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STABILITY OF SHOCK WAVES IN RELATIVISTIC HYDRODYNAMICS

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Stability against small perturbations of the discontinuity surface is investigated for shock waves in an arbitrary medium, described by relativistic equations for an ideal fluid.*

1. INTRODUCTION

THE concept of an ideal fluid is applicable in two limiting cases of relativistic hydrodynamics: at sufficiently low temperatures, when the mean number of produced pairs is much smaller than the number of virtual particles, and in the ultra-relativistic case of super-high temperatures, when the mean number of pairs is much larger than the number of virtual particles. In fact, it follows from the equations†

$$\partial T_i^k / \partial x^k = 0, \qquad T_{ik} = w u_i u_k + p g_{ik} \tag{1.1}$$

that the entropy flux density satisfies the equation

$$\frac{\partial \sigma^i}{\partial x^i} = -\frac{\mu}{T}\frac{\partial n^i}{\partial x^i}, \quad \sigma^i = \sigma u^i, \ n^i = n u^i. \tag{1.2}$$

In the first case, which we shall call relativistic, the equation of continuity holds for the number of particles in zero approximation of the ratio of the mean number of pairs to the number of particles

$$\partial n^i / \partial x^i = 0. \tag{1.3}$$

In the ultra-relativistic limit the chemical potential is equal to zero in zeroth approximation of the ratio of virtual particles to the number of pairs:

$$\mu = 0. \tag{1.4}$$

In both limiting cases (and only then), the entropy is conserved:

$$\partial \sigma^i / \partial x^i = 0. \tag{1.5}$$

It should be noted that, as shown by Khalatnikov,[3] it is possible to obtain the equations for the ultra-relativistic case, (1.1) and (1.5) from Eqs. (1.1) and (1.3) of the relativistic case by a simple substitution:

$$w \to T\sigma, \ n \to \sigma, \tag{1.6}$$

and putting $\mu = 0$.* We shall make use of this result later.

*In classical hydrodynamics this problem was solved by D'iakov.[1]

†Our notation follows Ch. XV of the book by Landau and Lifshitz.[2]

*Thermodynamical relations necessary for the completeness of the system (with exception of the equation of state) remain valid after the substitution (1.6), in view of Eq. (1.4): if $\mu = 0$, n does not enter into the thermodynamical identities, and $w = T\sigma$. The equations for the ultra-relativistic case can therefore be obtained at any stage from the relativistic equations if one does not use the equation of state explicitly. If the boundary conditions are obtained directly from the equations, or if there is no condition imposed on n at the boundary, then the above procedure permits us to obtain the corresponding solution for the ultra-relativistic case from the solution of the boundary problem.

If we note that conditions at hydrodynamic discontinuities do not follow from equations of the ideal fluid, but represent additional physical requirements (following from the equations with dissipation), it becomes clear that the substitution (1.6) is applicable to tangential and is inapplicable to normal discontinuities, since n enters the boundary conditions for the latter.

It should also be noted that considerations of stability of shock waves against motion of the surfact of discontinuity as a whole, which lead to the following inequalities in the non-relativistic case:

$$v < s, \quad \bar{v} > \bar{s}, \tag{1.7}$$

are fully valid in relativistic hydrodynamics. Here v and $\bar{v}$ denote the component normal to the surface of discontinuity of the three-dimensional hydrodynamical velocity in the system where the discontinuity is at rest, while s and $\bar{s}$ is the velocity of sound in the eigen-system; the bar denotes values in the part of the medium before the oncoming shock wave.

In fact, according to Einstein's formula, we have for the velocity of the disturbance in the system where the discontinuity is at rest,

$$(v+s)/(1+vs), \quad (\bar{v}+\bar{s})/(1+\bar{v}\bar{s}).$$

The sign of the above expressions is determined by the numerator only, i.e., the choice of the sign determining the possibility or impossibility of propagation of the disturbance from the surface of discontinuity leads to the same inequalities as obtained from the Galileo transformation. It is necessary for the sake of stability that the number of parameters describing the perturbation should not exceed the number of equations. The inequalities (1.7) lead to such a minimum number of parameters both in the non-relativistic[2] and relativistic and ultra-relativistic cases. It should be noted that, although the number of equations in the ultra-relativistic case is smaller by one, the number of parameters decreases also by the same amount since there are no separate sound and entropy disturbances.

2. VARIATIONAL EQUATIONS OF MOTION

According to (1.7), the motion of a shock wave should be supersonic in the part of the medium into which it enters, and subsonic in the region left behind. This means that disturbances of the surface of discontinuity will influence only the flow behind the wave. Let us choose the axis $y = x_2$ in the direction of the normal to the surface of discontinuity, aligned with the hydrodynamical velocity v, and the axis $x = x_1$ in the direction of the wave vector k_1 of the disturbance, which will be specified as a traveling wave of small amplitude η:

$$y = \eta \exp i(k_1 x^1 + k_0 x^0). \tag{2.1}$$

Variation δA of an arbitrary variable A behind the shock wave will be of the form of a traveling wave with wave vectors k_1 and k_2 and frequency k_0

$$\delta A \sim \exp(ik_l x^l), \quad \partial \delta A/\partial x_l = ik_l \delta A.$$

Next we shall write linearized equations for the amplitudes:

$$k_l \delta T_i^l = 0; \quad \delta T_i^l = \delta w u_i u^l + w\delta u_i u^l + w u_i \delta u^l; \tag{2.2}$$

$$k_l \delta n^l = 0; \ \delta n^l = \delta n u^l + n \delta u^l. \tag{2.3}$$

Multiplying Eq. (2.2) by u^i, and making use of Eq. (2.3) and of the thermodynamical identity

$$d\frac{w}{n} = Td\frac{\sigma}{n} + \frac{1}{n}dp, \tag{2.4}$$

we obtain easily

$$k_l u^l \delta \frac{\sigma}{n} = 0. \tag{2.5}$$

It is convenient to transform Eq. (2.2) into

$$k_l u^l \{u_i \delta p + w\delta u_i\} + k_i \delta p = 0. \tag{2.6}$$

It can be seen from Eq. (2.5) that there are two types of solutions. The first represents turbulent entropy waves carried by the liquid current

$$k_l^{(1)} u^l = 0. \tag{2.7}$$

It follows immediately from (2.3) and (2.6) that

$$k_l^{(1)} \delta u^{(1)l} = 0, \quad \delta p^{(1)} = 0. \tag{2.8}$$

The second type of solutions (disturbances of the sound-wave type) is

$$k_l^{(2)} u^l \neq 0. \tag{2.9}$$

Omitting the superscripts in $k_l^{(2)}$ and $\delta p^{(2)}$, we have

$$\delta u_i^{(2)} = -\delta p \frac{u_i k_l u^l + k_i}{w k_l u^l}. \tag{2.10}$$

Multiplying Eq. (2.6) by k_i we obtain an equation for $k^{(2)}$

$$(k_l u^l)^2 (1 - s^{-2}) + k_l k^l = 0. \tag{2.11}$$

In the ultra-relativistic case, all relations are obtained from the above through the substitution (1.6). The solutions of the first type represent turbulent waves. In Eq. (2.11), s denotes the velocity of sound: $s^{-2} = (\partial e/\partial p)_{\sigma/n}$.

3. BOUNDARY CONDITIONS AT THE SURFACE OF DISCONTINUITY

The continuity condition

$$[T_i'^f] = 0, \quad i = f, \tau, 0. \tag{3.1}$$

is satisfied at a normal discontinuity. The prime denotes a system of coordinates in which the disturbed discontinuity is at rest, f is the normal, and τ the tangent to the surface of discontinuity. In addition, we have in the relativistic case the following condition:

$$[n'^f] = 0. \tag{3.2}$$

The Rankine-Hugoniot equation, relating the thermodynamical variables at the discontinuity, is a consequence of Eqs. (3.1) and (3.2). It follows from the equation of state and the Rankine-Hugoniot equation that only one thermodynamical variable is independent at the discontinuity (we shall choose the pressure p). It should be noted that we deal with variables on one side of the surface of discontinuity only — behind the shock wave.

In the ultra-relativistic case it is clear from thermodynamical identities that there is only one independent thermodynamical variable at all. In both cases, therefore, we have at the discontinuity

$$\delta w = q \delta p, \tag{3.3}$$

where, in the relativistic case

$$q = (\partial w / \partial p)_{\mathrm{H}} \tag{3.4}$$

is the derivative along the Hugoniot adiabatic. In ultra-relativistic case, when there is no Hugoniot equation, but $\mu = 0$,

$$q = dw / dp. \tag{3.5}$$

The system (3.1) and (3.3) is the complete system of boundary conditions at the perturbed discontinuity in linear approximation for both limiting cases.

We shall write down certain relations at the discontinuity, omitting the primes. From Eq. (3.1) we have

$$[wu_f^2] = -[p], \quad [wu_f u_\tau] = 0, \quad [wu_f u_0] = 0. \tag{3.6}$$

It follows that the discontinuities of the components of the three-dimensional velocity are:

$$[u_\tau / u_0] = 0, \quad [u_f / u_0] = -[p] / wu_f u_0. \tag{3.7}$$

Furthermore,

$$[wu_\tau^2] = \frac{u_\tau^2}{u_0^2}[wu_0^2], \quad [wu_0^2] = \frac{[w]-[p]}{1-u_\tau^2/u_0^2}; \tag{3.8}$$

$$\left[\frac{u_0}{u_f}\right] = \frac{1}{wu_f^2}\frac{\bar{u}_0}{\bar{u}_f}[p], \qquad [wu_0^2] = \frac{u_0\bar{u}_0}{u_f\bar{u}_f}[p]; \tag{3.9}$$

$$[e] = [w]-[p] = \left(1-\frac{u_\tau^2}{u_0^2}\right)\frac{u_0\bar{u}_0}{u_f\bar{u}_f}[p]. \tag{3.10}$$

The last relation yields for $u_\tau = 0$

$$[e]/[p] = u_0\bar{u}_0 / u_f\bar{u}_f. \tag{3.11}$$

All the above relations are valid for the ultra-relativistic case as well. Making use of Eq. (3.2), $u_f = jV$, where $V = 1/n$ and $[j] = 0$, we obtain

$$j^2 = -[p]/[wV^2]. \tag{3.12}$$

Since, according to (3.6) we have

$$[w^2u_f^2u_\tau^2] = [w^2u_f^2u_0^2] = 0,$$

then it follows that

$$j^2 = -[w^2V^2]/[w^2V^4]. \tag{3.13}$$

Eliminating j^2 from Eqs. (3.12) and (3.13), we obtain the Rankine-Hugoniot equation:[4]

$$[p] = [w^2V^2]/(wV^2 + \bar{w}\bar{V}^2). \tag{3.14}$$

4. DERIVATION OF THE CHARACTERISTIC EQUATION

In order to write down the perturbed boundary conditions it is necessary to carry out a transformation from the system (I) in which the unperturbed discontinuity surface is at rest, to the system (II) where the normal velocity of the perturbed discontinuity vanishes. The Lorentz formulae of transformation to the moving (primed) system (the motion along the i-axis with a 4-velocity U^i) are:

$$A_i' = U^0A_i + U^iA_0, \quad A_0' = U^0A_0 + U^iA_i, \tag{4.1}$$

where there is no summation over double indices.

From the equation of the perturbed surface (2.1) we find the normal $f_\alpha(-ik\eta, 1, 0)$ and the tangent $\tau_\alpha(1, ik\eta, 0)$, $k \equiv k_1$. The velocity of the surface of discontinuity in (I) is

$$D_y = ik_0\eta, \quad D^0 = 1. \tag{4.2}$$

The perturbed hydrodynamical velocities in (I) are

$$u_y + \delta u_y, \quad \delta u_x, \quad u_0 + \delta u_0; \quad \bar{u}_y, \ 0, \ \bar{u}_0. \tag{4.3}$$

We next project on the directions of f and τ:

$$D_f = ik_0\eta; \quad D_\tau = 0; \quad u_f = u_y + \delta u_y; \quad \bar{u}_f = \bar{u}_y;$$
$$u_\tau = \delta u_x + ik\eta u_y; \quad \bar{u}_\tau = ik\eta\bar{u}_y.$$

By means of Eq. (4.1) we go over to system (II), assuming that $U_i = D_f = ik_0\eta$:

$$\begin{gathered} u_f' = u_y + \delta u_y + ik_0\eta u_0, \quad u_\tau' = \delta u_x + ik\eta u_y, \\ u_0' = u_0 + \delta u_0 + ik_0\eta u_y, \\ \bar{u}_f' = \bar{u}_y + ik_0\eta\bar{u}_0, \quad \bar{u}_\tau' = ik\eta\bar{u}_y, \quad \bar{u}_0' = \bar{u}_0 + ik_0\eta\bar{u}_y. \end{gathered} \tag{4.4}$$

It follows from $u_iu^i = -1$ that

$$u_y\delta u_y = u_0\delta u_0. \tag{4.5}$$

We shall denote by ~ the perturbed variables

$$\tilde{w} = w + \delta w, \quad \tilde{\bar{w}} = \bar{w}, \quad \tilde{p} = p + \delta p, \quad \tilde{\bar{p}} = \bar{p}. \tag{4.6}$$

Boundary conditions (3.1) become then

$$[\widetilde{w}u_f'^2] = -[\widetilde{p}], \quad [\widetilde{w}u_f'u_\tau'] = 0, \quad [\widetilde{w}u_f'u_n'] = 0,$$

substituting Eqs. (4.4) — (4.6) into the above and eliminating η, we find

$$\delta u_x = \frac{k}{k_0} P\delta p, \quad P = \frac{1 + u_y^2(2-q)}{2u_0wu_y^2([e]/[p] - 1)}; \qquad (4.7)$$

$$\delta u_y = Q\delta p, \quad Q = -(1 + qu_y^2)/2wu_y. \qquad (4.8)$$

In carrying out the transformations it is convenient to use the formulae of Sec. 3.

We shall now make use of the variational equations of motion

$$\delta u_i = \delta u_i^{(1)} + \delta u_i^{(2)}.$$

According to Eqs. (4.5) and (2.7), the first of conditions (2.8) becomes

$$ku_yu_0\delta u_x^{(1)} + k_0\delta u_y^{(1)} = 0.$$

Multiplying Eq. (4.7) by ku_yu_0 and Eq. (4.8) by k_0, adding them together, and then substituting $\delta u_i^{(2)}$ from Eq. (2.1) we obtain, since $\delta p \neq 0$

$$k^2k_0u_yu_0 + k_0^2(u_yk_lu^l + k_y) = -wk_lu^l(k^2u_yu_0P + k_0^2Q). \qquad (4.9)$$

Equations (4.9) and (2.11) constitute a system of characteristic equations for k_0 and k_y as functions of k in both limits of relativistic hydrodynamics. The variables P and Q which are independent of k^l are given by Eqs. (4.7) and (4.8) and the parameter q by Eqs. (3.4) and (3.5) for the relativistic and ultra-relativistic cases respectively.

5. THE CHARACTERISTIC EQUATION

The dispersion relations obtained above describe, in the coordinate system I in which the medium as a whole moves with the 4-velocity u_i, the possible motions of the medium, due to perturbations of the boundary conditions at the discontinuity, taking place behind the shock wave. Let us now go over to a system III where the undisturbed medium behind the shock wave is at rest (we denote variables in III by $''$). We have

$$k_y = u^0k_y'' - k_0''u_y, \quad k_x \equiv k = k'', \quad k_0 = u^0k_0'' - k_y''u_y,$$

$$u_x'' = u_y'' = 0; \quad u_0'' = -1.$$

Let us put $k^{0''} = \Omega$. Then $k_lu^l = -\Omega$.

Equation (2.11) becomes in III

$$k^2 + k_y''^2 = \Omega^2/s^2. \qquad (5.1)$$

Let us introduce polar coordinates

$$k'' = (\Omega/s)\sin\varphi; \quad k_y'' = (\Omega/s)\cos\varphi. \qquad (5.2)$$

Equation (5.1) becomes then an identity. In system I we have

$$k_y = \Omega\left(\frac{u^0}{s}\cos\varphi + u_y\right), \quad k = \frac{\Omega}{s}\sin\varphi,$$
$$k_0 = -\Omega\left(u^0 + \frac{u_y}{s}\cos\varphi\right). \qquad (5.3)$$

Substituting into Eq. (4.9) we obtain the characteristic equation for $\cos\varphi$:

$$\cos^2\varphi\left\{\frac{u^{02}u_y}{s^2} - \frac{u_yu^0}{s^2}wP - \frac{u_y^2wQ}{s^2}\right\}$$
$$+ \cos\varphi\left\{\frac{u^{03}}{s} + \frac{u^0u_y^2}{s^3} - 2\frac{u^0u_y}{s}wQ\right\} \qquad (5.4)$$
$$+ \left\{\frac{u_yu^{02}}{s^2} - u^{02}wQ + \frac{u_yu^0}{s^2}wP\right\} = 0.$$

Equation (5.4) corresponds, in non-relativistic hydrodynamics, to D'iakov's characteristic equation.[1] Like that equation, Eq. (5.4) is quadratic, which makes it possible to study it by the method of Ref. 1, as indicated by Landau.

6. INVESTIGATION OF THE CHARACTERISTIC EQUATION

Using Eq. (5.2), we can express Ω in terms of k (assumed to be real):

$$\Omega = sk/\sin\varphi; \qquad (6.1)$$

$$k_y = \frac{sk}{\sin\varphi}\left(\frac{u^0}{s}\cos\varphi + u_y\right), \quad k_0 = -\frac{sk}{\sin\varphi}\left(u^0 + \frac{u_y}{s}\cos\varphi\right). \qquad (6.2)$$

The conditions of instability are

$$\operatorname{Im} k_0 < 0, \quad \operatorname{Im} k_y > 0. \qquad (6.3)$$

Let us introduce real variables ρ and ψ

$$\cot(\varphi/2) = \rho e^{i\psi} \qquad (6.4)$$

Using the inequalities

$$u^0/s - u_y > 0, \quad u^0 - u_y/s > 0,$$

we can write Eq. (6.3) in the following form:

$$|x| > 1, \text{ where } x = \frac{1+\cos\varphi}{1-\cos\varphi}\cdot\frac{u^0 + u_y/s}{u^0 - u_y/s}. \qquad (6.5)$$

Since it is necessary for instability that k_0 and k_y were complex, then $\cos\varphi$ and $\sin\varphi$ cannot be real simultaneously. We have therefore

$$|\cos\varphi| > 1 \quad \text{for } \operatorname{Im}\cos\varphi = 0. \qquad (6.6)$$

Rearranging Eq. (5.4) in terms of x we obtain

$$x^2(u^{02} - u_y^2/s^2)(u^0/s - wQ)$$
$$+ 2x\{2(u^0u_y/s^2)wP - wQ(u^{02} - u_y^2/s^2)\} \qquad (6.7)$$
$$-(u^{02} - u_y^2/s^2)(u^0/s + wQ) = 0.$$

It is necessary to find such relations between the coefficients that inequality (6.5) is satisfied for at least one root of Eq. (6.7). Let us denote the roots by x_1 and x_2; we have then the following well-known inequalities for the quadratic equation $ax^2 + bx + c = 0$:

$$|a|>|c|, \quad |b|<|a+c|, \quad |x_1|, \quad |x_2|<1; \qquad (6.8)$$

$$|a|<|c|, \quad |b|<|a+c|, \quad |x_1|, \quad |x_2|>1; \qquad (6.9)$$

$$|a|\leqq|c|, \quad |b|>|a+c|, \quad |x_1|<1, \; |x_2|>1. \qquad (6.10)$$

It is easy to see that inequalities (6.9) are in contradiction with Eq. (6.7). The conditions of instability are therefore represented by (6.10) where x_1 and x_2 are real.

According to Eq. (6.5) $\cos \varphi$ is then real as well, i.e., relation (6.6) should be satisfied. This is possible only for $x < 0$. Consequently, for instability it is necessary that one root of Eq. (6.7) lies in the interval $(-\infty, -1)$. The condition for this is

$$a>0, \quad a-b+c<0; \qquad (6.11)$$

$$a<0, \quad a-b+c>0. \qquad (6.12)$$

We find $a - b + c$

$$a-b+c = -4u^0 u_y s^{-2} wP. \qquad (6.13)$$

It follows that for the regions of absolute instability

$$q > 2 + u_y^{-2}, \qquad (6.14)$$

or

$$q < -u_y^{-2}(1 + 2u^0 u_y/s). \qquad (6.15)$$

Relations (6.14) and (6.15) correspond, in non-relativistic hydrodynamics, to the regions of instability obtained by D'iakov.

Inequality (6.14) can be written in a more revealing form using the identity $q \equiv (\partial e/\partial p)_H + 1$

$$v^2 > s_H^2, \text{ where } s_H^{-1} = \sqrt{\left(\frac{\partial e}{\partial p}\right)_H} \qquad (6.16)$$

is the "sound velocity" on the Hugoniot adiabatic.

In the ultra-relativistic case one should substitute in Eq. (6.7) s^2 for s_H^2 according to Eq. (3.5). Condition (6.16) becomes then identical with that of the instability region (1.7), that is, it does not represent an additional limitation. Inequality (6.15) cannot be satisfied since $q > 0$.

For ultra-relativistic shock waves, therefore, there is no region of absolute instability.

7. SPONTANEOUS EMISSION OF SOUND FROM THE DISCONTINUITY

An interesting result of the work of D'iakov is the discovery of regions in which the solution is of the form of undamped (in linear approximation) waves propagating from the discontinuity. We cannot exclude the possibility that, in spontaneous emission of sound, the energy of shock wave is fed into the emitted waves during such a long period that we have to consider the phenomenon as separate from the cases of stable and unstable motion.

We shall find the condition for spontaneous emission of sound from the discontinuity. In system (I) where the unperturbed discontinuity is at rest (and the fluid moves with velocity u_i), the velocity V_α of waves emitted from the discontinuity should have a positive normal component V_y (in contrast to waves incident upon the discontinuity, for which V_y is negative). The condition for emission is

$$\operatorname{Im} k^0 = 0, \quad \operatorname{Im} k_y = 0, \quad V_y > 0. \qquad (7.1)$$

Velocity V_α can be found, for example, by differentiating Eq. (2.11)

$$V_\alpha = \partial k^0/\partial k^\alpha, \quad \alpha = 1, 2, 3. \qquad (7.2)$$

We obtain:

$$V_\alpha = \frac{k_i u^i u_\alpha (1-s^{-2}) + k_\alpha}{k_i u^i u^0 (1-s^{-2}) + k^0}. \qquad (7.3)$$

Using Eq. (5.3) we shall express V_y through variables measured in system (III) where the fluid behind the discontinuity is at rest:

$$V_y = \frac{u_y/s + u^0 \cos\varphi}{u^0/s + u_y \cos\varphi}. \qquad (7.4)$$

The condition of spontaneous emission of sound (7.1) becomes

$$-M < \cos\varphi < 1, \qquad (7.5)$$

where M is the Mach number; $M = v/s = u_y/su^0$.

The waves emitted by the discontinuity can move, with respect to fluid at rest, in the direction opposite to the motion of the shock wave ($0 < \cos\varphi < 1$) as well as following it ($-M < \cos\varphi < 0$). In the latter case the emitted sound wave lags behind the shock wave, while continuing to propagate in the same direction as seen by observer situated in system (III).

Inequality (7.5) remains valid in non-relativistic hydrodynamics.[5] We shall find now by means of Sturm's theorem the condition that the roots of Eq. (5.4) lie outside the instability region (6.14) — (6.15) and that at least one of them satisfies the relation (7.5). For the region of spontaneous emission of sound we obtain the double inequality

$$-\frac{1}{u_y^2}\left(1+2u^0\frac{u_y}{s}\right)<q$$
$$<-\frac{1}{u_y^2}\,\frac{1-M^2-(M^2/u_y^2\alpha)(1+2u_y^2)}{1-M^2+M^2/u_y^2\alpha} \qquad (7.6)$$

where

$$\alpha=[w]/[p]-2=(1-v\bar{v})/v\bar{v}.$$

In the ultra-relativistic case $q = 4$, $\alpha = 2$, $s^{-2} = 3$.

The left-hand side of inequality (7.6) is satisfied for all values of v. The right-hand side can be written in the following form:

$$v^4-{}^4/_9 v^2+{}^1/_{27}>0,$$

which is satisfied by $v < 1/3$ and $v > 1/\sqrt{3}$. The latter case corresponds to instability and should be excluded. Making use of the relation $v\bar{v} = 1/3$ which follows for the ultra-relativistic case from Eq. (3.11) we obtain that for emission it is necessary that $\bar{v} > 1$, i.e., the velocity of propagation of shock waves in the part of the medium ahead of the front should be greater than velocity of light. The spontaneous emission of sound is therefore impossible in ultra-relativistic hydrodynamics and, when the relation (1.7) is satisfied, shock waves are absolutely stable.

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Note added in proof (December 22, 1957). In an article published in November, 1957, Iordanskii[6] treats by a somewhat different method the stability of non-relativistic shock waves with respect to small perturbations of the fluid behind the wave. The regions of absolute instability found by him are identical with those of D'iakov. Since D'iakov [who required for the waves emitted by the discontinuity that $0 < \cos\varphi < 1$ instead of relation (7.5)] did not find explicitly the region of spontaneous sound emission, there is some discrepancy with results of Iordanskii, which disappears when the latter are compared with those of Ref. 5. Iordanskii explained the discrepancy by the fact that the perturbations considered by him were more general than perturbations of the surface of discontinuity only. It can be easily seen, however, that only the perturbations from the region of compression which were reflected by the shock wave can contribute to the solution (otherwise an unstable flow without shock waves would occur). If that is the case, however, perturbations can be always considered as originating at the shock wave, as was indeed assumed by D'iakov.

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