ON THE THEORY OF THE ANOMALOUS SKIN EFFECT IN METALS

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The anomalous skin effect is considered, using Landau's theory of a Fermi liquid, as extended by the author[1] to the case of a degenerate electron fluid. We show that the information we get about the Fermi surface by measuring the surface impedance does not depend on whether we consider the conduction electrons to be a gas or to be a degenerate fluid. We discuss the problem of how to define the parameters for an isotropic model of a metal.

1. It is well known that in the region of the anomalous skin effect[2] it is impossible to use such macroscopic characteristics of the metal as conductivity, and that it becomes necessary to use some model for the behavior of the electron conductivity in a metal. It is therefore usual in the theory of the anomalous skin effect[3] to use the model of an electron gas, although in actual fact the interaction between the conduction electrons in a metal is relatively strong. In this connection one should consider this phenomenon by starting from the theory of Fermi-liquids.[4]

If we confine our interest to the case of weak fields, we may assume that the electron distribution differs only slightly from its equilibrium value,

$$f = f_0 + f_1(\mathbf{p}, \mathbf{r}), \quad f_1 \ll f_0 = \frac{2}{(2\pi\hbar)^3}\left\{\exp\left[\frac{\varepsilon_0(\mathbf{p}) - \mu}{kT}\right] + 1\right\}^{-1},$$

where $\epsilon_0(\mathbf{p})$ is the electron energy in the equilibrium state. We find then from Eq. (14) of Ref. 1 (see also Ref. 4) that

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_0\frac{\partial f_1}{\partial \mathbf{r}} - \frac{\partial f_0}{\partial \mathbf{p}}\int d\mathbf{p}'\,\frac{\partial f_1(\mathbf{p}', \mathbf{r})}{\partial \mathbf{r}}\Phi(\mathbf{p}, \mathbf{p}') + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = -\frac{1}{\tau}f_1, \tag{1}$$

where $\mathbf{v}_0 = \partial\epsilon_0/\partial\mathbf{p}$, τ is the relaxation time and $\Phi(\mathbf{p}, \mathbf{p}')$ describes the correlation of the particles.

To develop a theory of the anomalous skin effect it is necessary to solve Eq. (1) combined with the Maxwell equations. In that case the current density is, according to Landau,

$$\mathbf{j} = e\int\frac{\partial\varepsilon}{\partial\mathbf{p}}f d\mathbf{p} = e\int\left(\frac{\partial\varepsilon_0}{\partial\mathbf{p}}f_1 + \frac{\partial\varepsilon_1}{\partial\mathbf{p}}f_0\right)d\mathbf{p} = e\int d\mathbf{p}\left\{\mathbf{v}_0 f_1 - \frac{\partial f_0}{\partial\mathbf{p}}\int d\mathbf{p}'\,\Phi(\mathbf{p}, \mathbf{p}')f_1(\mathbf{p}')\right\}. \tag{2}$$

2. Let the metal occupy the half-space $z \geq 0$. The function f_1 can then be written as follows

$$f_1(\mathbf{p}, z, t) = e\frac{\partial f_0}{\partial\varepsilon}\psi(\mathbf{n}, z)e^{i\omega t} \quad (\mathbf{n} = \mathbf{v}_0/v_0). \tag{3}$$

The transport equation (1) and the equation for the electric field can now be written

$$\frac{1}{l_{\text{eff}}}\psi(\mathbf{n}, z) + \mathbf{kn}\frac{\partial}{\partial z}\left\{\psi(\mathbf{n}, z) + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}', z)\right\} + \mathbf{nE} = 0, \tag{4}$$

$$\mathbf{E}'' + \varepsilon_0\left(\frac{\omega}{c}\right)^2\mathbf{E} = -\frac{8\pi i\omega e^2}{c^2(2\pi\hbar)^3}\int dS\mathbf{n}\left\{\psi(\mathbf{n}, z) + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}', z)\right\}, \tag{5}$$

where $\mathbf{k}$ is a unit vector in the direction of the z-axis, dS is an element of the Fermi surface, ϵ_0 the dielectric constant, excluding the contribution from the conduction electrons, $l_{\text{eff}} = v_0\tau/(1 + i\omega\tau)$, and finally

$$F(\mathbf{n}, \mathbf{n}') = [2/(2\pi\hbar)^3]\,\Phi(\mathbf{p}, \mathbf{p}')/v_0(\mathbf{n}) \tag{6}$$

Solving (4) and (5), we can find an experimentally measured quantity, namely the surface impedance of the metal.

3. In the region of the normal skin effect, where the inhomogeneity of the electron distribution in space is not important, the solution of Eq. (4) is of the form $\psi = -\ell_{eff}(\mathbf{n}\cdot\mathbf{E})$. This leads to the following expression for the current,

$$j_\alpha = 2e^2(2\pi\hbar)^{-3}\int dS l_{\text{eff}}\left\{n_\alpha n_\beta + \int dS' F(\mathbf{n}', \mathbf{n})\, n'_\alpha n_\beta\right\}E_\beta = \sigma_{\alpha\beta}E_\beta. \tag{7}$$

Using expression (7) we can write down an expression for the conductivity tensor or, more generally, the complex dielectric constant ($\epsilon' = \epsilon_0 - i4\pi\sigma/\omega$).

Assuming the metal to be isotropic, we can write Eq. (7) in the following form,

$$\mathbf{j} = \sigma\mathbf{E} = \frac{8\pi e^2 p_0^2 l_{\text{eff}}}{3(2\pi\hbar)^3}\left\{1 + p_0^2\int d\Omega\cos\chi F(\cos\chi)\right\}\mathbf{E}. \tag{8}$$

In the region of low frequencies ($\omega\tau \ll 1$) the complex dielectric constant is determined by static conductivity,

$$\sigma_0 = \frac{8\pi e^2 p_0^2 v_0\tau}{3(2\pi\hbar)^3}\left\{1 + p_0^2\int d\Omega\cos\chi F(\cos\chi)\right\} \equiv \frac{8\pi e^2 p_0^2 l}{3(2\pi\hbar)^3}, \tag{9}$$

where $\ell = v_0\tau\{1 + p_0^2\int d\Omega\cos\chi\, F(\cos\chi)\}$ plays the role of the electron mean free path. On the other hand, in the region of high frequencies ($\omega\tau \gg 1$), the dielectric constant has the form

$$\varepsilon = \varepsilon_0 - 4\pi e^2 N/m\omega^2, \tag{10}$$

where

$$N = \frac{8\pi p_0^2 v_0 m}{3(2\pi\hbar)^3}\left\{1 + p_0^2\int d\Omega\cos\chi F(\cos\chi)\right\}. \tag{11}$$

4. To solve the transport equation (4) in the region of the anomalous skin effect, where the spatial inhomogeneity plays an essential role, it is necessary to introduce a boundary condition at the metal surface.[5] We introduce such a condition for the function

$$\Psi(\mathbf{n}, z) = \psi(\mathbf{n}, z) + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}', z), \tag{12}$$

which determines the particle current as can be seen from Eq. (5). We assume that on the metal surface

$$\Psi(n, z=0)|_{\mathbf{kn}>0} = q\Psi(n_x, n_y, -n_z, z=0)|_{\mathbf{kn}<0}. \tag{13}$$

We can then easily obtain from Eq. (4) an integral equation for $\psi(\mathbf{n}, z)$, using Eq. (13) and taking into account the fact that as $z\to\infty$ $\Psi(\mathbf{n}, z)$ tends to zero. In the limit of a sharply expressed anomalous skin effect ($\ell_{eff}\to\infty$), this equation simplifies considerably and has the form

$$\Psi(\mathbf{n}, z) \equiv \psi(\mathbf{n}, z) + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}', z) = \psi_0(\mathbf{n}, z). \tag{14}$$

Here

$$\mathbf{kn} \geqslant 0:\quad \psi_0(\mathbf{n}, z) = -\int_0^z dz' \exp\left(-\frac{1}{l_{\text{eff}}}\left|\frac{z'-z}{\mathbf{kn}}\right|\right)\frac{\mathbf{nE}(z')}{\mathbf{nk}} - q\int_0^\infty dz' \exp\left(-\frac{z'+z}{l_{\text{eff}}\,\mathbf{kn}}\right)\frac{\mathbf{nE}(z')}{\mathbf{nk}}, \tag{15}$$

$$\mathbf{kn} \leqslant 0:\quad \psi_0(\mathbf{n}, z) = \int_z^\infty dz' \exp\left(-\frac{1}{l_{\text{eff}}}\left|\frac{z'-z}{\mathbf{kn}}\right|\right)\frac{\mathbf{nE}(z')}{\mathbf{nk}}, \tag{16}$$

where we have neglected the z-component of the electrical field, since it gives a negligibly small contribution. The function $\psi_0(\mathbf{n}, z)$ is the corresponding solution of the usual theory of the anomalous skin effect.[6]

In our considerations the function Ψ plays the same role in determining the current as the function ψ in the usual theory. We can therefore confirm, according to (15), that in the limit, as $\ell_{eff}\to\infty$, the expression which we obtain for the surface impedance coincides with the corresponding expression of the usual theory.

We must note that this result is obtained when we satisfy the boundary condition (13). This boundary condition for the function Ψ differs, generally speaking, from the usual boundary condition[3,5,6] imposed upon the function ψ. One can show, however, that if condition (13) is imposed upon the function $\psi(\mathbf{n}, z)$, the result just obtained is not changed.

The anomalous skin effect is an important means of studying the form of the Fermi surface. In that respect one can summarize the situation and say that if we consider instead of an electron gas a degenerate electron fluid, no complications arise in the analysis of the Fermi surface. Such an analysis can also be conducted using the methods of Kaganov and Azbel'.

5. We consider now some problems connected with the application of the above theory to an analysis of experimental results in the case where we can use an isotropic model of the metal to discuss the experiments (see, for instance, Ref. 7). The expression for the surface impedance in the case of a sharply expressed anomalous skin effect ($\ell_{\text{eff}} \to \infty$) is of the form[3]

$$Z_\infty = (\sqrt{3}\pi\omega^2 l/c^4\sigma_0)(1 + \sqrt{3}i), \tag{17}$$

where ℓ is the mean free path of the electrons and where

$$\sigma_0/l = {}^2/_3\, 4\pi e^2 p_0^2 (2\pi\hbar)^{-3}. \tag{18}$$

Using the isotropic model, we can try to determine from the experimental data the quantity

$$A = p_0^2 \int d\Omega \cos\chi F(\cos\chi), \tag{19}$$

by which (9) to (11) differ from the corresponding formulae of the usual theory. We should also use experimental data on the electronic heat capacity, on the dielectric constant (in the region of infrared radiation), and on σ_0/ℓ as determined from the anomalous skin effect. The measurement of the electronic heat capacity ($c_e = \gamma T$) gives us the quantity[4]

$$\gamma = \frac{2\pi^2 k^2}{3(2\pi\hbar)^3} \int \frac{dS}{v(\mathbf{n})} = \frac{2\pi^2}{3} k^2 \frac{4\pi p_0^2}{(2\pi\hbar)^3 v_0}, \tag{20}$$

where k is Boltzmann's constant. The determination of the dielectric constant in the infrared region of the spectrum, at frequencies much larger than the collision frequencies of the electrons but much smaller than the frequencies of the natural absorption, enables us to find the quantity N determined by equation (11).

The state of the electrons in the isotropic model of a metal is characterized by three parameters, a natural choice for which is the velocity v_0 of the electrons at the Fermi surface, their momentum p_0, and the quantity A [see Eq. (19)]. These quantities can all be determined with the above mentioned measurements, according to the following equations,

$$v_0 = \frac{\pi^2 k^2 (\sigma_0/l)}{e^2\gamma}; \quad p_0^2 = \frac{3(2\pi\hbar)^3 \sigma_0}{8\pi e^2 l}; \quad 1 + A = \frac{e^4}{\pi^2 k^2 m} \frac{\gamma N}{(\sigma_0/l)}. \tag{21}$$

In the table we give the values of N, γ and σ_0/ℓ for several metals[7,8,9] and the value of the parameter A determined from Eq. (21); this parameter characterizes the difference between a Fermi fluid and a gas. As can be seen from the table, this parameter is not small compared to unity. If we consider the use of an isotropic model for a metal as legitimate, this fact can be considered to indicate that the conduction electrons are considerably different from a gas. We note that among the different equations given in Ref. 7 for the velocity v_0 of the electrons at the Fermi surface, assuming the metal to be isotropic, the correct value under those assumptions agrees with our determination of v_0 according to Eq. (21). The corresponding value of v_0 is given in the table.

	$\gamma \cdot 10^{-3}$ (erg-cm^{-1}-deg.$^{-2}$)	$N \cdot 10^{-22}$	$(\sigma_0/l)\ 10^{-11}$ (Ref. 8)	A according to (21)	$v_0 \cdot 10^{-8}$ from (21)
Cu	1.02	3.3	13.9±0.4	−0.46	1.11
Ag	0.65	5.2	7.8±0.4	+0.9	1.04
Au	∼0.65	5.1	7.6±0.4	+0.8	∼1.0
Sn	1.03	<3.5	8.6±0.7	<+0.5	0.68
Al	1.46	5.4	18.4±1.4	+0.28	0.31

6. In this way we can estimate effects caused by the difference between an electron fluid and a gas, if we assume that the isotropic model of a metal is justified. In those cases, however, where the Fermi surface differs appreciably from a sphere, it is considerably more complicated to obtain a similar estimate. At any rate, we can say that, in order to determine the role of the correlation between particles, it is necessary to study the dielectric permeability tensor in the infrared range of the spectrum,

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^{(0)} - \frac{1}{\omega^2}\frac{8\pi e^2}{(2\pi\hbar)^3} \int dS v(\mathbf{n}) \left\{ n_\alpha n_\beta + \int dS' F(\mathbf{n}, \mathbf{n}') n'_\alpha n_\beta \right\}. \tag{22}$$

One can now hope that one can also elucidate the role of the function $F(\mathbf{n}, \mathbf{n}')$ in the case of an anisotropic metal, by using also data on the electronic heat capacity and on the form of the Fermi surface, obtained in particular from the anomalous skin effect.

The role of the correlation between particles can also be elucidated by studying phenomena in a magnetic field. The transport equation is in that case

$$\frac{\partial}{\partial t}\psi + \left\{\mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\frac{\partial}{\partial \mathbf{p}}\right\}\left\{\psi + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}', \mathbf{r}')\right\} + e\mathbf{v}\mathbf{E} = I(\psi). \qquad (23)$$

This equation is practically the quasi-classical Schrödinger equation for the electron state, which differs only slightly from its equilibrium state. In particular, we can thus determine the quasi-classical spectrum of the electronic energy levels in a magnetic field from the equation

$$i\omega\psi + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\frac{\partial}{\partial \mathbf{p}}\left\{\psi + \int dS' F(\mathbf{n}, \mathbf{n}')\psi(\mathbf{n}')\right\} = 0. \qquad (24)$$

The presence of the function F in Eq. (24) considerably affects the determination of the eigenfrequencies of the electrons. In the case of the isotropic model the solution of Eq. (24) can easily be found by expanding $e^{ik\varphi}P_n^k(\cos\theta)$ in spherical harmonics. The eigenfrequencies are then $\omega_n^k = k(eHv_0/cp_0)\{1 + 4\pi A_n/(2n+1)\}$, where A_n is the expansion coefficient of the function $F(\cos\chi)$ in Legendre polynomials. The fact that the spectrum of the electronic energy levels in a magnetic field depends strongly on the function F enables us to state that, for instance, the de Haas-Van Alphen effect can be used to determine the shape of this function. On the other hand, the determination of the Fermi surface is apparently very difficult. This is also true for the case of diamagnetic resonance (bismuth). The change in the eigenfrequencies of the electrons in a constant magnetic field, caused by the correlation between particles, can also influence considerably the interpretation of resonance frequencies in the case of cyclotron resonance, which is observed in the region of the anomalous skin effect. We note finally that in the case of galvanomagnetic phenomena the difference between Eq. (23) and the usual one consists in fact only in a redefinition of the collision integral. The results of the usual theory of galvanomagnetic and magnetothermal phenomena remain thus practically the same.

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