# APPLICATION OF DISPERSION RELATIONS FOR THE DETERMINATION OF BOUND STATES*

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Dispersion relations are discussed in the framework of nonrelativistic theory. It is shown that the energies, parities, and certain spectral coefficients characterizing the bound states of the system can be determined on the basis of scattering data. Some characteristics of the deuteron are determined by using the n — p scattering data.

THE dispersion relations for the scattering amplitude are usually formulated on the basis of relativistic field theory. This leads to a number of difficulties, which with few exceptions preclude the obtaining of applicable results. We shall discuss dispersion relations in the framework of nonrelativistic theory.

In order for dispersion relations to exist, it is necessary that the scattering amplitude be an analytic function of the complex energy and be bounded for $E \to \infty$. However, in the nonrelativistic case, the scattering amplitude behaves like $e^{-2ikR}$ for $k \to \infty$, where k is the wave number, and $R_0$ is a radius beyond which the interaction vanishes. Dispersion relations can be written only for the function $F(k) = f(k, \theta, \varphi)\, e^{2ikR}$, $R > R_0$. Let us denote

$$F_0(k) = \operatorname{Re} f(k, \theta, \varphi) \cos 2kR - \operatorname{Im} f(k, \theta, \varphi) \sin 2kR, \qquad F_1(k) = \operatorname{Re} f(k, \theta, \varphi) \sin 2kR + \operatorname{Im} f(k, \theta, \varphi) \cos 2kR.$$

From the Cauchy theorem we then have:

$$F_0(k) = \frac{1}{\pi} \int_0^\infty \frac{2k'dk'}{k'^2 - k^2} F_1(k') - \sum_\lambda \frac{\nu_\lambda}{k^2 + |k_\lambda|^2} e^{-2|k_\lambda|R},$$

$$F_1(k) = -\frac{1}{\pi} \int_0^\infty \frac{2kdk'}{k'^2 - k^2} F_0(k') + \sum_\lambda \frac{k\nu_\lambda}{|k_\lambda|(k^2 + |k_\lambda|^2)} e^{-2|k_\lambda|R}.$$

Here $k_\lambda$ are the poles of the scattering amplitude that lie on the upper half of the imaginary axis, and $\nu_\lambda = -\operatorname{Res} f$, the residues at the points $E = E_\lambda$. The following integral relations also hold

$$\frac{2}{\pi} \int_0^\infty dk F_0(k) = - \sum_\lambda \frac{\nu_\lambda}{|k_\lambda|} e^{-2|k_\lambda|R}; \qquad \frac{2}{\pi} \int_0^\infty k dk F_1(k) = - \sum_\lambda \nu_\lambda e^{-2|k_\lambda|R}.$$

The coefficient $\nu_\lambda$ is connected with the asymptotic part of the wave function of the stationary state. To show this we write the (elastic) scattering equation in the form

$$T = V + V(E - H)^{-1} V, \quad \operatorname{Im} E > 0. \tag{1}$$

Here T is the scattering operator introduced by Lippman and Schwinger.[1] We now investigate the scattering of a particle by a nucleus in state A. Let $\Phi_a$ be a normalized wave function for channel a

$$\Phi_a = (2\pi)^{-3/2} e^{i\mathbf{k}_a \mathbf{r}} \varphi_{Aa}, \quad (\Phi_{a'}, \Phi_a) = \delta(\mathbf{k}_a - \mathbf{k}_a') \delta_{AA'} \delta_{aa'}.$$

We designate by $\Psi_a$ the solution of the scattering problem

*The first conjecture that the bound states can be found from scattering data by analytic continuation of the S matrix is due to Heisenberg

$$\Psi_a = \Phi_a + (E - H_0)^{-1} V \Psi_a.$$

The scattering amplitude for the transition from state a into state a′ is given by

$$f_{a\to a'}(k, \theta, \varphi) = -2\pi^2 (\Phi_{a'}, T\Phi_a). \tag{2}$$

The integral equation (1) allows an analytical continuation into the complex E plane. From (1) and (2) one can easily obtain the residue of the scattering amplitude

$$[\mathrm{Res}\, f_{a\to a'}]_{E_\lambda} = -2\pi^2 \sum_{\lambda \in E_\lambda} (\Phi_{a'}, V\Psi_\lambda)(\Psi_\lambda, V\Phi_a).$$

Here the summation is over all quantum numbers $\lambda$ characterizing the state with energy $E_\lambda$. The functions $\Phi_{a'}$ and $\Phi_a$ are the appropriate analytical continuations of the respective plane waves. At large r, where the interaction vanishes and where the angular factor becomes insignificant, the wave function has the form $\exp(-|k_\lambda| r)\, g_\lambda / r$. The factor $g_\lambda$ depends on angular, spin, and other variables. For the scalar products we obtain

$$(\Phi_{a'}, V\Psi_\lambda) = -(2/\pi)^{1/2} g'_\lambda, \quad (\Psi_\lambda, V\Phi_a) = -(2/\pi)^{1/2} I_\lambda g_\lambda.$$

The number $I_\lambda$, which equals ± 1, characterizes the parity of the state $\lambda$. In the particular case of elastic forward scattering $a = a'$, $\bar{k} = \bar{k}'$ and

$$\nu_\lambda = \pi I_\lambda \sum_{\lambda \in E_\lambda} g_\lambda^2.$$

This shows that the sign of $\nu_\lambda$ determines the parity of the bound state $\lambda$. The absolute value of $\nu_\lambda$ is proportional to the virtual decay, i.e., to the probability that the bound particles of the system separate to a distance larger than the range of the forces.

We now apply the dispersion relations to the concrete problem of neutron-proton scattering. We utilize the fact that the n − p interaction can be well represented by the potential function $\hat{V} = V(\mathbf{r})\, \delta(\mathbf{r} - \mathbf{r}')$. One can show (we shall not give the proof here) that the forward-scattering amplitude for this potential, as a function of the complex energy, is bounded for $E \to \infty$ and approaches the limit

$$f(\infty) = -\frac{1}{4\pi} \int V(\mathbf{r})\, d\mathbf{r}.$$

Cauchy's theorem can be applied to the difference $f(k) - f(\infty)$, where $f(k)$ is the forward scattering amplitude. Let $f_t(k)$ and $f_s(k)$ be the triplet and singlet scattering amplitudes respectively. Let further

$$f_0(k) = \mathrm{Re}\left[\frac{3}{4} f_t(k) + \frac{1}{4} f_s(k)\right],$$

and let the cross section be $\sigma = 3/4\, \sigma_t + 1/4\, \sigma_s$. The dispersion relations then become

$$f_0(k) - f_0(0) = \frac{k^2}{2\pi^2} \int_0^\infty \frac{\sigma(k')\, dk'}{k'^2 - k^2} + \frac{3}{4} \frac{k^2 \nu_1}{k_1^2 (k^2 + k_1^2)}.$$

Here $\nu_1$ is the spectral coefficient and $k_1$ the reciprocal of the deuteron radius: $k_1 = 1/R_D$; $R_D = 4.314 \times 10^{-13}$ cm. The n − p scattering cross section is known with good accuracy up to high energies. We performed the integration to an upper limit corresponding to 70 Mev. The contribution from the discarded part of the integral does not exceed a fraction of one per cent. The computation was performed for values of k corresponding to 1, 2, and 5 Mev in the laboratory system. The coefficient $\nu_1$ was obtained from the dispersion relations and found to be roughly the same for all energies, $(7.5 \pm 0.4) \times 10^{12}\ \mathrm{cm}^{-1}$.

The positive sign of the spectral coefficient indicates the even parity of the wave function of the deuteron. The spectral coefficient can be obtained independently from a model of the deuteron. For example, in the approximation of effective-range theory one obtains the expression $\nu_1 = 2/(R_D - r_{0t})$ which gives a value $7.7 \times 10^{12}\ \mathrm{cm}^{-1}$. The discrepancy between this value and the value obtained from the dispersion relations is small and well within the limits of possible errors.

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[1] B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).

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[1]H. Oiglane, J. Exptl. Theoret. Phys. this issue, p. 1511 (Russian), p. 1167 (transl.).
[2]A. Salam and J. C. Polkinghorne, Nuovo cimento 2, 685 (1955).
[3]D. C. Peaslee, Nuovo cimento 6, 1 (1957).

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*ERRATA TO VOLUME 6*

| Page | Line | Reads | Should Read |
|---|---|---|---|
| 643 | 16 from bottom | where $\kappa = \pi a^2 \Omega - \ldots$ | where $\kappa = \pi a^2 \Omega \varphi - \ldots$ |
| 690 | 8 from bottom | $\ldots \sin [- \ldots$ | $\ldots \sin \vartheta [- \ldots$ |
| | 5 from bottom | $\ldots \sin 2\vartheta \sqrt{\frac{1}{3}} \ldots$ | $\ldots \sin 2\vartheta \left[ \sqrt{\frac{1}{3}} \ldots \right.$ |
| 809 | 9 from top | $\ldots \left( \frac{1}{2 \sinh u} + \ldots \right.$ | $\ldots \left( \frac{1}{\sinh u} + \ldots \right.$ |
| 973 | unnumbered equation | $\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} T^{(n)}_{\mu'-\mu'}.$ | $\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} < S' \Vert T^{(n)} \Vert S^{-1} > \times T^{(n)}_{\mu'-\mu}.$ |
| 975 | 5 from bottom | ...of a particle by a nucleus ... | ...of a particle in state a by a nucleus ... |
| 992 | Eq. (18) | $\ldots \tau_1 \tau_2^{-2} / 2\hbar^1 \ldots$ | $\ldots \tau_1 \tau_2^{-1} / 2\hbar^2 \ldots$ |