SELECTION RULES FOR ELECTROMAGNETIC TRANSITIONS IN DEFORMED NUCLEI

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Selection rules for electromagnetic transitions have been obtained for the following asymptotic quantum numbers of deformed nuclei: Ω, Λ, and Σ (the projections of the total, the orbital, and the spin angular momentum respectively of a particle on the axis of an elongated nucleus), n_z (the oscillator quantum number along the axis), and N (the principal oscillator quantum number.). Taking into account these selection rules allows an explanation of the discrepancies between theoretical and experimental values of transition probabilities. A first order forbiddenness in one of the asymptotic quantum numbers decreases the transition probability by a factor 10 — 100.

1. INTRODUCTION

LET a nuclear state be characterized by the energy E_i, the angular momentum I_i, its projection on an axis fixed in space M_i, and the parity π_i. It will be able to make a transition to a state with E_f, I_f, M_f, and π_f with emission of a γ-quantum of energy $\hbar\omega = E_i - E_f$ and multipolarity λ. The probability of such a transition is given[1] by

$$T(\alpha\lambda, I_i \rightarrow I_f) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2\lambda+1} B(\lambda), \tag{1}$$

where $B(\lambda)$ is the reduced transition probability

$$B(\lambda) = \sum_{\mu, M_f} |\Psi_f^* \mathfrak{M}(\alpha\lambda, \mu)\Psi_i d\tau|^2, \tag{1a}$$

and $\mathfrak{M}(\alpha\lambda, \mu)$ —the operator of the multipole moment. The α denotes the electric or magnetic character of the transition.

For a multipole transition to be possible the following selection rules must be fulfilled

$$|I_i - I_f| \leqslant \lambda \leqslant I_i + I_f, \tag{2}$$

$$M_f - M_i = \mu, \tag{2a}$$

$$\pi_i\pi_f = \begin{cases}(-1)^\lambda & \text{for } E\lambda,\\ (-1)^{\lambda+1} & \text{for } M\lambda.\end{cases} \tag{3}$$

The dependence of the transition probability on the nuclear structure is contained in (1) in the matrix element of the multipole operator. One consequently has to use in actual computations a nuclear model of one kind or another. Because of its simplicity one usually takes the shell model. The emission of γ-rays is then treated as a single-body process. Utilizing the known expressions for the multipole-moment operators, one can derive formulae[1] for the probability of emission of electromagnetic radiation of multipolarity λ. In the region of validity of the shell model the agreement between experiment and theory is in general not bad. The ratio between the observed and the calculated transition probability (denoted in the following by F) is close to unity. However, in the region of highly deformed nuclei ($150 < A < 190$ and $222 < A < 256$) there is a strong disagreement between the experiment and the values calculated by the shell model. This is not unexpected since in this region the unified model of Bohr and Mottelson[2,3] applies.

Nilsson and Mottelson[4,5] worked out single particle states in deformed nuclei. They classified the states according to the quantum numbers Ω, Λ, Σ (projections of the total angular momentum, the orbital angular momentum and the spin of the particle respectively on the elongated nuclear axis) n_z, $n_\perp$

(oscillator quantum numbers along the elongated axis and in a plane perpendicular to it respectively) and N (principal oscillator quantum number).

Alaga et al.[6] give an additional selection rule for electromagnetic transitions in deformed nuclei. Let K be the projection of the total (collective and internal) angular momentum on the elongated nuclear axis. Then

$$|K_f - K_i| \equiv |k| \leqslant \lambda. \quad (4)$$

The degree of K-forbiddenness is characterized by the number ν

$$\nu = |k| - \lambda \quad (5)$$

The experimental data show that in strongly deformed nuclei K is a "good" quantum number and $\nu = 1$ decreases the transition probability by a factor of about 100. However, selection rules (2)—(4) are still insufficient to explain the discrepancies between experiment and theory. It frequently turns out that the F-factor for transitions allowed with respect to these selection rules is very small. In this connection it is of interest to investigate selection rules with respect to the asymptotic quantum numbers and to determine how they are obeyed in deformed nuclei.

2. ASYMPTOTIC SELECTION RULES

We consider the case of strong nucleon-nuclear surface coupling. Here the spin-orbit coupling is destroyed and the projections of the orbital and spin angular momentum on the nuclear axis, Λ and Σ, are separately constants of the motion. Furthermore, the oscillations of the nucleon along the nuclear axis and perpendicularly to it proceed independently. A natural choice here seems the n_z, $n_\perp$, Λ, Σ, taking as eigenfunctions the functions of an anisotropic harmonic oscillator in cylindrical coordinates. However, the particle states in deformed nuclei are commonly characterized by the quantum numbers N, n_z, Λ, Ω $(= \Lambda + \Sigma)$ and it is therefore necessary to work in this representation.

We now consider the transition of a single particle from the state I_i, $K_i = \Omega_i$, π_i, N_i, n_{zi}, Λ_i, Σ_i into the state I_f, $K_f = \Omega_f$, π_f, N_f, n_{zf}, Λ_f, Σ_f with the emission of electromagnetic radiation of multipolarity λ, k. We assume that the transition is allowed with respect to (2)—(4) with respect to total angular momentum I, its projection K, and parity π. We write the operator of the electric multipole moment in the form

$$\mathfrak{M}(E\lambda, k) = \left[e + (-1)^\lambda \frac{Z}{A^\lambda}\right]^2 \left(\frac{\hbar}{M\omega_0}\right)^{\lambda/2} r^\lambda Y_{\lambda k}. \quad (6)$$

Here the contribution of the particle magnetic moment and of the nuclear collective motion has not been taken into account. The latter can have a considerable influence on E2 transitions. We write for the magnetic multipole moment operator[3,4]

$$\mathfrak{M}(M\lambda, k) = (e\hbar / 2Mc)\,(\hbar / M\omega_0)^{(\lambda-1)/2} \nabla (r^\lambda Y_{\lambda k})\,[g_s \mathbf{s} + g_l \mathbf{l}]. \quad (7)$$

Here the contribution due to the collective motion has also been omitted. It can be taken into account for M1 transitions by replacing g_s and g_l by $g_s - g_R$ and $g_l - g_R$ respectively; $g_R \approx Z/A$.

The selection rules in (7) are essentially given by the terms $\nabla(r^\lambda Y_{\lambda k})\mathbf{s}$ and $\nabla(r^\lambda Y_{\lambda k})\mathbf{l}$. We rewrite them in the following form

$$\nabla (r^\lambda Y_{\lambda k})\,\mathbf{s} = \sqrt{(2\lambda+1)/(2\lambda-1)}\, r^{\lambda-1} \Big\{-\tfrac{1}{2}\sqrt{(\lambda+k)(\lambda+k-1)}\, Y_{\lambda-1,\, k-1}(s_x + is_y)$$
$$+ \tfrac{1}{2}\sqrt{(\lambda-k)(\lambda-k-1)}\, Y_{\lambda-1,\, k+1}(s_x - is_y) + \sqrt{\lambda^2 - k^2}\, Y_{\lambda-1,\, k} s_z\Big\}. \quad (8)$$

A similar expression for $\nabla(r^\lambda Y_{\lambda k})\mathbf{l}$ can be obtained by replacing in (8) $(s_x \pm is_y)$ and s_z by $(l_x \pm il_y)$ and l_z respectively. The operator $s_x + is_y$ in (8) corresponds to the transition $\Sigma_i = -1/2 \rightarrow \Sigma_f = +1/2$; $s_x - is_y$ corresponds to the transition $\Sigma_i = 1/2 \rightarrow \Sigma_f = -1/2$, and s_z, together with the operator for the orbital magnetic moment $\nabla(r^\lambda Y_{\lambda k})\mathbf{l}$, corresponds to transitions with $\Sigma_i = \Sigma_f$. Utilizing the wave functions of an ellipsoidal harmonic oscillator without spin-orbit coupling one can determine for which values of N, n_z, Ω, Λ, Σ the matrix elements of the electric and magnetic multipole moments differ from zero. The results of such a calculation are given in Tables I and II. The tables are arranged identically. The first column gives the difference $k = \Omega_f - \Omega_i$ between final and initial Ω. It

TABLE I. Selection rules for N, n_z, Λ, Σ in electric transitions of multipolarity λ

$\Delta\Omega = \Omega_f - \Omega_i$	$\Delta\Lambda = \Lambda_f - \Lambda_i$	$\Delta\Sigma = \Sigma_f - \Sigma_i$	$\Delta N = N_f - N_i$	$\Delta n_z = n_{zf} - n_{zi}$	Additional Conditions
$\pm\lambda$	$\pm\lambda$	0	$\lambda, \lambda-2, \ldots -\lambda$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda, \lambda-2, \ldots -\lambda$	±1	
0	0	0	±2	±2	$\lambda = 2$
0	0	0	$0, \pm2$	0	$\lambda = 2$

TABLE II. Selection rules for N, n_z, Λ, Σ in magnetic transitions of multipolarity λ

$\Delta\Omega = \Omega_f - \Omega_i$	$\Delta\Lambda = \Lambda_f - \Lambda_i$	$\Delta\Sigma = \Sigma_f - \Sigma_i$	$\Delta N = N_f - N_i$	$\Delta n_z = n_{zf} - n_{zi}$	Additional Conditions
$\pm\lambda$	$\pm\lambda$	0	$\lambda+1, \lambda-1, \ldots -\lambda-1$	±1	
$\pm\lambda$	$\pm(\lambda-1)$	±1	$\lambda-1, \lambda-3, \ldots -\lambda+1$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda-1, \lambda-3, \ldots -\lambda+1$	0	
$\pm(\lambda-1)$	$\pm(\lambda-1)$	0	$\lambda+1, \lambda-1, \ldots -\lambda-1$	±2	$\lambda > 1$
$\pm(\lambda-1)$	$\pm(\lambda-2)$	±1	$\lambda-1, \lambda-3, \ldots -\lambda+1$	±1	$\lambda > 1$
0	0	0	$\pm1, \pm3$	±1	$\lambda = 2$
0	±1	∓1	±1	0	$\lambda = 2$

suffices to consder for a certain polarity λ transitions with $\Delta\Omega = \pm\lambda, \pm(\lambda - 1)$. An exception is quadrupole radiation in transitions with $\Omega_i = \Omega_f$. In the second and third columns the selection rules for Λ and Σ are given. They have to be considered together with $\Delta\Omega$. The condition $\Delta\Omega = \Delta\Lambda + \Delta\Sigma$ has to be fulfilled. We note that electric multipole transitions can occur only without change of the projection of the spin on the nuclear axis (neglecting the magnetic moment of the particle). Magnetic transitions can occur both with and without change of the projection of the spin on the nuclear axis. The fourth and fifth columns give the selection rules of N and n_z. If $\Delta n_z \neq 0$, the sign of Δn_z is restricted to the sign of ΔN for the largest $|\Delta N|$. The smaller $|\Delta N|$ can occur with either sign of Δn_z. For example, in an E1 transition with $\Delta\Omega = 0$, $\Delta\Lambda = 0$; the value $\Delta N = +1$ corresponds to $\Delta n_z = +1$, and for $\Delta N = -1$ only $\Delta n_z = -1$ is allowed. Or, in a magnetic transition with $\Delta\Omega = \pm(\lambda - 1)$, $\Delta\Lambda = \pm(\lambda - 1)$, we have $\Delta n_z = 2$ for $\Delta N = \lambda + 1$, and $\Delta n_z = -2$ for $\Delta N = -(\lambda + 1)$. For the other possible $\Delta N = \lambda - 1, \lambda - 3, \ldots, -\lambda + 1$, $\Delta n_z = \pm 2$ is allowed.

It should be emphasized that the obtained selection rules hold only for transitions of the type $\Omega_f + \Omega_i > \lambda$. In transitions $\Omega_f + \Omega_i \leq \lambda$ the particular case with $\Omega = 1/2$ appears. In this state the interaction with the surface vanishes and the spin-orbit interaction is not weakened. The quantum numbers n_z, Λ, Σ here have no meaning.

3. COMPARISON WITH EXPERIMENT

We shall now investigate how well the selection rules established in the previous section are obeyed in transitions in actual nuclei which have only finite deformations. For this it is necessary to compute transition probabilities using wave functions of deformed nuclei and to compare these with corresponding transition probabilities in spherical nuclei and with experiment. Such computations were performed for transitions in a number of rare-earth and transuranic elements for which there exist data on experimental half-lives.[7] The transition probabilities in deformed nuclei were calculated using Nilsson's[4] formulae and wave functions. The transition probabilities for spherical nuclei were computed with formulae given by Blatt and Weisskopf.[1] The results are given in Tables III and IV. Table III gives the results for electric multipole transitions and Table IV for magnetic multipole transitions. The first columns of the tables list the elements; the second the transition energy in kev, and the third the spin and parity of the initial (top) and final (bottom) state. Only states with $I = K = \Omega$ were considered. Transitions into rotational levels can easily be eliminated by use of the intensity rule given by Alaga et al.[6] The fourth column also gives two sets of data: the upper line has N, n_z, Λ, Σ for the initial state and the lower for the final state. In the fifth column the character of the transition is indicated. Transitions which are allowed with respect to all quantum numbers are uninhibited and denoted by u; transitions forbidden because of n_z, Λ, or Σ are hindered and denoted by h. In order to facilitate the determination of the character of the transition the table is broken up into parts according to the values of λ and k. The head of each subdivision lists the changes in the quantum numbers N, n_z, Λ, Σ which are allowed according to Table II. The sixth column lists the values of F_1 — the ratios of the theoretical transition probabilities in deformed nuclei to similar ones in spherical nuclei. And, finally, the last column shows the ratios of the experimental[7,8] to the theoretical transition probabilities in deformed nuclei, F_2. The ratio of the experimental transition probability to the transition probability in spherical nuclei (the F-factor) is equal to the product F_1F_2.

From Tables III and IV it is first evident that the quantum numbers n_z, Λ, and Σ play a significant part in real nuclei. If the transition is uninhibited the unified model and the shell model lead to similar

TABLE III. Electric multipole transitions ($E\lambda$)

Element	Transition energy kev	$\Omega_i\ \pi_i$ $\Omega_f\ \pi_f$	$N_i\ n_{zi}\Lambda_i\ \Sigma_i$ $N_f\ n_{zf}\Lambda_f\ \Sigma_f$	Transition character	$F_1 = \frac{\tau_{def}}{\tau_{sph}}$	$F_2 = \frac{\tau_{exp}}{\tau_{def}}$
$\lambda = 1$; $\Delta\Omega = 0$; $\Delta N = \pm 1$; $\Delta n_z = \pm 1$; $\Delta\Lambda = 0$; $\Delta\Sigma = 0$						
Eu^{153}_{63}	97.3	$5/2-$ $5/2+$	$5\,3\,2 + 1/2$ $4\,1\,3 - 1/2$	h	$0.38\cdot 10^{-4}$	
Np^{237}_{93}	60	$5/2-$ $5/2+$	$5\,2\,3 - 1/2$ $6\,4\,2 + 1/2$	h	$0.70\cdot 10^{-4}$	0.13
Pu^{239}_{94}	106.2	$5/2-$ $5/2+$	$5\,0\,3 - 1/2$ $6\,2\,2 + 1/2$	h	$1.1\cdot 10^{-6}$	0.5
$\lambda = 1$; $\Delta\Omega = -1$; $\Delta N = \pm 1$; $\Delta n_z = 0$, $\Delta\Lambda = -1$; $\Delta\Sigma = 0$						
Lu^{177}_{71}	146	$9/2-$ $7/2+$	$5\,1\,4 + 1/2$ $4\,0\,4 - 1/2$	h	$0.40\cdot 10^{-4}$	$0.7\cdot 10^{-2}$
$\lambda = 2$; $\Delta\Omega = 1$; $\Delta N = 0$; $\Delta n_z = \pm 1$; $\Delta\Lambda = 1$; $\Delta\Sigma = 0$						
Ta^{181}_{73}	482	$5/2+$ $7/2+$	$4\,0\,2 + 1/2$ $4\,0\,4 - 1/2$	h	$0.60\cdot 10^{-4}$	
$\lambda = 2$; $\Delta\Omega = 2$; $\Delta N = 0$; $\Delta n_z = 0$; $\Delta\Lambda = 2$; $\Delta\Sigma = 0$						
Ta^{181}_{73}	133	$1/2+$ $5/2+$	$4\,1\,1 - 1/2$ $4\,0\,2 + 1/2$	h	$1.8\cdot 10^{-3}$	0.99
Eu^{153}_{63}	172	$1/2+$ $5/2+$	$4\,1\,1 - 1/2$ $4\,1\,3 - 1/2$	u	0.16	
$\lambda = 3$; $\Delta\Omega = 3$; $\Delta N = \pm 1$; $\Delta n_z = 0$; $\Delta\Lambda = 3$; $\Delta\Sigma = 0$						
Dy^{165}_{66}	108	$1/2-$ $7/2+$	$5\,2\,1 - 1/2$ $6\,3\,3 + 1/2$	h	$0.85\cdot 10^{-3}$	0.60*
Er^{167}_{68}	208	$1/2-$ $7/2+$	$5\,2\,1 - 1/2$ $6\,3\,3 + 1/2$	h	$0.80\cdot 10^{-3}$	2.7*

* Computed by Iu. I. Kharitonov

results ($F \sim 1$). But if the transition is hindered then the unified model gives smaller transition probabilities than the shell model. The size of the decrease depends on the degree of forbiddenness. One sees from the given data that a first-order violation of the selection rules for one of the asymptotic quantum numbers decreases the transition probability by a factor 10 — 100. This allows an explanation of the disagreement between the experimental transition probabilities and the shell model (compare F_2 and F_1F_2). Unfortunately only little experimental data exists. However, even the presently-available data clearly indicate the usefulness of the unified model as can be seen from the last column of the tables. The unified model describes the experimental data much better than the shell model.

The case Ir^{191}_{77} is of particular interest. This nucleus is on the boundary of the region of large deformations and the question of its equilibrium shape is presently open. An M1 transition of 82.6 kev was considered, with the deformation parameter taken as 0.14. The obtained result (Table IV) shows that the assumption of a deformed nucleus leads to a considerable improvement.

In conclusion we remark that the simple physical model of a single particle moving in a deformed harmonic oscillator potential on which this calculation is based gives a fundamentally correct description

TABLE IV. Magnetic multipole transitions (Mλ)

Element	Transition energy, kev	$\Omega_i\pi_i$ / $\Omega_f\pi_f$	$N_i\, n_{zi}\, \Lambda_i\, \Sigma_i$ / $N_f\, n_{zf}\, \Lambda_f\, \Sigma_f$	Transition character	$F_1 = \frac{T_{def}}{T_{sph}}$	$F_2 = \frac{T_{exp}}{T_{def}}$
$\lambda = 1$;	$\Delta\Omega = 1$;	$\Delta N = 0$;	$\Delta n_z = \pm{0 \atop 1}$;		$\Delta\Lambda = {0 \atop 1}$;	$\Delta\Sigma = {1 \atop 0}$
Eu_{63}^{153}	69	$1/2+$ / $3/2+$	$4\,1\,1-1/2$ / $4\,1\,1+1/2$	u	0.72	0.09
Eu_{63}^{153}	103	$3/2+$ / $5/2+$	$4\,1\,1+1/2$ / $4\,1\,3-1/2$	h	$1.2\cdot10^{-4}$	13
Ta_{73}^{181}	482	$5/2+$ / $7/2+$	$4\,0\,2+1/2$ / $4\,0\,4-1/2$	h	$1.4\cdot10^{-3}$	$0.87\cdot10^{-2}$
Ir_{77}^{191}	82.6	$1/2+$ / $3/2+$	$4\,0\,0+1/2$ / $4\,0\,2-1/2$	h	$0.56\cdot10^{-3}$	1.8
$\lambda = 2$	$\Delta\Omega = -1$;	$\Delta N = -1$;	$\Delta n_z = -1$;		$\Delta\Lambda = 0$;	$\Delta\Sigma = -1$
Lu_{71}^{177}	146	$9/2-$ / $7/2+$	$514+1/2$ / $404-1/2$	u	0.1	

of the laws of the electromagnetic transitions in deformed nuclei. This is a rather remarkable fact.

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[1] J. Blatt and V. Weisskopf, Theoretical Nuclear Physics, Ch. XII.

[2] A. Bohr, Kgl. Dansk. Vid. Selsk. Math. Fys. Medd. **26**, 14 (1952), transl. in Probl. of Contemp. Phys. #9 (1955).

[3] A. Bohr and B. Mottelson, Kgl. Dansk. Vid. Selsk. Math. Fys. Medd. **27**, 16 (1953), transl. in Probl. of Contemp. Phys. #9 (1955).

[4] S. Nilsson, Kgl. Dansk. Vid. Selsk. Math. Fys. Medd. **29**, 16 (1955).

[5] B. Mottelson and S. Nilsson, Phys. Rev. **99**, 1615 (1955), transl. in Probl. of Contemp. Phys. #1 (1956).

[6] Alaga, Alder, Bohr, and Mottelson, Kgl. Dansk. Vid. Selsk. Math. Fys. Medd. **29**, 9 (1955), transl. in Probl. of Contemp. Phys. #1 (1956).

[7] B. S. Dzhelepov and L. K. Peker, Схемы распада радиоактивных изотопов (Decay Schemes of Radioactive Isotopes), Academy of Sciences, U.S.S.R. (1957).

[8] L. A. Sliv and I. M. Band, Tables of Internal Conversion Coefficients of Gamma Rays, Academy of Sciences, U.S.S.R. (1956).

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