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THEORY OF GALVANOMAGNETIC AND THERMOMAGNETIC EFFECTS IN METALLIC FILMS

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The electron distribution function, the electric conductivity, the thermal conductivity, and the Thomson coefficients have been determined for a metallic film located in a constant magnetic field directed at an arbitrary angle with respect to the surface of the film. No special assumptions are made regarding the electron dispersion law. The region of strong magnetic fields is studied in detail. Comparison of theory with experiment shows excellent agreement.

1. INTRODUCTION

THE study of the effect of shape and dimensions of a sample on the electric conductivity, thermal conductivity, and other kinetic coefficients of metals permits us to obtain data on the magnitude of the mean free path, the character of the electron energy spectrum, etc.

In Refs. 1 — 6, the conductivity of thin metallic films and wires was calculated for the absence of a magnetic field. Englman and Sondheimer[3] showed that the conductivity of an anisotropic monocrystalline film, whose thickness was much less than the mean free path length, depends not only on the angles between the current direction and the crystallographic axes, but also on the orientation of the latter with respect to the surface of the film. In Refs. 4 and 5, the conductivity of wires of circular and rectangular cross section was computed.

The conductivity of films and wires in a magnetic field was determined in Refs. 5 — 8 for various orientations of a constant magnetic field (parallel to the plane of the film and to the axis of the wire) relative to the current.

In the work of MacDonald and Sarginson,[5] the distribution function and the conductivity of the film in crossed electric and magnetic fields were found improperly (which is also noted in Ref. 7). Azbel'[7] determined the conductivity of a film in a longitudinal magnetic field, while Sondheimer[7a] has shown that the resistance of a thin film in a perpendicular magnetic field oscillates on change in the field. In the researches of Chambers[7] and Königsberg[8] on the determination of the conductivity of films and wires in a magnetic field, simple kinetic considerations were employed. These were based on a study of the trajectory of an isolated electron which enabled one to avoid the cumbersome solution of the kinetic equation (see Ref. 7).

In all these researches, with the exception of the work of Kaganov and Azbel',[2] the calculations were carried out under the assumption that the electrons in the metal were free, that is, a square law of dispersion was assumed (sometimes with anisotropic effective masses). Moreover, the anisotropy of the time of free flight was never taken into account. Also, the kinetic coefficients in thin films depend on the form of the dispersion law and the anisotropy of the collision integral. This circumstance can, in principle, be used for the study of the form of the limiting Fermi surface and the character of the electron-lattice interaction.

Insofar as the thermomagnetic effects in films are concerned, they have remained virtually unstudied.

In connection with experimental researches (for example, Refs. 5, 10), it must be noted that in almost all of them, the samples were polycrystalline. So far as we know, only in the research of Borovik and Lazarev,[9] where the effect of the shape of the specimen on the electric conductivity of bismuth was studied, were the specimens monocrystalline. In that case a plate of thickness $\ell \sim 10^{-2}$ cm was used. The plate was connected with a massive single crystal; the orientation of the crystallographic axes in the film and in the large sample were the same. The measurements were carried out simultaneously on the film and on the large monocrystal in order to eliminate the effect of the natural anisotropy of Bi and to isolate the effect of shape in pure form. The study of the rotational diagram (the dependence of the resistance of the film on the angle between the magnetic field and the surface of the film) allowed one to determine the order of magnitude of the free path length. The results of these experiments are completely explained by the theory developed below.

The purpose of the present paper was the calculation of the kinetic coefficients of the metallic film (located in a constant magnetic field) for an arbitrary electron dispersion law. The possibility of the introduction of a relaxation time $t_0(\mathbf{p})$ was assumed, where $\mathbf{p}$ is the quasi-momentum of the electron.

2. STATEMENT OF THE PROBLEM AND SOLUTION OF THE KINETIC EQUATION

Let us first consider the electric conductivity of a layer of metal of thickness d ($0 \leq z \leq d$, the z axis being parallel to the inwardly drawn normal to the surface $z = 0$), located in a constant magnetic field **H**. We choose the x axis along the projection of **H** on the plane of the film.

We shall begin with the linearized kinetic equation[11] for the contribution of f to the equilibrium Fermi distribution function

$$v_z \partial f / \partial z + \Omega \partial f / \partial \tau + f / t_0(\mathbf{p}) = e\mathbf{E}\mathbf{v}\, \partial f_0 / \partial \varepsilon, \quad f_0\left(\frac{\varepsilon - \zeta}{T}\right) = \left[\exp\left(\frac{\varepsilon - \zeta}{T}\right) + 1\right]^{-1} \tag{2.1}$$

Here, we choose as variables the coordinate z, the energy of the electrons ϵ, the conserved component of the quasi-momentum $p_H = \mathbf{p}\mathbf{H}/H$ and the dimensionless period of the electron in its orbit $\tau = \Omega t$. The "cyclotron" frequency $\Omega = eH/mc$; $-e$ = charge, $m = (1/2\pi)\partial S(\epsilon, p_H)/\partial\epsilon$ = effective mass, $\mathbf{v} = \nabla_p \epsilon(\mathbf{p})$ = velocity, $\zeta(T)$ = chemical potential of the electrons; $S(\epsilon, p_H)$ = cross sectional area of the surface, $\epsilon(\mathbf{p}) = \epsilon$ on the surface p_H = const; t = actual period of electron in orbit; $\mathbf{E}(z)$ = direction of the electric field inside the metal. It follows from the unperturbed equation of motion of the electron in a magnetic field $d\mathbf{p}/dt = -(e/c)[\mathbf{v} \times \mathbf{H}]$ that the Jacobian of the transformation from the variables p_x, p_y, p_z to the variables ϵ, τ, p_H is equal to m.

The boundary conditions for Eqs. (2.1) are the periodicity of $f(z, \epsilon, \tau, p_H)$ in the variable τ with period 2π and the condition of diffuse reflection of the electrons from the boundaries of the film:

$$f(0; \mathbf{v})\big|_{v_z>0} = 0; \quad f(d; \mathbf{v})\big|_{v_z<0} = 0. \tag{2.2}$$

Knowledge of the distribution function $f(z; \epsilon, \tau, p_H)$ permits us to compute the current density $\mathbf{j}(z)$

$$\mathbf{j}(z) = -(2e/h^3)\int \mathbf{v} f\,(d\mathbf{p}), \tag{2.3}$$

whence we find

$$\bar{\mathbf{j}} = \frac{1}{d}\int_0^d \mathbf{j}(z)\,dz$$

and the tensor of effective conductivity σ_{ik}. In this case, it must be taken into consideration that the components of the electric field E_x and E_y are constant along the film, while the Hall field E_z depends on z. This dependence ought to be obtained from the equation $j_z = 0$, which corresponds to the evident fact that there is an "open circuit" along the z axis. However, in what follows we shall neglect the dependence of the Hall field E_z on the coordinate z, since we shall be interested in two limiting cases: strong ($\gamma \ll 1$) and a weak ($\gamma \gg 1$) magnetic field ($\gamma = 1/\Omega t_0$) for $d \sim l = vt_0$. In the strong magnetic field ($\gamma \ll 1$), E_z changes appreciably only close to the boundaries of the film in a narrow range of z, $d - z \sim r \ll l \sim d$* (the contribution from which we can neglect), remaining practically constant over the thickness of the film. In the weak magnetic field ($\gamma \gg 1$), because of the small value of E_z, the Hall field can simply be neglected in the computation of the conductivity.

Starting out from physical considerations similar to those with the help of which Chambers[6] derived the formula for the conductivity of films and wires, it is not difficult to write down at once the solution of the kinetic equation (2.1), which satisfies the boundary condition (2.2):

$$f(z; \tau; \varepsilon, p_H) = \frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\int_{\lambda(z;\tau)}^{\tau} \mathbf{v}(\tau_1).\mathbf{E}\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2\right) \exp\left(\int_{\tau}^{\tau_1}\gamma\, d\tau_2\right) d\tau_1, \tag{2.4}$$

where $\lambda(z; \tau)$ denotes the closet preceding τ root of one of the following equations

$$z + \frac{1}{\Omega}\int_{\tau}^{\lambda(z;\tau)} v_z\, d\tau_2 = 0,\ d; \quad \lambda(z; \tau) \leqslant \tau. \tag{2.5}$$

From the periodicity requirement of f in τ with period 2π, it follows that $\lambda(z; \tau + 2\pi) = \lambda(z, \tau) + 2\pi$. If Eqs. (2.5) do not have a solution (massive

*$r = pc/eH$ is the radius of the electron orbit in the magnetic field, $\gamma = r/l$.

metal), then we must set $\lambda(z;\tau)$ equal to $-\infty$. We can show by a direct demonstration that (2.4) satisfies all the required conditions.

In the case in which the magnetic field is parallel to the surface of the film, the distribution function $f(z;\tau)$ is discontinuous, where the discontinuity occurs along the characteristic of Eq. (2.1), i.e., along the line

$$0 \leqslant z = \frac{1}{\Omega}\int^{\tau} v_z\, d\tau_2 + \text{const} \leqslant d. \tag{2.6}$$

The presence of the discontinuity corresponds to the fact that the electrons (with given τ) reaching one of the surfaces of the film cannot penetrate directly (i.e., without collisions inside the metal) to a depth greater than $2r(\tau)$.* At the same time, the electrons (with the same value of τ) capable of penetrating to depths greater than $2r(\tau)$ do not collide directly with the boundaries of the film. Therefore, all the electrons are divided in a natural way into two groups: electrons which undergo collisions with the boundaries, and electrons which do not reach the boundaries (without collisions inside the film). The distribution function for these two groups of electrons are essentially different, which also corresponds to the presence of a discontinuity.

It is not difficult to show that the solution of Eq. (2.1) with the boundary conditions (2.2) is unique in the class of functions which achieve discontinuities along the characteristic (2.6). For this purpose it suffices to note that Eq. (2.1) with the right hand side equal to zero has its own general solution

$$f = \exp\left(-\int_0^{\tau}\gamma d\tau_2\right)\Phi\left(z - \frac{1}{\Omega}\int_0^{\tau} v_z d\tau_2\right)$$

(Φ is an arbitrary function which can have a discontinuity), which, however, does not satisfy the requirement of periodicity in τ and consequently is identically equal to zero. Thus, for finding the jump in the function f at the discontinuity, no additional conditions are required.

We can obtain (2.4) directly from (2.1). For this purpose, we continue the function $f(z;\tau)$ by setting it equal to zero for $z < 0$ and $z > d$, and apply a Laplace transformation:

$$F(p;\tau) = \int_0^{d} f(z;\tau)\, e^{-pz}\, dz;$$

$$G(p;\tau) = \int_0^{d} g(z;\tau)\, e^{-pz}\, dz \equiv \int_0^{d} \frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\mathbf{v}\mathbf{E}(z)\, e^{-pz}\, dz,$$

then

$$\frac{\partial F}{\partial \tau} + \left(\gamma + \frac{p v_z}{\Omega}\right)F = G(p;\tau) + \frac{v_z}{\Omega}\left[f(0;\tau) - f(d;\tau)\, e^{-pd}\right].$$

The periodic integral of this equation is equal to

$$F(p;\tau) = \int_{-\infty}^{\tau} d\tau_1 \cdot \exp\left[\int_{\tau}^{\tau_1}\left(\gamma + \frac{p v_z}{\Omega}\right)d\tau_2\right]$$
$$\times \left\{G(p;\tau_1) + \frac{v_z(\tau_1)}{\Omega}\left[f(0;\tau_1) - f(d;\tau_1)\, e^{-pd}\right]\right\},$$

whence

$$f(z;\tau) \equiv \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} F(p;\tau)\, e^{pz}\, dp$$
$$= \int_{-\infty}^{\tau} d\tau_1 \exp\left(\int_{\tau}^{\tau_1}\gamma d\tau_2\right)\left\{g\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2;\, \tau_1\right)\right.$$
$$+ \frac{v_z(\tau_1)}{\Omega}\left[f(0;\tau_1)\,\delta\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2\right)\right. \tag{2.7}$$
$$\left.\left. - f(d;\tau_1)\,\delta\left(z - d + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2\right)\right]\right\}.$$

It follows from the boundary conditions (2.2) that

$$f(0;\tau) = -\operatorname{sgn} v_z f(0;\tau); \quad f(d;\tau) = \operatorname{sgn} v_z f(d;\tau);$$
$$\operatorname{sgn} x = \begin{cases} 1\ (x > 0) \\ -1\ (x < 0). \end{cases} \tag{2.8}$$

Introducing

$$u(z;\tau) = \exp\left(\int_0^{\tau}\gamma\, d\tau_2\right) f(z;\tau)$$

and making use of (2.8), we get

$$u(z;\tau) + \int_{-\infty}^{\tau} d\tau_1 u\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2;\, \tau_1\right)$$
$$\times \frac{|v_z(\tau_1)|}{\Omega}\left\{\delta\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2\right) + \delta\left(z - d + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2\right)\right.$$
$$= \int_{-\infty}^{\tau} \exp\left(\int_0^{\tau_1}\gamma\, d\tau_2\right) g\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2;\, \tau_1\right) d\tau_1.$$

Because of the presence of the δ-function, the left side of this equation reduces to

$$u(z;\tau) + \sum_s u\left(z + \frac{1}{\Omega}\int_{\tau}^{\lambda_s(z;\tau)} v_z\, d\tau_2;\, \lambda_s(z;\tau)\right)$$

*For simplicity in the given case, we assume that the thickness of the film d is greater than the diameter of the maximum orbit max $2r(\tau)$.

$$= \int_{-\infty}^{\tau} d\tau_1 \exp\left(\int_0^{\tau_1} \gamma\, d\tau_2\right) g\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2;\ \tau_1\right). \qquad (2.9)$$

The summation in (2.9) extends over all the roots $\lambda_s(z;\ \tau)$ of the equation

$$z + \frac{1}{\Omega}\int_{\tau}^{\lambda_s(z;\tau)} v_z\, d\tau_2 = 0,\ d;\quad \tau \geqslant \lambda_1 > \lambda_2 > \ldots \geqslant -\infty.$$

The solution of the functional equation (2.9), as is easy to verify, is

$$u(z;\ \tau) = \int_{\lambda_1(z;\,\tau)}^{\tau} d\tau_1 \exp\left(\int_0^{\tau_1} \gamma\, d\tau_2\right) g\left(z + \frac{1}{\Omega}\int_{\tau}^{\tau_1} v_z\, d\tau_2;\ \tau_1\right),$$

which gives Eq. (2.4) for $f(z;\ \tau)$. In this case, we need to take into account that, by definition,

$$\lambda_1\left(z + \frac{1}{\Omega}\int_{\tau}^{\lambda_s} v_z\, d\tau_2;\ \lambda_s\right) \equiv \lambda_{s+1}(z;\ \tau).$$

3. GENERAL FORMULA FOR THE EFFECTIVE CONDUCTIVITY TENSOR

To find the effective conductivity tensor, we must compute the mean value of the distribution function in terms of which the average current is expressed [see Eq. (2.3)]. From (2.1), we have

$$\frac{\partial \bar f(\tau)}{\partial \tau} + \gamma \bar f = \frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\mathbf{v}\bar{\mathbf{E}} - \frac{v_z}{\Omega d}\left[f(d;\ \tau) - f(0;\ \tau)\right], \qquad (3.1)$$

$$\bar f(\tau) = \frac{1}{d}\int_0^d f(z;\ \tau)\, dz;\quad \gamma = 1/\Omega t_0.$$

Making use of (2.8), we find

$$\bar f(\tau) = \int_{-\infty}^{\tau} d\tau_1 \exp\left(\int_{\tau}^{\tau_1} \gamma\, d\tau_2\right)\left\{\frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\cdot \mathbf{v}(\tau_1)\bar{\mathbf{E}} - \frac{|v_z(\tau_1)|}{\Omega d}\left[f(d;\ \tau_1) + f(0;\ \tau_1)\right]\right\}. \qquad (3.2)$$

After substitution for $f(0;\ \tau)$ and $f(d;\ \tau)$ from (2.4), we get

$$\bar f(\tau) = \frac{e}{\Omega}\frac{\partial f_0}{\partial \varepsilon}\int_{-\infty}^{\tau} d\tau_1 \exp\left(\int_{\tau}^{\tau_1} \gamma\, d\tau_2\right)\left\{\mathbf{v}(\tau_1)\bar{\mathbf{E}} - \frac{|v_z(\tau_1)|}{\Omega d}\int_{s(\tau_1)}^{\tau_1} d\tau' \exp\left(\int_{\tau_1}^{\tau'} \gamma\, d\tau_2\right)\mathbf{v}(\tau')\,\mathbf{E}\left(\frac{1}{\Omega}\int_{\tau_1}^{\tau'} v_z\, d\tau_2\right)\right\}, \qquad (3.3)$$

where $s(\tau) = \lambda_1(d-0;\ \tau)$ for $v_z(\tau) > 0$, $s(\tau) = \lambda_1(+0;\ \tau)$ for $v_z(\tau) < 0$, where $s(\tau) < \tau$. It is easy to see that $s(\tau)$ coincides with the nearest preceding τ root of one of the equations

$$\frac{1}{\Omega}\left|\int_{s(\tau)}^{\tau} v_z\, d\tau_2\right| = 0;\quad \frac{1}{\Omega}\left|\int_{s(\tau)}^{\tau} v_z\, d\tau_2\right| = d.$$

Taking $\mathbf{E}(z)$ to be constant (see Sec. 2) and substituting (3.3) in (2.3), we get the final expression for the effective conductivity tensor in the very general form

$$\sigma_{ik} = \sigma_{ik}^{(0)} - \delta\sigma_{ik} \quad (\bar j_i = \sigma_{ik}\bar E_k);$$

$$\sigma_{ik}^{(0)} = -\frac{2e^2}{h^3}\int_0^{\infty}\frac{\partial f_0}{\partial \varepsilon}\, d\varepsilon \int \frac{m}{\Omega}\, dp_H \times \int_0^{2\pi} v_i(\tau)\, d\tau \int_{-\infty}^{\tau} \exp\left(\int_{\tau}^{\tau_1} \gamma\, d\tau_2\right) v_k(\tau_1)\, d\tau_1; \qquad (3.4)$$

$$\delta\sigma_{ik} = -\frac{2e^2}{h^3 d}\int_0^{\infty}\frac{\partial f_0}{\partial \varepsilon}\, d\varepsilon \cdot \int \frac{m}{\Omega^2}\, dp_H \int_0^{2\pi} v_i(\tau)\, d\tau \int_{-\infty}^{\tau} |v_z(\tau_1)|\, d\tau_1 \times \int_{s(\tau_1)}^{\tau_1} \exp\left(\int_{\tau}^{\tau'} \gamma\, d\tau_2\right) v_k(\tau')\, d\tau'.$$

In the case of several bands in (3.4), it is necessary to sum the corresponding expressions over all bands.

The tensor $\sigma_{ik}^{(0)}$ coincides with the conductivity tensor for the massive metal; the presence of $\delta\sigma_{ik}$ is brought about by the finite thickness of the film. In the limiting case of zero magnetic field, Eqs. (3.4) reduce to the corresponding equations of the researches of Fuchs[1] and Kaganov and Azbel'.[2] If the magnetic field is parallel to the surface of the film, (3.4) coincides with the results of Azbel'[7] and Königsberg.[8]

It should be noted that the Onsager relations $\sigma_{ik}(\mathbf{H}) = \sigma_{ki}(-\mathbf{H})$ for the effective conductivity tensor of the film are, generally speaking, violated [it appears that $\sigma_{ik}^{(0)}(\mathbf{H}) = \sigma_{ki}^{(0)}(-\mathbf{H})$]. This is not unexpected, since in the case of the film, the conductivity tensor, strictly speaking, is an operator, because of the dependence of the electric field on the coordinate z. However, in several cases (for example, a strong magnetic field parallel to the surface) we can neglect the nondiagonal components of $\delta\sigma_{ik}$ in comparison with the corresponding components in $\sigma_{ik}^{(0)}$, and then the Onsager relations will be satisfied asymptotically.

4. INVESTIGATION OF LIMITING CASES

Having in mind the comparison of the results of the present theory with the experiments of Borovik and Lazarev,[9] we consider in this section the region of strong magnetic fields ($\gamma \ll 1$) in which we consider that $d \sim l$. Just this case is studied in detail in Ref. 9. With the aim of simplifying the final formulas, we shall assume the dispersion law to be isotropic and quadratic, and the relaxation time t_0 to be constant and not to depend on **p** (the residual resistance). We can show that the qualitative results (in particular, the dependence of H) do not depend on the form of the dispersion law and the collision integral.

The asymptoticity of the tensor $\sigma_{ik}^{(0)}$ in strong fields was studied in detail by Lifshitz, Azbel', and Kaganov[11] for an arbitrary form of dispersion law $\epsilon(\mathbf{p})$ and collision integral. It was shown in their work that for closed Fermi surfaces, the dispersion law and the collision integral have no effect on the dependence of $\sigma_{ik}^{(0)}$ on strong magnetic fields.

The basic difficulty in the calculation of $\delta\sigma_{ik}$ is contained in the determination of the function $s(\tau)$. Unfortunately, we did not succeed in finding a suitable analytic expression for $s(\tau)$ for arbitrary angle of inclination of **H** to the surface of the film. Simple formulas for $s(\tau)$ were obtained in a constant magnetic field parallel to and perpendicular to the surface of the film.

In the perpendicular field, v_z does not depend on τ and

$$s(\tau) = \tau - \Omega d / |v_z|. \tag{4.1}$$

In the parallel field two cases are distinguished: a) $d > 2r$ ($r = mvc/eH$, sufficiently strong magnetic field). In this case, the electrons which are colliding with one of the boundaries of the film cannot directly reach the other boundary (i.e., without collisions inside the film). b) $d = 2r$ (weak magnetic field or sufficiently thin film). The calculations are especially simple in case a) and are considerably complicated in case b).

For $d = 2r$, all the components of the tensor $\delta\sigma_{ik}$ are continuous. Finite jumps have a third derivative δs_{xx} with respect to H, second derivatives $\delta\sigma_{xy}$, $\delta\sigma_{yx}$, $\delta\sigma_{xz}$ and first derivatives of all the remaining components of $\delta\sigma_{ik}$.

In the perpendicular field, the nonzero components of the effective conductivity tensor are equal to

$$\frac{\sigma_{xx}}{\sigma_0} = \frac{\sigma_{yy}}{\sigma_0} = \frac{\gamma^2}{1+\gamma^2}\left\{1 - \frac{3}{8k}\frac{(\gamma^2-1)A + 2\gamma B}{1+\gamma^2}\right\};$$

$$\frac{\sigma_{zz}}{\sigma_0} = 1 - \frac{3}{4k}C; \tag{4.2}$$

$$\frac{\sigma_{xy}}{\sigma_0} = -\frac{\sigma_{yx}}{\sigma_0} = -\frac{\gamma}{1+\gamma^2}\left\{1 - \frac{3}{8k}\frac{\gamma}{1+\gamma^2}[2\gamma A + (1-\gamma^2)B]\right\},$$

where $n = 8\pi p^3/3\hbar^3$ is the density of electrons,

$$\sigma_0 = ne^2 t_0/m, \quad \gamma = r/l = 1/\Omega t_0; \quad \alpha = d/r; \quad k = d/l;$$

$$A + iB = 1 - 4E_3(k + i\alpha) + 4E_5(k + i\alpha); \; C = 1 - 4E_5(k);$$

$$E_n(x) = \int_1^\infty \xi^{-n} e^{-x\xi}\, d\xi.$$

Equations (4.2) are accurate (and coincide with the corresponding equations in Ref. 7a), as far as the equation $j_z = 0$ is satisfied by E_z, which is equal to zero, and we can introduce the tensor (and not the operator!) of the conductivity. It is obvious that in this case the Onsager relations are satisfied.

For $\gamma \ll 1$ and $k \sim 1$, Eqs. (4.2) yield

$$\sigma_{xx}/\sigma_0 = \sigma_{yy}/\sigma_0 = \gamma^2(1 + 3/8k); \quad \sigma_{zz}/\sigma_0 = 1 - 3C/4k;$$

$$\sigma_{xy}/\sigma_0 = -\sigma_{yx}/\sigma_0 = -\gamma(1 - \gamma^2 + 3\gamma^2/4k). \tag{4.3}$$

a) In a sufficiently strong parallel field ($d > 2r$), the function $s(\tau)$ is a solution of the equation

$$\frac{1}{\Omega}\int_{s(\tau)}^{\tau} v_z(\tau_2)\, d\tau_2 \equiv \frac{1}{m\Omega}[p_y(s(\tau)) - p_y(\tau)] = 0$$

(the equation

$$\frac{1}{\Omega}\left|\int_{s(\tau)}^{\tau} v_z\, d\tau_2\right| = d$$

has no solution).

Setting

$$\mathbf{v} = \mathbf{p}/m = v(\cos\vartheta, \quad \sin\vartheta\sin\tau,$$

$$\sin\vartheta\cos\tau), \quad -\pi/2 \leqslant \tau \leqslant 3\pi/2,$$

we find

$$s(\tau) = -\pi - \tau \ \text{ for } -\pi/2 \leqslant \tau \leqslant \pi/2, \; s(\tau)$$

$$= \pi - \tau \ \text{ for } \pi/2 \leqslant \tau \leqslant 3\pi/2. \tag{4.4}$$

The nonzero components of the tensor σ_{ik} have the form

$$\frac{\sigma_{xx}}{\sigma_0} = 1 - \frac{3}{8k}\left(1 - \frac{1}{2}\frac{1 + e^{-2\pi|\gamma|}}{1 + 4\gamma^2}\right);$$

$$\frac{\sigma_{yy}}{\sigma_0} = \frac{\gamma^2}{1+\gamma^2}\left\{1 - \frac{3}{8k(1+\gamma^2)}\left[\gamma^2 - 2 - \frac{3(6 - 5\gamma^2 + 4\gamma^4)}{2(1+4\gamma^2)(9+4\gamma^2)}\left(1 + e^{-2\pi|\gamma|}\right)\right]\right\};$$

$$\frac{\sigma_{zz}}{\sigma_0} = \frac{\gamma^2}{1+\gamma^2}\left\{1 - \frac{3}{8k(1+\gamma^2)}\left[2\gamma^2 - 1 + \frac{9(1+6\gamma^2)(1+e^{-2\pi|\gamma|})}{2(1+4\gamma^2)(9+4\gamma^2)}\right]\right\}; \quad \frac{\sigma_{yz}}{\sigma_0} = -\frac{\sigma_{zy}}{\sigma_0} \tag{4.5}$$

$$= -\frac{\gamma}{1+\gamma^2}\left\{1 - \frac{9}{8k}\frac{\gamma^2}{(1+\gamma^2)}\left[1 + \frac{(7-8\gamma^2)(1+e^{-2\pi|\gamma|})}{2(1+4\gamma^2)(9+4\gamma^2)}\right]\right\}.$$

Equations (4.5) have meaning only for $r \ll \ell$, $r < d/2$, when we can neglect the dependence of the Hall field E_z on the coordinates. For $\gamma \ll 1$,

$$\frac{\sigma_{xx}}{\sigma_0} = 1 - \frac{3\pi}{k}|\gamma|; \quad \frac{\sigma_{yy}}{\sigma_0} = \gamma^2\left\{1 + \frac{3}{2k} - \frac{3\pi}{4k}|\gamma|\right\}; \quad (4.6)$$

$$\frac{\sigma_{zz}}{\sigma_0} = \gamma^2\left\{1 + \frac{3\pi|\gamma|}{8k}\right\}; \quad \frac{\sigma_{yz}}{\sigma_0} = -\frac{\sigma_{zy}}{\sigma_0} = -\gamma\left\{1 - \gamma^2 - 2\frac{\gamma^2}{k}\right\}.$$

b) In the case of a weak magnetic field ($d < 2r$), the function $s(\tau)$ is determined differently for the regions $\sin\vartheta < \alpha/2$ and $\sin\vartheta > \alpha/2$. For $\sin\vartheta < \alpha/2$, $s(\tau)$ is given by Eq. (4.4), while if $\sin\vartheta > \alpha/2$, then

$$s(\tau) = -\pi - \tau \text{ for } -\pi/2 < \tau < \tau_0;$$

$$s(\tau) = \arcsin(\sin\tau - \alpha/\sin\vartheta) \text{ for } \tau_0 < \tau < \pi/2;$$

$$s(\tau) = \pi - \tau \text{ for } \pi/2 < \tau < \pi + \tau_0;$$

$$s(\tau) = \pi - \arcsin(\sin\tau + \alpha/\sin\vartheta) \text{ for } \pi + \tau_0 < \tau \leqslant 3\pi/2;$$

$$\tau_0 = \arcsin(-1 + \alpha/\sin\vartheta).$$

Let us cite the result for the conductivity of the film in a weak longitudinal magnetic field ($\mathbf{E} \parallel \mathbf{H} \parallel x$), when $\gamma \gg 1/k + k$, $\alpha \ll 1$:

$$\sigma_{xx} = \sigma_F - \sigma_0 M/\gamma^2, \quad \sigma_F/\sigma_0$$

$$= 1 - {}^3/_8 k + ({}^3/_2 k)(E_3(k) - E_5(k)). \quad (4.7)$$

Here σ_F is the conductivity of the film in the absence of a magnetic field, obtained by Fuchs,[1] while the coefficient M depends on the thickness of the film:

$$M = \frac{3}{64k}\left\{-1 + 2\left[1 + k + \frac{k^2}{6}(1-k)\right]e^{-k} + \frac{k^4}{3}E_1(k)\right\}.$$

5. SPECIFIC RESISTANCE AND THE HALL FIELD. COMPARISON WITH EXPERIMENT

By experiment one usually obtains not the direction of the electric field, but the current density and, consequently, the resistivity tensor $\rho_{ik} = \sigma_{ik}^{-1}$ rather than its inverse, the conductivity tensor. Employing the results obtained above for σ_{ik}, it is not difficult to find the dependence of the electrical resistance on **H**.

Let us obtain the formulas for the resistivity and the Hall field in a film located in a strong magnetic field, in the presence of two types of current carriers (electrons and "holes"). The indices 1 and 2 will refer to electrons and "holes," respectively:

1) $n_1 \neq n_2$;

$$\rho_\perp = \frac{n_1u_1 + n_2u_2 + {}^3/_8(n_1u_1/k_1 + n_2u_2/k_2)}{(n_1-n_2)^2e^2} + O(H^{-2});$$

$$\rho_\parallel^\parallel = \rho_0\left\{1 + \frac{3\pi}{8}\rho_0\frac{ec}{H}\left(\frac{n_1}{k_1} + \frac{n_2}{k_2}\right)\right\} + O(H^{-2});$$

$$\rho_0 = \left[e^2\left(\frac{n_1}{u_1} + \frac{n_2}{u_2}\right)\right]^{-1}$$

$$\rho_\parallel^\perp = \frac{n_1u_1 + n_2u_2 e + (3\pi c/8H)(n_1u_1^2/k_1 + n_2u_2^2/k_2)}{(n_1-n_2)^2e^2} + O(H^{-2});$$

$$\left(\frac{E_y}{E_x}\right)_\perp = -\frac{eH}{c}\frac{n_1-n_2}{n_1u_1 + n_2u_2 + {}^3/_8(n_1u_1/k_1 + n_2u_2/k_2)} + O(H^{-1});$$

$$\left(\frac{E_z}{E_y}\right)_\parallel = -\frac{eH(n_1-n_2)}{c(n_1u_1+n_2u_2)} + O(H^{-1}). \quad (5.1)$$

Here, $u_i = m_i/t_{0i}$ is the mobility of the carriers; $k_i = d/\ell_i$; the subscript symbols in all the quantities indicate the orientation of **H** relative to the surface of the film, while the superscript symbols for $\rho_\parallel$ denote the orientation of the current relative to the magnetic field.

2) $n_1 = n_2 = n$;

$$\rho_\perp = \frac{H^2}{nc^2[u_1 + u_2 + {}^3/_8(u_1/k_1 + u_2/k_2)]};$$

$$\rho_\parallel^\parallel = \rho_0\left\{1 + \frac{3\pi}{8}\rho_0\frac{nec}{H}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)\right\};$$

$$\rho_\parallel^\perp = \frac{H^2}{nc^2[u_1 + u_2 + {}^3/_2(u_1/k_1 + u_2/k_2)]}; \quad (5.2)$$

$$\left(\frac{E_y}{E_x}\right)_\perp = \frac{c}{eH}\frac{u_1^2 - u_2^2 - {}^3/_4(u_1^2/k_1 - u_2^2/k_2)}{u_1 + u_2 + {}^3/_8(u_1/k_1 + u_2/k_2)};$$

$$\left(\frac{E_z}{E_y}\right)_\parallel = \frac{c}{eH}\frac{u_1^2 - u_2^2 + 2(u_1^2/k_1 - u_2^2/k_2)}{u_1 + u_2}.$$

The Hall constant $R = \overline{E}_z/jH = -\delta V/jHd$ in strong fields (δV = potential difference between the sides of the film) for $n_1 \neq n_2$ becomes identical asymptotically with that for the massive metal

$$R \cong 1/ec(n_1 - n_2),$$

and for $n_1 = n_2$ depends on the thickness of the film:

$$R_\parallel = \frac{u_1^2 - u_2^2 + 2(u_1^2/k_1 - u_2^2/k_2)}{nec(u_1+u_2)[u_1 + u_2 + {}^3/_2(u_1/k_1 + u_2/k_2)]};$$

$$R_\perp = \frac{u_1^2 - u_2^2 - {}^3/_4(u_1^2/k_1 - u_2^2/k_2)}{nec[u_1 + u_2 + {}^3/_8(u_1/k_1 + u_2/k_2)]^2} \quad (5.3)$$

We proceed to a comparison of theory with experiment[9] on the effect of the shape of the specimen on the resistance of single crystals of Bi. From the fact of the increase in resistance in a strong magnetic field for massive Bi, we can draw the conclusion that $n_1 = n_2$, while it follows from experiments on the de Haas — van Alphen effect[12] that the Fermi surface for electrons in Bi is represented by a set of three uniaxial ellipsoids, located in a binary plane and turned one to the other by 120° about an axis of third order. The Fermi surface for the "holes" in Bi was little investigated, but usually it is spherical. For simplicity, we replace the set of three ellipsoids by a single sphere (qualitative results are not changed for this case). From the equality of electrons and "holes" follows the equality of the boundary momenta and their orbital radii. The minimum resistance on the rotation diagram for $\varphi = 0$ (φ = angle between $\mathbf{H}$ and the film surface, $\mathbf{H} \perp \mathbf{j}$) discovered in Ref. 9 is satisfactorily explained by the formulas (5.2):

$$\frac{\rho_{\parallel}^{\perp}}{\rho_{\perp}} = \frac{d+3\bar{l}/8}{d+3\bar{l}/2} < 1; \quad 2/\bar{l} = 1/l_1 + 1/l_2; \ u_i = p/l_i.$$

It is seen from (5.2) that the increase in the resistance of the film in a strong transverse magnetic field takes place according to the same law as for the massive metal ($k_i = \infty$), but more slowly than in the latter:*

$$\rho_{\parallel}^{\perp}(d)/\rho_{\parallel}^{\perp}(\infty) = d/(d+3\bar{l}/2) < 1;$$

$$\rho_{\perp}(d)/\rho_{\perp}(\infty) = d/(d+3\bar{l}/8) < 1.$$

The dependence of the quantity β on the magnetic field

$$\beta = \rho_{\parallel}^{\perp}/\rho_{\perp} = \frac{d+3\bar{l}/8}{d+3\bar{l}/2}\left\{1+\frac{3\pi}{4}\frac{r}{d+3\bar{l}/2}\right\}$$
$$\equiv \frac{d+3\bar{l}/8}{d+3\bar{l}/2}\left\{1+\frac{H_0}{H}\right\}, \ H_0 = \frac{3\pi}{4}\frac{pc}{e(d+3\bar{l}/2)} \ll H, \quad (5.4)$$

agrees with that observed by Borovik and Lazarev for the decrease in this ratio in a strong field.

The presence of a discontinuity (jump in the derivative) for the quantity β at $r = d/2$ is clearly evident in Fig. 6 of Ref. 9.

Change of β with temperature is connected with the temperature dependence of the mean free path $\bar{l}$. For $\bar{l} \ll d$, (5.4) gives unity while for $\bar{l} \gg d$, $\beta \approx 1/4$. In Ref. 9, at low temperatures, when we can assume that $l \gg d$, the quantity $\beta \sim 1/2$. This difference can be connected both with the anisotropies of the dispersion law and the mean free path length, and also with the fact that $\bar{l} \sim d$ in the region of residual resistance, where $\bar{l}$ no longer depends on the temperature.

It is obvious that in such experiments only the mean free path length $\bar{l}$ can be measured, and not l_1 and l_2 separately.

Comparison with experiments on the dependence of the Hall field in the film on H is not possible because of the absence of experimental data.

*An exception is the specimen Bi-3 investigated in Ref. 9, the massive part of which (according to the way of obtaining it) was more "rougher" than the film, i.e. $\bar{l}_{\text{mass}} < d(3/2\,\bar{l}_f + d)/\bar{l}_f$.

6. THERMOMAGNETIC EFFECTS IN FILMS

The effect of the magnetic field on the thermal conductivity, the coefficients of Thomson, Peltier and others, in a massive metal, was studied by Azbel', Kaganov, and Lifshitz.[13] Here we shall briefly consider the thermomagnetic effects in films and establish the connection of the tensors of thermal conductivity and the Thomson coefficients with the electrical conductivity tensor.

If a temperature gradient exists in the metal; then a heat current arises, the density of which, $\mathbf{w}$, is equal to

$$\mathbf{w} = 2h^{-3}\int \varepsilon \mathbf{v} f\,(d\mathbf{p}).$$

To find the kinetic coefficients, we must compute the density of the electric current $\mathbf{j}$ and the energy flux density $\mathbf{w}$, which arises as a result of the electric field and the temperature gradient. In this case, in the right side of the kinetic equation (2.2), we have, in place of $e\mathbf{E}\cdot\mathbf{v}\,\partial f_0/\partial\epsilon$:

$$e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial \varepsilon} - \mathbf{v}\frac{\partial f_0}{\partial T}\nabla T. \quad (6.1)$$

In those cases in which we can introduce the effective conductivity tensor (see Sec. 2—4), we have

$$\bar{j}_i = \sigma_{ik}\bar{E}_k + S_{ik}^{(0)}\partial\bar{T}/\partial x_k; \ \bar{w}_i = \xi_{ik}\bar{E}_k + S_{ik}^{(1)}\partial\bar{T}/\partial x_k, \quad (6.2)$$

where

$$\sigma_{ik} = -\int_0^\infty \frac{\partial f_0}{\partial \varepsilon}\sigma_{ik}(\varepsilon)\,d\varepsilon; \quad \xi_{ik} = e^{-1}\int_0^\infty \varepsilon\frac{\partial f_0}{\partial \varepsilon}\sigma_{ik}(\varepsilon)\,d\varepsilon;$$
$$S_{ik}^{(n)} = (-1)^n e^{-1-n}\int_0^\infty \varepsilon^n\frac{\partial f_0}{\partial T}\sigma_{ik}(\varepsilon)\,d\varepsilon, \quad (6.3)$$

while $\sigma_{ik}(\epsilon)$ is determined by the formula

$$\sigma_{ik}(\varepsilon) = \frac{2e^2}{h^3}\int_{\varepsilon(\mathbf{p})=\varepsilon}\frac{m}{\Omega}dp_H\int_0^{2\pi} v_i(\tau)\,d\tau\int_{-\infty}^{\tau}\exp\left(\int_{\tau}^{\tau_1}\gamma d\tau_2\right)d\tau_1$$
$$\times\left\{v_k(\tau_1) - \frac{|v_z(\tau_1)|}{\Omega d}\int_{s(\tau_1)}^{\tau_1}\exp\left(\int_{\tau_1}^{\tau'}\gamma d\tau_2\right)v_k(\tau')\,d\tau'\right\}. \quad (6.4)$$

From the law of conservation of energy, we get [and also from (6.2)]:

$$\frac{\partial Q}{\partial t} = j_i E_i - \frac{\partial w_i}{\partial x_i} \equiv \rho_{ik} j_i j_k + \frac{\partial}{\partial x_i}\left(\varkappa_{ik}\frac{\partial T}{\partial x_k}\right) - \mu_{ik} j_i \frac{\partial T}{\partial x_k}.$$

Here Q is the internal energy density; $\rho_{ik} = \sigma_{ik}^{-1}$ the resistivity tensor;

$$\varkappa_{ik} = \xi_{ip}\rho_{pq}S_{qk}^{(0)} - S_{ik}^{(1)} \ \text{и} \ \mu_{ik} = \rho_{ip}S_{pk}^{(0)} + \frac{\partial}{\partial T}(\xi_{ip}\rho_{pk}) -$$

are the thermal conductivity and Thomson coefficients.

It was shown in Ref. 13 that if the collision operator was a δ-function in the energy, then for each of the components of the tensors of conductivity and thermal conductivity, the Wiedmann-Franz law is satisfied. The calculations carried out in Ref. 13 are not connected with the concrete form of σ_{ik}, μ_{ik}, κ_{ik}, and therefore we can make direct use of the results of Ref. 13 and write down at once:

$$\sigma_{ik} = \sigma_{ik}(\zeta_0; d); \quad \varkappa_{ik} = {}^1/_3\pi^2k^2T\sigma_{ik};$$
$$\mu_{ik} = \frac{\pi^2k^2T}{3e}\left(\rho_{ip}\frac{\partial\sigma_{pk}}{\partial\zeta_0} - 2\rho_{pi}\frac{\partial\sigma_{kp}}{\partial\zeta_0}\right). \qquad (6.5)$$

Here $\zeta_0 = \zeta(0)$ is the chemical potential of the electron gas at absolute zero (the limiting Fermi energy), k is Boltzmann's constant. Thus, the tensors κ_{ik} and μ_{ik} are expressed in terms of the tensor σ_{ik} studied above.

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STATIONARY CONVECTIVE FLOW OF AN ELASTICALLY CONDUCTING LIQUID BETWEEN PARALLEL PLATES IN A MAGNETIC FIELD

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A study is made of the stationary convection of an electrically conducting liquid in the space between two parallel plates, heated to different temperatures, in the presence of a magnetic field. The distribution of velocity, temperature, and induced fields are found, and the convective heat flow is calculated.

IT is well known that currents are induced in a conducting liquid which moves in a magnetic field. The interaction of these currents with the magnetic field is the cause of the various magnetohydrodynamic effects which have been intensively studied in recent years. The magnetic field will,