rightarrow d + \pi^+ + \pi^- + \pi^0) \geqslant 2\sigma^t (\gamma + d \rightarrow d + \pi^0 + \pi^0 + \pi^0). \quad (25)$$

Close to the threshold of the reaction under consideration, $B^0_{0S} = f^0_{1V} = 0$. therefore, the following equations hold between the differential cross sections σ and between the total cross sections σ^t:

$$9\sigma (\gamma + d \rightarrow d + \pi^+ + \pi^- + \pi^0) = \sigma (\gamma + d \rightarrow d + \pi^0 + \pi^0 + \pi^0); \quad (26)$$

$$3\sigma^t (\gamma + d \rightarrow d + \pi^+ + \pi^- + \pi^0) = 2\sigma^t (\gamma + d \rightarrow d + \pi^0 + \pi^0 + \pi^0). \quad (27)$$

The relations obtained between the differential cross sections (18), (22) and (26) also hold away from the threshold of the reaction if the three pions in the reactions are formed with identical momenta $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3$.

I thank Prof. I. M. Shmushkevich for discussion of the results and advice on completing the present research.

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Translated by R. T. Beyer
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