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TRANSITION BETWEEN HYPERFINE STRUCTURE LEVELS IN MU-MESIC HYDROGEN

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It is shown that, through the mechanism of "jumping" of the μ^- meson from one proton to another, which was proposed by Ia. B. Zel'dovich, mesic hydrogen atoms convert completely to the ground state of the hyperfine structure during the lifetime of the μ meson. As a result, there is complete depolarization of μ mesons in hydrogen, and the neutrons which are formed from capture of μ^- mesons by protons via $\mu^- + p = n + \nu$ will be completely polarized along their direction of motion.

THE separation between the upper ($F = 1$, where F is the total spin of the mesic atom) and lower ($F = 0$) levels of the hyperfine structure for a μ meson in the K orbit of a mesic hydrogen atom, is[1]

$$\Delta\varepsilon = \frac{16\pi}{3}\beta_\mu\beta_N g_i |\psi(0)|^2 = 0.25 \text{ ev} \tag{1}$$

(β_μ is the μ-mesonic and β_N the nuclear Bohr magneton, $g_i = 2 \times 2.79$ is the gyromagnetic ratio of the proton).

Because of the smallness of this separation, the radiative transition to the lower state is extremely improbable ($\tau_{rad} \sim 10^6$ sec). However, because of the neutrality of mesic hydrogen there is a very effective mechanism via which there is a complete transition into the lower hyperfine structure state during the lifetime of the μ meson. This mechanism is the "jump" of the μ meson from one proton to another with simultaneous transition into the lower state of the hyperfine structure.* Since the hyperfine splitting is much greater than the thermal energy in collisions of a proton and a mesic atom, the process is irreversible. In the present paper we give an estimate of the cross section for this transition.

*This was called to the attention of the author by Ia. B. Zel'dovich.

In mesic units ($e = 1$, $\hbar = 1$, $m_\mu = 1$), the Hamiltonian for the interaction of a meson with a pair of protons, including the interaction of the spins of the meson and the protons is

$$\hat{H} = -\frac{1}{2M}\Delta_{\mathbf{R}_1} - \frac{1}{2M}\Delta_{\mathbf{R}_2} - \frac{1}{2}\Delta_{\mathbf{r}} - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{R} + \frac{4}{3}g_i\beta_\mu\beta_N\left(\frac{\delta(r_1)}{r_1^2}\mathbf{s}\mathbf{i}_1 + \frac{\delta(r_2)}{r_2^2}\mathbf{s}\mathbf{i}_2\right) \tag{2}$$

($\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{r}$ are the coordinates and $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{s}$ are the spins of the protons and the meson, $R = |\mathbf{R}_1 - \mathbf{R}_2|$ is the distance between the protons, while $r_1 = |\mathbf{r} - \mathbf{R}_1|$ and $r_2 = |\mathbf{r} - \mathbf{R}_2|$ are the distances of the meson from the two protons). We are neglecting the spin-spin interaction between the protons and between the meson and the second proton when the meson is at the position of the first proton.

At the velocities we are considering, the relative motion of the protons is described by an s wave, so that the total spin is conserved. The spin of the system consisting of two protons and a μ

meson can take on the values $^3/_2$ and $^1/_2$. However, transitions to the lower hyperfine structure state are possible only for the state with spin $^1/_2$. The wave function for such a state, antisymmetric under interchange of the protons, is expressible as

$$\Psi = G(R)\Sigma_g(R, \mathbf{r})S^0_{1/2, 1/2}(1, 2; \mu) + H(R)\Sigma_u(R, \mathbf{r})S^1_{1/2, 1/2}(1, 2; \mu). \quad (3)$$

(where, to be specific, we consider a state with projection of the total spin on the arbitrary z axis equal to $+^1/_2$). Σ_g and Σ_u are the wave functions of the μ meson in the field of the two fixed protons; they contain the distance between the protons as a parameter and are, respectively, symmetric and antisymmetric under interchange of the protons:

$$\left(-\frac{1}{2}\Delta_r - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{R}\right)\Sigma_{g,u} = E_{g,u}(R)\Sigma_{g,u}. \quad (4)$$

$S^0_{1/2,1/2}(1, 2; \mu)$ and $S^1_{1/2,1/2}(1, 2; \mu)$ are spin functions corresponding to total spin of 0 and 1 for the particles whose indices are separated off by the semicolon; the total spin of the system is $^1/_2$ and the total spin projection on the z axis is $+^1/_2$. Introducing the usual spin functions $\alpha = \begin{pmatrix}1\\0\end{pmatrix}$ and $\beta = \begin{pmatrix}0\\1\end{pmatrix}$ for the two protons and the μ meson, we can write

$$S^0_{1/2,1/2}(1,2; \mu) = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)]\alpha(\mu),$$

$$S^1_{1/2,1/2}(1,2; \mu) = \frac{1}{\sqrt{6}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)]\alpha(\mu) - \sqrt{\frac{2}{3}}\alpha(1)\alpha(2)\beta(\mu). \quad (5)$$

On the other hand, for sufficiently large distance between the protons, the wave function of the system with total spin $^1/_2$ and spin projection $+^1/_2$, for the lower hyperfine structure state (F = 0) is

$$\Psi^0 = K(R)\psi(r_1)S^0_{1/2,1/2}(1, \mu; 2) - K(R)\psi(r_2)S^0_{1/2,1/2}(2, \mu; 1), \quad (6)$$

and for the upper state (F = 1)

$$\Psi^1 = L(R)\psi(r_1)S^1_{1/2,1/2}(1, \mu; 2) - L(R)\psi(r_2)S^1_{1/2,1/2}(2, \mu; 1), \quad (7)$$

where $\psi(\mathbf{r}_1)$ and $\psi(\mathbf{r}_2)$ are the wave functions of the μ meson at the first and second protons, respectively.

At large distances between the protons

$$\Sigma_{g,u} \approx (\psi(r_1) \pm \psi(r_2))/\sqrt{2}, \quad (8)$$

and the wave function (3) of the system must be a linear combination of (6) and (7), so we easily find the relations between the functions G(R), H(R) and K(R), L(R):

$$\begin{aligned} G(R) &= (K - \sqrt{3}L)/\sqrt{2}, \\ K(R) &= (G - \sqrt{3}H)/2\sqrt{2}, \\ H(R) &= (-\sqrt{3}K - L)/\sqrt{2}, \\ L(R) &= (-\sqrt{3}G - H)/2\sqrt{2}. \end{aligned} \quad (9)$$

Substituting the wave function (3) in the Schrödinger equation $\hat{H}\Psi = E\Psi$ with the Hamiltonian (3) and using (4), we find, after multiplying the equations for the functions G(R) and H(R) by $\Sigma_g S^0_{1/2,1/2}(1, 2; \mu)$ and $\Sigma_u S^1_{1/2,1/2}(1, 2; \mu)$, respectively, and integrating over the μ-meson coordinates:

$$\begin{aligned} &-\frac{1}{M}\Delta_R G + E_g G - \frac{1}{2M}K_{gg}G + \frac{4\pi}{3}g_i\beta_\mu\beta_N\sqrt{3}|\psi(0)|^2 H = EG, \\ &-\frac{1}{M}\Delta_R H + E_u H - \frac{1}{2M}K_{uu}H + \frac{4\pi}{3}g_i\beta_\mu\beta_N|\psi(0)|^2(\sqrt{3}G - 2H) = EH, \end{aligned} \quad (10)$$

where

$$\begin{aligned} K_{gg} &= \int \Sigma_g(\Delta_{\mathbf{R}_1} + \Delta_{\mathbf{R}_2})\Sigma_g\, d\mathbf{r}, \\ K_{uu} &= \int \Sigma_u(\Delta_{\mathbf{R}_1} + \Delta_{\mathbf{R}_2})\Sigma_u\, d\mathbf{r}. \end{aligned} \quad (11)$$

For sufficiently large distance between the protons (neglecting exponentially small terms and terms of order R^{-4}),

$$K_{gg} = 2\int \Sigma_g\Delta_{R_1}\Sigma_g\, d\mathbf{r} \approx \int \psi(r_1)\Delta_{r_1}\Psi(r_1)\, d\mathbf{r} = -1.$$

In this same approximation, $K_{uu} \approx K_{gg}$. The expressions $-K_{gg}/2M$ and $-K_{uu}/2M$ are thus the correction to the kinetic energy of motion of the proton in the mesic atom, which is usually taken into account by introducing the reduced mass. The equations for K(R) and L(R) follow from (9) and (10):

$$\begin{aligned} &\frac{1}{M}\Delta_R L + \varepsilon L - {}^1/_4(3E_g + E_u + 2)L + \frac{\sqrt{3}}{4}(E_g - E_u)K = 0, \\ &\frac{1}{M}\Delta_R K + (\varepsilon + \Delta\varepsilon)K - {}^1/_4(3E_u + E_g + 2)K + \frac{\sqrt{3}}{4}(E_g - E_u)L = 0, \end{aligned} \quad (12)$$

where

$$\varepsilon = E + \frac{1}{2} - \frac{1}{2M} - \frac{1}{4}\Delta\varepsilon,$$

and $\Delta\epsilon$ is the hyperfine structure energy (1).

The relative motion of the mesic atom and the proton is described by the function $L(R)$ in the upper hyperfine structure state, and by $K(R)$ in the lower state. Equation (12) thus describes the transfer of hyperfine structure energy into energy of relative motion (ϵ and $\epsilon + \Delta\epsilon$, for $R \to \infty$, are equal to the kinetic energy of relative motion in the upper and lower hyperfine structure states, respectively).

Finding the cross section for "jumping" of the μ meson is essentially a problem of inelastic scattering of slow particles.[2] Because of the smallness of the quantity $k_0 = \sqrt{M\Delta\epsilon} = 0.02$ (mesic units), there is a quite large region of free motion ($1 \ll R \ll 1/k_0$) in which we can neglect both the energies E_g and E_u as well as the energies ϵ and $(\epsilon + \Delta\epsilon)$. The cross section for the transition into the lower hyperfine structure state will be determined by the asymptotic form of the functions $L(R)$ and $K(R)$ in this region.

For sufficiently small R we can neglect ϵ and $(\epsilon + \Delta\epsilon)$ compared to E_g and E_u. In this region, Eqs. (10) for the radial functions $g(R) = RG(R)$ and $h(R) = RH(R)$ have the form

$$\frac{1}{M}\frac{d^2g}{dR^2} - \left(E_g + \frac{1}{2}\right)g = 0, \tag{13}$$

$$\frac{1}{M}\frac{d^2h}{dR^2} - \left(E_u + \frac{1}{2}\right)h = 0. \tag{14}$$

In the neighborhood of the minimum ($R_0 = 2$), the function E_g is well approximated by the Morse potential[3]

$$E_g = -{}^1\!/_2 + A\left[E^{-2\alpha(R-R_0)} - 2e^{-\alpha(R-R_0)}\right].$$

The parameters A (the depth of the well $E_g + \frac{1}{2}$ at the minimum) and α are determined from the exact values of $E_g(R)$, which are found by numerical integration:[4]

$$A = 0.1027;\ 2\alpha^2 A = d^2E_g/dR^2 = 0.0976;\ \alpha \approx 0.69.$$

For energy $\epsilon > 0$, the Schrödinger equation with the Morse potential has the exact solution

$$g = e^{-\xi/2}\left[\xi^{is}e^{-is\varphi_0}F(-n,\ 1+2is,\ \xi) - \xi^{-is}e^{is\varphi_0}F(-n^*, 1-2is,\ \xi)\right],$$

where

$$\xi = (2\sqrt{MA}/\alpha)\,e^{-\alpha(R-R_0)};\quad is = \sqrt{-M\varepsilon/\alpha};$$

$$n = \sqrt{MA}/\alpha - {}^1\!/_2 - is;$$

$$e^{2is\varphi_0} = \Gamma(1+2is)\,\Gamma(-n^*)/\Gamma(1-2is)\,\Gamma(-n).$$

In the region $1 \ll R - R_0 \ll 1/s$, for $\epsilon \to 0$, the solution becomes

$$g \sim R + \vartheta, \tag{15}$$

where

$$\vartheta = \frac{\varphi_0}{\alpha} - R_0 - \frac{1}{\alpha}\ln\frac{2\sqrt{MA}}{\alpha},$$

$$\varphi_0 = 2\psi(1) - \psi\left({}^1\!/_2 - \frac{\sqrt{MA}}{\alpha}\right);$$

$\psi(x)$ is the logarithmic derivative of the Γ-function and has poles at the points $0, -1, -2, \ldots$. In addition, the quantity $(\frac{1}{2} - \sqrt{MA}/\alpha) \approx -0.88$ is near to -1, which leads to a quite large value of $\varphi_0 \approx 11$. This is related to the fact that there is a virtual level of the mesic molecule near to zero energy, which results in resonance.[5]

To find the asymptotic form of $h(R)$, we note that at sufficiently large distances [for which, however, Eq. (14) is still valid]

$$E_{g,u} \approx -\frac{1}{2} \mp \frac{2}{e}Re^{-R}$$

[The solution of Eq. (4) in the approximation (9) gives $E_{g,u} \approx {}^1\!/_2 \mp {}^2\!/_3 Re^{-R}$.]

At distances R for which the expression $(2M/e)Re^{-R}$ ($M = 8.87$) is still important, it can be approximated well by the function

$$(2M/e)\,Re^{-R} \approx e^{-\alpha'(R-R_1)};\quad R_1 \approx 3;$$

$$\alpha' = 1 - 1/R_1 \approx 0.7.$$

In this case, Eq. (14) is reduced to the Bessel equation by the substitution $\xi = \exp\{-\alpha'(R-R_1)/2\}$. Since for a repulsive state there should be no solution for which $h(R)$ increases exponentially with decreasing R, we find

$$h(K) = K_0\left(\frac{2}{\alpha'}e^{-\alpha'(R-R_1)/2}\right),$$

where $K_0(x)$ is a Bessel function of imaginary argument.

Since $K_0(x) \to \ln(2/x)$ for $x \to 0$, we have, for $R - R_1 \gg 1$,

$$h(R) \sim R + \omega;\quad \omega = -R_1 - (2/\alpha')\ln(1/\alpha'). \tag{16}$$

Using (9), (15), and (16) we easily find the asymptotic form of the radial functions $p(R) = R \times K(R)$ and $q(R) = RL(R)$:

$$p = (\lambda - \sqrt{3}\,\nu)R + (\lambda\vartheta - \sqrt{3}\,\nu\omega),$$

$$q = -(\lambda\sqrt{3} + \nu)R - (\lambda\sqrt{3}\,\vartheta + \nu\omega).$$

The constants λ and ν are to be fixed by using the condition that $p(R)$ contain only an outgoing wave e^{ik_0R} for $R \to \infty$, and from the normalization to unity of the probability density in the incoming wave q(R).

Thus

$$p \approx \frac{\sqrt{3}\,(\vartheta-\omega)}{4-ik_0\,(3\omega+\vartheta)}\,(1+ik_0R),$$
$$q \approx R + \frac{(3\vartheta+\omega)-4ik_0R}{4-ik_0\,(3\omega+\vartheta)}\,. \qquad (17)$$

The cross section for transition to the lower hyperfine structure state is

$$\sigma = 4\pi\,\frac{k_0}{k}\,\frac{3\,(\vartheta-\omega)^2}{16+k_0^2\,(3\omega+\vartheta)^2}\,, \qquad (18)$$

or, inserting the numerical values of the parameters,

$$\sigma \approx \frac{3}{4}\,\frac{\varphi_0^2}{\alpha^2}\,\pi\,\frac{k_0}{k} \approx 150\,\pi\,\frac{k_0}{k}\,. \qquad (19)$$

The statistical weight of the spin ½ state of the system consisting of a μ meson and two protons is equal to ⅓. The probability of transition per unit time to the lower hyperfine structure state is

$$W = {}^1/_3\,\sigma v N \approx 50\pi a_\mu^2 v_0 N \approx 2\cdot 10^9\ \mathrm{sec}^{-1} \qquad (20)$$

$$(N = 4\cdot 10^{22}\ \mathrm{cm}^{-3};\ v_0 = \sqrt{\Delta\varepsilon/M} \approx 5\cdot 10^5\ \mathrm{cm/sec};$$
$$a_\mu = \hbar^2/m_\mu e^2 = 2.55\cdot 10^{-11}\ \mathrm{cm}).$$

This value may be somewhat high. Inclusion of terms $\sim m_\mu/M$ in the interaction potential energy and deviations E_g from the Morse function may cause a shift from resonance, and consequently, reduce the value of σ. However, since the transition probability (20) is at least three orders of magnitude greater than the probability of decay of the μ meson ($W_{dec} = 0.5\times 10^6\ \mathrm{sec}^{-1}$), the conclusion that there is an almost complete transfer of the μ-mesic atom into the lower hyperfine structure state undoubtedly remains valid.*

*Since the wavelength of the μ meson is comparable to the dimensions of the H_2 molecule, it is necessary to treat the scattering of the μ meson by the H_2 molecule, and not by a free proton. This treatment leads to a difference in the cross sections for the process in para- and orthohydrogen, but does not change the basic conclusions of the present paper.

This result has two unusual consequences.

First, μ^- mesons entering hydrogen should be completely depolarized.

Second, neutrons produced from μ^- capture by protons in the reaction $\mu^- + p = n + \nu$ should be completely polarized in the direction of their motion. For, since the capture of the μ meson occurs from the lower hyperfine structure state (the total angular momentum of the $\mu + p$ system is zero), and the neutrino is polarized along its direction of motion, our statement follows from the conservation laws.*

In conclusion, I express my deep gratitude to L. D. Landau and Ia. B. Zel'dovich for interest in the work and valuable remarks.

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*For capture of the μ meson from both hyperfine structure levels, the polarization of the neutrons along their direction of motion would be a relativistic effect $\sim v_n/c$.