3. The interactions of a shock wave with a weak and a weak-tangential discontinuity have been considered. As a result of these, outgoing weak and weak-tangential discontinuities are formed.

4. A singular phenomenon of "resonance" has been discovered, which leads to a significant amplification of the intensity of the incoming discontinuities. The connection is pointed out of this phenomenon with the possibility of splitting of the shock wave.

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INTERACTION OF SHOCK WAVES WITH SMALL PERTURBATIONS II.

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Consideration is given to the general case of the interaction of small perturbations with a shock wave of arbitrary intensity within the framework of a two dimensional stationary problem in the case of subsonic flow behind the shock wave. A solution is given for the interaction of weak and weak-tangential discontinuities with the shock wave. In both cases, the weak-tangential discontinuity formed behind the shock wave possesses a specific logarithmic discontinuity.

1. INTRODUCTION

IN the previous paper,[1] the perturbation theory was given for the shock wave with supersonic flow behind it; with the aid of perturbation methods, problems were considered of the interaction of weak singularities with the shock wave. The purpose of the present work is the consideration of analogous interactions in the case of subsonic flow behind the shock wave.

For subsonic flow behind the shock wave, the equations of hydrodynamics are elliptic and the method of characteristics used in the previous paper are not applicable.

Let us consider a plane shock wave in one dimensional flow. As the y axis we choose the perpendicular to the wave in the direction of the normal component of the velocity. Let the x axis lie in the plane of the wave and be directed along the tangent to the component of the velocity.

Inasmuch as the motion is supersonic for $y < 0$, all the incident perturbations are described by solutions obtained in Sec. 1 of the previous paper. For $y > 0$, the system of hydrodynamic equations will possess only one set of real characteristics — flow lines along which the perturbation of the entropy and the curl of the velocity are "transferred." Therefore the entropy-vortical solution is preserved here also. Two sets of characteristics become imaginary. To find the solutions corresponding to the elliptic equation it is expedient to reduce it to Laplace's equation.

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The linearized hydrodynamic equations have the form

$$(\mathbf{v}\nabla)\,\delta\mathbf{v} + V\nabla\delta p = 0, \tag{1.1}$$

$$\operatorname{div}\delta\mathbf{v} + (V/c^2)\,\mathbf{v}\nabla\delta p = 0, \tag{1.2}$$

$$(\mathbf{v}\nabla)\,\delta s = 0. \tag{1.3}$$

Taking the divergence of (1.1) and eliminating div $\delta\mathbf{v}$ with the help of (1.2), we obtain the following elliptic equation for δp:

$$\Delta\delta p - c^{-2}(\mathbf{v}\nabla)^2\,\delta p = 0. \tag{1.4}$$

For the reduction of this equation to Laplace's equation, we undertake a transformation to a new variable by the substitutions

$$\sqrt{1-M^2}\,\xi = (1 - v_y^2/c^2)\,x + (v_x v_y/c^2)\,y, \quad \eta = y, \tag{1.5}$$

after which, as is easily shown, Eq. (1.4) takes the form

$$\partial^2\delta p/\partial\xi^2 + \partial^2\delta p/\partial\eta^2 = 0. \tag{1.6}$$

Thus δp is a harmonic function of the variables ξ, η.

It is appropriate to subject the independent solution of the hydrodynamic equations with $\delta p \neq 0$ to the conditions

$$\delta s = 0, \; I_0 = \mathbf{v}\delta\mathbf{v} + V\delta p = 0 \tag{1.7}$$

similarly to what was done in the previous paper. Taking the gradient of the latter condition

$$\nabla(\mathbf{v}\delta\mathbf{v}) + V\nabla\delta p \equiv (\mathbf{v}\nabla)\,\delta\mathbf{v} + [\mathbf{v}\operatorname{curl}\delta\mathbf{v}] + V\nabla\delta p = 0$$

and comparing with (1.1), we can establish the fact that

$$\operatorname{curl}\delta\mathbf{v} = 0, \quad \text{i.e.} \quad \delta\mathbf{v} = \nabla\varphi, \tag{1.8}$$

where φ is the velocity potential. It is easy to show that φ satisfies the equation (1.4), i.e., it is a harmonic function of the variables ξ, η.

Thus the independent solution of the hydrodynamic equations with $\delta p \neq 0$ is characterized by conditions of a potential and isentropic character, where δp and φ satisfy Eq. (1.4).

The position of the characteristics relative to the shock wave is shown in Fig. 1. As in the previous paper, the illustrations are shown for a gas with $\gamma = 1.4$, in which the pressure increases by a factor of 10 in the shock. However, the horizontal velocity v_x is less and amounts to 0.5 of the velocity of sound in front of the wave (for $y < 0$). Here, for $y < 0$, $c/v = 0.333$, and the flow is supersonic; for $y < 0$: $c/v = 1.75$ and the flow is subsonic. From the side $y < 0$, any of the perturbations calculated in Ref. 1 can fall on the shock wave. As a result, the entropy-vortical perturbation arises for $y > 0$, and is propagated along the characteristics; the perturbation is described by the solution of the elliptic equations. To speak here of perturbations "arriving" from the side $y > 0$ has no meaning because of the elliptic character of the equations. For a given type of perturbation incident from the side $y < 0$, the solution of the problem will be a linear combination of independent solutions that are determined from the boundary conditions on the discontinuity.

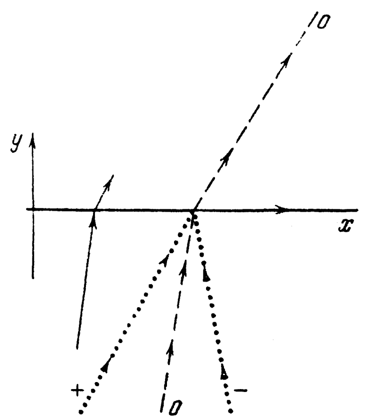

Fig. 1.———— shock wave, --- — characteristics of the entropy-vortical perturbation, which coincide with the flow lines, ... — characteristics of the acoustic perturbation,

The boundary conditions on the discontinuity were derived in Ref. 1. They are:

$$\delta v_{0x} - \delta v_x = (v_y - v_{0y})\,\eta'; \tag{1.9}$$

$$\frac{\delta v_{0y} - \delta v_y}{v_{0y} - v_y} = \frac{1}{2}\left[\frac{\delta p - \delta p_0}{p - p_0} + \frac{\delta V_0 - \delta V}{V_0 - V}\right]; \tag{1.10}$$

$$\delta V = \frac{\partial V}{\partial p}\delta p + \frac{\partial V}{\partial p_0}\delta p_0 + \frac{\partial V}{\partial V_0}\delta V_0; \tag{1.11}$$

$$\frac{2}{v_{0y}}(\delta v_{0y} - v_{0x}\eta') = \frac{2\delta V_0}{V_0} + \frac{\delta p - \delta p_0}{p - p_0} - \frac{\delta V_0 - \delta V}{V_0 - V}. \tag{1.12}$$

Furthermore, the solution of the elliptical equations must satisfy the condition:

$$\delta p = 0;\ \nabla\varphi = 0 \text{ at infinity} \tag{1.13}$$

2. SOLUTION OF THE EQUATIONS

Let us proceed to finding the solutions (of interest to us) set up by the incident perturbation.

To find δp and φ, it is necessary to solve Laplace's equation with boundary conditions on the shock wave ($y = \eta = 0$).

Transforming the boundary conditions (1.9) — (1.12) in the same way as in Ref. 1, we get

$$2\cot\varphi\,(v_y\delta v_x - v_x\delta v_y) - \left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\delta p = F(x); \tag{2.3}$$

$$F \equiv 2\cot\varphi(v_y\delta v_{0x} - v_x\delta v_{0y}) - \left[V_0\cot^2\varphi\left(1 + j^2\frac{\partial V}{\partial p_0}\right) - V\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0 + 2j\frac{V}{V_0}(V_0 - V)\,\delta v_{0y}$$
$$+ j^2\left[\left(1 - \frac{\partial V}{\partial V_0}\right)(V + V_0\cot^2\varphi) - 2\frac{V}{V_0}(V_0 - V)\right]\delta V_0. \tag{2.4}$$

Now, differentiating (2.3) with respect to x (along the shock wave) and, making use of the equations of hydrodynamics, we get

$$v_x\frac{\partial\delta v_y}{\partial x} - v_y\frac{\partial\delta v_x}{\partial x} = \frac{v_xv_y}{c^2}V\frac{\partial\delta p}{\partial x} - \left(1 - \frac{v_y^2}{c^2}\right)V\frac{\partial\delta p}{\partial y}.$$

As a result, we obtain the boundary condition in which there enters only the derivatives of δp along the shock wave and normal to it; transforming to ξ, η in it, we finally get

$$-\left[V_0\cot^2\varphi\left(1 - j^2\frac{\partial V}{\partial p}\right) - V\left(1 + j^2\frac{\partial V}{\partial p}\right)\right]\frac{\partial\delta p}{\partial\xi} + 2V\cot\varphi\sqrt{1 - M^2}\,\frac{\partial\delta p}{\partial\eta} = \frac{\partial F}{\partial\xi}. \tag{2.5}$$

To find the harmonic functions which satisfy (2.5), we introduce the quantity Φ by the definition

$$\Phi = \partial\delta p/\partial\eta + i\,\partial\delta p/\partial\xi. \tag{2.6}$$

By virtue of Laplace's equation for δp the quantity Φ, which is considered as a function of the complex variable

$$\zeta = \xi + i\eta = \frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}}x + \left(\frac{v_xv_y/c^2}{\sqrt{1 - M^2}} + i\right)y, \tag{2.7}$$

is analytic in the upper half plane.

Thus the problem reduces to finding an analytic function along the given connection of its imaginary and real parts for $\eta = 0$ (the Riemann–Hilbert problem). If we now define the function $\Omega(\zeta)$ by the expression

$$\Omega(\zeta) = \int^{\zeta}\Phi(\zeta)\,d\zeta,$$

where the integration is carried out along any contour lying in the upper half plane and diverging to infinity, then δp is connected with $\Omega(\zeta)$ by the obvious relation

$$\delta p(\xi, \eta) = \operatorname{Im}\Omega(\zeta). \tag{2.8}$$

For the determination of δv, we note that $\delta v = \delta v^{(1)} + \delta v^{(2)}$, where $\delta v^{(1)}$ is determined by the entropy-vortical solution, and $\delta v^{(2)} = \nabla\varphi$ by the solution of the elliptic equations. It is easy to see that a determination of $\delta v^{(2)}$ reduces again to a solution of the Riemann–Hilbert problem. Substituting in Eq. (2.3) for δp its expression in terms of $\delta v^{(2)}$ [Eq. (1.7)] and in place of δv the sum $\delta v^{(1)} + \delta v^{(2)}$, we find that $\delta v^{(1)}$ drops out of the expression and we obtain the following relation between the components $\delta v_x^{(2)}$ and $\delta v_y^{(2)}$ on the shock wave:

$$j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi)\,\delta v_y^{(2)}-j\cot\varphi\left[(V-V_0)+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)\right]\delta v_x^{(2)}=-F(x). \tag{2.9}$$

Transforming to the variables ξ, η, and introducing the notation

$$\delta v_\xi=\partial\varphi/\partial\xi,\quad \delta v_\eta=\partial\varphi/\partial\eta;$$
$$\delta v_x^{(2)}=\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}\,\delta v_\xi,\quad \delta v_y^{(2)}=\delta v_\eta+\frac{v_xv_y/c^2}{\sqrt{1-M^2}}\,\delta v_\xi,$$

we finally obtain:

$$j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi)\,\delta v_\eta-\frac{j}{\sqrt{1-M^2}}\left\{\cot\varphi\left[(V-V_0)+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)\right]\right.$$
$$\left.-\frac{j^3V}{c^2}\left[V\cot\varphi\,(V-V_0)+2V_0(V+V_0\cot^2\varphi)\right]\right\}\delta v_\xi=-F(\xi). \tag{2.10}$$

From this it is evident that the function

$$\Psi=\delta v_\eta+i\delta v_\xi,$$

considered as a function of the complex variable ζ, is analytic in the upper half plane; the determination of $\Psi(\zeta)$ also reduces to the Riemann—Hilbert problem.

The components of $\delta v^{(2)}$ are connected with $\Psi(\zeta)$ by the following obvious relations:

$$\delta v_x^{(2)}=(1-v_y^2/c^2)(1-M^2)^{-1/2}\,\mathrm{Im}\,\Psi(\zeta),\quad \delta v_y^{(2)}=\mathrm{Re}\,\Psi(\zeta)+(v_xv_y/c^2)(1-M^2)^{-1/2}\,\mathrm{Im}\,\Psi(\zeta). \tag{2.11}$$

As is well known,[2] the solution of the Riemann—Hilbert problem for the function $\Phi(\zeta)=u(\xi,\eta)+iv(\xi,\eta)$ satisfying the condition $au-bv=f(\xi)$ on the line $\eta=0$, where a and b are constants, and $f(\xi)$ is an arbitrary function, is given by the following integral of the Cauchy type:

$$\Phi(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{f(\tau)\,d\tau}{\tau-\zeta}.$$

Therefore, for the function $\Phi(\zeta)$ of interest to us, we have:

$$\Phi(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{\partial F(\tau)}{\partial\tau}\frac{d\tau}{\tau-\zeta},\quad a=2V\cot\varphi\sqrt{1-M^2},\quad b=V_0\cot^2\varphi\left(1-j^2\frac{\partial V}{\partial p}\right)-V\left(1+j^2\frac{\partial V}{\partial p}\right). \tag{2.12}$$

We shall consider that the function $F\to 0$ for $\tau\to\pm\infty$. Then, at infinity, $\Phi(\zeta)$ behaves as $1/\zeta^2$, so that the integral $\int\Phi(\zeta)\,d\zeta$ along the contour which goes off to infinity does have meaning; in this case,

$$\Omega(\zeta)=\int^{\zeta}\Phi(\zeta)\,d\zeta=-\frac{b+ia}{\pi(a^2+b^2)}\int d\zeta\int\frac{dT}{\tau-\zeta}\frac{\partial F(\tau)}{\partial\tau}.$$

Reversing the order of integration in the latter integral, and integrating over τ by parts, we find

$$\Omega(\zeta)=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{d\tau}{(\tau-\zeta)^2}\frac{\partial F(\tau)}{\partial\tau}=-\frac{b+ia}{\pi(a^2+b^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}. \tag{2.13}$$

The latter integration by parts is valid in the case in which $\partial F(\tau)/\partial\tau$ has finite discontinuities. Thus

$$\delta p(x,y)=-\mathrm{Im}\,\frac{b+ia}{\pi(a^2+b^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}=-\frac{1}{\pi(a^2+b^2)}\int\frac{F(t)}{G(t,x,y)}\left[a\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\,y\right)+b\,\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}\,y\right]dt, \tag{2.14}$$

where we use the notation

$$G(t,x,y)=\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\right)y^2+\frac{c^4(1-M^2)}{(c^2-v_y^2)^2}\,y^2.$$

In what follows we need the value of $\delta p(x,y)$ on the shock itself. As is well known (see, for example, Ref. 2), the value of the integral (2.13) at any point ξ on the boundary is equal to

$$\lim_{\zeta\to\xi}\int\frac{F(\tau)\,d\tau}{\tau-\zeta}=\pi iF(\xi)+\int\frac{F(\tau)\,d\tau}{\tau-\xi}, \tag{2.15}$$

where the diverging integral on the right side is taken in the sense of its principal value. We then have

$$\delta p(x,\,0)=-\frac{1}{\pi(a^2+b^2)}\left[b\pi F(x)+a\int\frac{F(t)\,dt}{t-x}\right]. \tag{2.16}$$

In similar fashion, we have for $\Psi(\zeta)$:

$$\Psi(\zeta)=\frac{\beta+i\alpha}{\pi(\alpha^2+\beta^2)}\int\frac{F(\tau)\,d\tau}{\tau-\zeta},\quad \alpha=j\left(1+j^2\frac{\partial V}{\partial p}\right)(V+V_0\cot^2\varphi),$$

$$\beta=j(1-M^2)^{-1/2}\left\{\cot\varphi\left[V-V_0+\frac{V_0}{V}(V+V_0\cot^2\varphi)\left(1-j^2\frac{\partial V}{\partial p}\right)-\frac{j^2V}{c^2}[V\cot\varphi\,(V-V_0)+2V_0(V+V_0\cot^2\varphi)]\right]\right\}. \tag{2.17}$$

The components of $\delta v^{(2)}$, in accord with (2.11), are expressed by the following formulas:

$$\delta v_x^{(2)}(x,y)=\frac{1-v_y^2/c^2}{\pi\sqrt{1-M^2}\,(\alpha^2+\beta^2)}\int\frac{F(t)}{G(t,\,x,\,y)}\left[\alpha\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\,y\right)+\beta\,\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}\,y\right]dt,$$

$$\delta v_y^{(2)}(x,\,y)=\frac{v_xv_y}{c^2-v_y^2}\,\delta v_x^{(2)}+\frac{1}{\pi(\alpha^2+\beta^2)}\int\frac{F(t)}{G(t,\,x,\,y)}\left[\beta\left(t-x-\frac{v_xv_y}{c^2-v_y^2}\,y\right)-\alpha\,\frac{c^2\sqrt{1-M^2}}{c^2-v_y^2}\,y\right]dt. \tag{2.18}$$

We take the values of $\delta v_x^{(2)}$ and $\delta v_y^{(2)}$ on the shock wave itself:

$$\delta v_x^{(2)}(x,\,0)=\frac{1-v_y^2/c^2}{\pi\sqrt{1-M^2}\,(\alpha^2+\beta^2)}\left[\pi\beta F(x)+\alpha\int\frac{F(t)\,dt}{t-x}\right],$$

$$\delta v_y^{(2)}(x,\,0)=\frac{1}{\pi(\alpha^2+\beta^2)}\left[-\pi\alpha F(x)+\beta\int\frac{F(t)\,dt}{t-x}\right]+\frac{v_xv_y}{c^2-v_y^2}\,\delta v_x^{(2)}(x,\,0). \tag{2.19}$$

We now proceed to finding the contribution of the entropy-vortical solution. The value of δV consists of two parts:

$$\delta V=\delta V_1+\delta V_2,\ \delta V_1=(\partial V/\partial s)_p\,\delta s,\ \delta V_2=-(V/c)^2\,\delta p.$$

Condition (1.11) gives for δV_1 on the shock wave:

$$\delta V_1=\left(\frac{V^2}{c^2}+\frac{\partial V}{\partial p}\right)\delta p+\frac{\partial V}{\partial p_0}\,\delta p_0+\frac{\partial V}{\partial V_0}\,\delta V_0. \tag{2.20}$$

If we then substitute the expression for δp [Eq. (2.16)] and replace the argument x by $x-y/k$, we obtain:

$$\delta V_1(x,\,y)=-\frac{V^2/c^2+\partial V/\partial p}{\pi(a^2+b^2)}\left[b\pi F\left(x-\frac{y}{k_0}\right)+a\int\frac{F(t)\,dt}{t-x+y/k_0}\right]+\frac{\partial V}{\partial p_0}\,\delta p_0\left(x-\frac{y}{k_0}\right)+\frac{\partial V}{\partial V_0}\,\delta V_0(x-y/k_0). \tag{2.21}$$

The divergent integral here, as in subsequent formulas, is understood in the sense of the principal value.

The contribution of the entropy-vortical solution to the velocity δv_1, is determined by the quantity $I_0=v\delta v_1$. Repeating the work of the previous paper, we obtain the value of I_0 on the shock wave:

$$I_0=v_0\delta v_0-\frac{1}{2}(V_0-V)\left(1+j^2\frac{\partial V}{\partial p}\right)\delta p+\frac{1}{2}\left[V\left(1+j^2\frac{\partial V}{\partial p_0}\right)+V_0\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0+\frac{1}{2}\,j^2(V_0-V)\left[2\frac{V}{V_0}-1+\frac{\partial V}{\partial V_0}\right]\delta V_0. \tag{2.22}$$

Replacing δp by its value (2.16), replacing the argument x by $x-y/k_0$, we find

$$I_0(x,\,y)=v_0\delta v_0(x-y/k_0)+\frac{1}{2}\frac{(V_0-V)(1+j^2\partial V/\partial p)}{\pi(a^2+b^2)}\left[b\pi F\left(x-\frac{y}{k_0}\right)+a\int\frac{F(t)\,dt}{t-x+y/k_0}\right]$$

$$+\frac{1}{2}\left[V\left(1+j^2\frac{\partial V}{\partial p_0}\right)+V_0\left(1-j^2\frac{\partial V}{\partial p_0}\right)\right]\delta p_0\left(x-\frac{y}{k_0}\right)+\frac{1}{2}\,j^2(V_0-V)\left[2\frac{V}{V_0}-1+\frac{\partial V}{\partial V_0}\right]\delta V_0\left(x-\frac{y}{k_0}\right). \tag{2.23}$$

We recall that δv_1 is connected with I_0 by the relation

$$\delta v_1=(v/M^2c^2)\,I_0(x,\,y).$$

The complete solution is obtained by summation of the entropy-vortical solution and the solution of the elliptic equation.

Finally, let us compute the form of the perturbed sound wave which we shall characterize by the curvature $K(x) \approx \eta''(x)$, for which we make use of the condition (1.12). Differentiating with respect to x and taking it into account that

$$\frac{\partial \delta p}{\partial x} = \frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}} \operatorname{Im} \Phi(\zeta),$$

we find

$$K(x) \Rightarrow \eta''(x) = \tan\varphi \left\{ \frac{\delta v'_{0y}(x)}{v_{0y}} + \frac{V/V_0 - \partial V/\partial V_0}{V_0 - V} \delta V'_0(x) + \frac{1 - j^2 \partial V/\partial p_0}{2(p - p_0)} \delta p'_0(x) - \frac{1 + j^2 \partial V/\partial p}{2(p - p_0)} \frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}} \operatorname{Im} \Phi\left(\frac{1 - v_y^2/c^2}{\sqrt{1 - M^2}} x\right)\right\}. \tag{2.24}$$

We see that the curvature of the front of the shock wave is determined by the first derivatives of hydrodynamic quantities.

The formulas obtained in this section have a general character, inasmuch as no assumptions have been made regarding the character of the incident perturbation. If we set

$$\delta s_0 = 0;\ \delta V_0 = -\frac{V_0^2}{c_0^2}\delta p_0,\ \delta v_{0x} = \frac{\mp v_{0y}\sqrt{M_0^2 - 1} - v_{0x}}{M_0^2 c_0^2} V_0 \delta p_0,\quad \delta v_{0y} = \frac{\pm v_{0x}\sqrt{M_0^2 - 1} - v_{0y}}{M_0^2 c_0^2} V_0 \delta p_0, \tag{2.25}$$

this will mean that the incident perturbation is of an acoustic type and is connected with the characteristics k_{+0} (upper sign; the shock wave "overtakes" the perturbation) or k_{-0} (lower sign: the shock wave "encounters" the perturbation).

If we are interested in the interaction of the shock wave with the entropy-vortical perturbation, we must set

$$\delta p_0 = 0;\ \delta V_0 = (\partial V/\partial s)_{p_0} \delta s_0;\ v_{0x}\delta v_{0y} - v_{0y}\delta v_{0x} = 0. \tag{2.26}$$

This perturbation is connected with the characteristics k_{00}.

3. INTERACTION OF A WEAK DISCONTINUITY WITH A SHOCK WAVE

We now proceed to a study of the interaction of a weak discontinuity with a shock wave, for which we will be interested in the intensity of a weak tangential discontinuity which passes "through" the shock wave, the behavior of all quantities near the point of intersection of the weak discontinuity and the shock wave, and also the form of the shock wave close to the point of intersection.

Let a small perturbation of arbitrary type, with a weak discontinuity, fall on the shock wave from the side $y < 0$; we consider the most interesting case, in which the weak discontinuity is a discontinuity of derivatives of hydrodynamical quantities. The intensity of the incident weak discontinuity is characterized by the jumps Δ, Δ_1, and Δ_2 in the derivatives of the pressure (for weak discontinuities of the acoustic type), of the entropy Δ_1, and of the quantity I_0 (for weak tangential discontinuities) respectively along the normal to the characteristic bearing the discontinuity. We choose the point of intersection of the discontinuity with the shock wave as the origin of the coordinates. We denote the jumps in the derivatives of the hydrodynamical quantities at the origin, along the shock wave, as Δp_0, Δv_0, etc. In correspondence with what has been shown, the decomposition of all the quantities on the shock wave from the side $y < 0$ near the origin will have the form

$$f(x) = \begin{cases} f(0) + (A_f - \Delta_f)x & \text{for } x < 0, \\ f(0) + A_f x & \text{for } x > 0. \end{cases}$$

Let us determine tentatively the behavior of integrals of the Cauchy type, which are encountered in our problem, near the point $\zeta = 0$.

As is well known,[1] the integral (2.12), in the neighborhood of the point $\zeta = 0$, and for the presence of a discontinuity in $\partial F/\partial \tau$, will be logarithmic:

$$\Phi(\zeta) \cong \frac{b + ia}{\pi(a^2 + b^2)} \frac{\sqrt{1 - M^2}}{1 - v_y^2/c^2} \Delta_F (\ln \zeta + A), \tag{3.1}$$

where A is a constant dependent on the behavior of the function $F(\tau)$ everywhere on the x axis.

In what follows, we need the values of (3.1) on the real axis

$$\Phi(\xi) \cong \begin{cases} \frac{\Delta_F}{\pi(a^2+b^2)}[(b\ln|\xi| - a\pi) + i(a\ln|\xi| + b\pi) + A] & \text{for } \xi < 0 \\ \frac{\Delta_F}{\pi(a^2+b^2)}[(b+ia)\ln\xi + A] & \text{for } \xi > 0. \end{cases} \quad (3.2)$$

For the determination of the behavior of the function $\Omega(\zeta)$ close to the origin, we integrate (3.1):

$$\Omega(\zeta) = \Omega(0) + \frac{(b+ia)\Delta_F\sqrt{1-M^2}}{\pi(a^2+b^2)(1-v_y^2/c^2)}\zeta(\ln\zeta + A - 1). \quad (3.3)$$

This formula gives an expression for δp behind the shock wave in the region about the origin. Actually, taking the imaginary part of the expression (3.3), and transforming to the original variables x, y, we will have

$$\delta p(x,y) = \delta p(0,0) + \frac{\sqrt{1-M^2}}{1-v_y^2/c^2}\frac{\Delta_F}{\pi(a^2+b^2)}\left[\left(a\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x + \left(\frac{av_xv_y/c^2}{\sqrt{1-M^2}} + b\right)y\right)\frac{1}{2}\ln\left[\left(x + \frac{v_xv_y}{c^2-v_y^2}y\right)^2 + \frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right]\right.$$
$$\left. + \left(b\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x + \left(\frac{bv_xv_y/c^2}{\sqrt{1-M^2}} - a\right)y\right)\tan^{-1}\frac{\sqrt{1-M^2}y}{(1-v_y^2/c^2)x + (v_xv_y/c^2)y}\right] + A_px + B_py. \quad (3.4)$$

Here $\delta p(0,0)$ is determined by Eq. (2.16); A_p and B_p are constants which are connected in a definite way with A. On the shock wave itself, the following relations hold:

$$\delta p(x,0) = \delta p(0,0) + \frac{\Delta_F a}{\pi(a^2+b^2)}x\ln|x| + (A_p - \Delta_p)x \quad \text{for } x<0,$$
$$\delta p(x,0) = \delta p(0,0) + \frac{\Delta_F a}{\pi(a^2+b^2)}x\ln x + A_px \quad \text{for } x>0, \quad (3.5)$$

where

$$\Delta_p = -\Delta_F b/(a^2+b^2).$$

The function $\Psi(\zeta)$ which is necessary for the calculation of $\delta v^{(2)}$ differs from $\Omega(\zeta)$ by the replacement of a by $-\alpha$ and of b by $-\beta$. Making use of Eqs. (2.11), we find (close to the origin):

$$\delta v_x^{(2)}(x,y) = \delta v_x^{(2)}(0,0) - \frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\left[\left(\alpha\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x + \left(\frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}} + \beta\right)y\right)\frac{1}{2}\ln\left[\left(x + \frac{v_xv_y}{c^2-v_y^2}y\right)^2 + \frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right]\right.$$
$$\left. + \left(\beta\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x + \left(\frac{\beta v_xv_y/c^2}{\sqrt{1-M^2}} - \alpha\right)y\right)\tan^{-1}\frac{y\sqrt{1-M^2}}{(1-v_y^2/c^2)x + (v_xv_y/c^2)y}\right] + A_{v_x}x + B_{v_x}y.$$

$$\delta v_y^{(2)}(x,y) = \delta v_y^{(2)}(0,0) + \frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\cdot\frac{\sqrt{1-M^2}}{1-v_y^2/c^2}\left\{\left[\left(\alpha\left(1 - \frac{v_x^2v_y^2/c^4}{1-M^2}\right) - \frac{2\beta v_xv_y/c^2}{\sqrt{1-M^2}}\right)y - \left(\beta + \frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x\right]\frac{1}{2}\right.$$
$$\times\ln\left[\left(x + \frac{v_xv_y}{c^2-v_y^2}y\right)^2 + \frac{1-M^2}{(1-v_y^2/c^2)^2}y^2\right] + \left[\left(\alpha - \frac{\beta v_xv_y/c^2}{\sqrt{1-M^2}}\right)\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}x + \left(\frac{2\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right.\right.$$
$$\left.\left.\left. + \beta\left(1 - \frac{v_x^2v_y^2/c^4}{1-M^2}\right)\right)y\right]\tan^{-1}\frac{y\sqrt{1-M^2}}{(1-v_y^2/c^2)x + (v_xv_y/c^2)y}\right\} + A_{v_y}x + B_{v_y}y. \quad (3.6)$$

On the shock wave itself, we have

$$\delta v_x^{(2)}(x,0) = \delta v_x^{(2)}(0,0) - \frac{\Delta_F\alpha}{\pi(\alpha^2+\beta^2)}\left(1 - \frac{v_y^2}{c^2}\right)x\ln|x| + (A_{v_x} - \Delta_{v_x})x \quad \text{for } x<0,$$
$$\delta v_x^{(2)}(x,0) = \delta v_x^{(2)}(0,0) - \frac{\Delta_F\alpha}{\pi(\alpha^2+\beta^2)}(1 - v_y^2/c^2)x\ln x + A_{v_x}x \quad \text{for } x>0,$$
$$\delta v_y^{(2)}(x,0) = \delta v_y^{(2)}(0,0) - \frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\left(\beta + \frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)x\ln|x| + (A_{v_y} - \Delta_{v_y})x, \quad \text{for } x<0,$$
$$\delta v_y^{(2)}(x,0) = \delta v_y^{(2)}(0,0) - \frac{\Delta_F}{\pi(\alpha^2+\beta^2)}\left(\beta + \frac{\alpha v_xv_y/c^2}{\sqrt{1-M^2}}\right)x\ln x + A_{v_y}x \quad \text{for } x>0,$$

where

$$\Delta_{v_x} = \frac{\Delta_F \beta}{\alpha^2+\beta^2}\frac{1-v_y^2/c^2}{\sqrt{1-M^2}}, \quad \Delta_{v_y} = \frac{\Delta_F}{\alpha^2+\beta^2}\left(\frac{\beta v_x v_y/c^2}{\sqrt{1-M^2}} - \alpha\right).$$

We now find the contribution of the entropy-vortical solution. The quantity $\delta V^{(1)}$ is determined from Eq. (2.20); substituting $x - y/k_0$ for x in this expression, we get the value of $\delta V^{(1)}$ in the vicinity of the flow line $x - y/k_0 = 0$, which will be the bearer of a real tangential discontinuity of very unique character:

$$\delta V^{(1)}(x, y) = \delta V^{(1)}(0,0) + \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p}\right)\frac{\Delta_F a}{\pi(a^2+b^2)}\left(x - \frac{y}{k_0}\right)\ln\left(\frac{y}{k_0} - x\right) + (A_V - \Delta_V)\left(x - \frac{y}{k_0}\right) \quad \text{for } x < y/k_0,$$

$$\delta V^{(1)}(x, y) = \delta V^{(1)}(0,0) + \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p}\right)\frac{\Delta_F a}{\pi(a^2+b^2)}\left(x - \frac{y}{k_0}\right)\ln\left(x - \frac{y}{k_0}\right) + A_V\left(x - \frac{y}{k_0}\right) \quad \text{for } x > y/k_0,$$

$$\Delta_V = \frac{\partial V}{\partial p_0}\Delta_{p_0} + \frac{\partial V}{\partial V_0}\Delta_{V_0} - \left(\frac{V^2}{c^2} + \frac{\partial V}{\partial p}\right)\frac{\Delta_F b}{a^2+b^2}. \tag{3.8}$$

Of similar character is the special feature possessed by the quantity I_0. According to (2.2), we have

$$I_0(x, y) = I_0(0,0) - \frac{1}{2}(V_0 - V)\left(1 + j^2\frac{\partial V}{\partial p}\right)\frac{\Delta_F a}{\pi(a^2+b^2)}\left(x - \frac{y}{k_0}\right)\ln\left(\frac{y}{k_0} - x\right) + (A_{I_0} - \Delta_{I_0})\left(x - \frac{y}{k_0}\right) \quad \text{for } x < y/k_0,$$

$$I_0(x, y) = I_0(0,0) - \frac{1}{2}(V_0 - V)\left(1 + j^2\frac{\partial V}{\partial p}\right)\frac{\Delta_F a}{\pi(a^2+b^2)}\left(x - \frac{y}{k_0}\right)\ln\left(x - \frac{y}{k_0}\right) + A_{I_0}\left(x - \frac{y}{k_0}\right) \quad \text{for } x > y/k_0,$$

$$\Delta_{I_0} = v_0\Delta_{v_0} + \frac{1}{2}\left[V\left(1 + j^2\frac{\partial V}{\partial p_0}\right) + V_0\left(1 - j^2\frac{\partial V}{\partial p_0}\right)\right]\Delta_{p_0}$$

$$+ \frac{1}{2}j^2(V_0 - V)\left[2\frac{V}{V_0} - 1 + \frac{\partial V}{\partial V_0}\right]\Delta_{V_0} + \frac{1}{2}(V_0 - V)\left(1 + j^2\frac{\partial V}{\partial p}\right)\frac{\Delta_F b}{a^2+b^2}. \tag{3.9}$$

We see from Eqs. (3.8) and (3.9) that a weak tangential discontinuity, which passes through a shock wave, differs in the subsonic case in an unusual character: in addition to the jump in the coefficients in the linear terms of the expansion of hydrodynamical quantities in the vicinity of a weak discontinuity, this expansion is characterized by the present of a logarithmic singularity.

For the calculation of the curvature of the shock wave in the vicinity of the origin, we make use of Eq. (2.24):

$$K(x) = \tan\varphi\left\{A_K - \Delta_K - \frac{1}{2}\frac{1 + j^2\frac{\partial V}{\partial p}}{p - p_0}\frac{\Delta_F a}{\pi(a^2+b^2)}\ln|x|\right\} \quad \text{for } x < 0,$$

$$K(x) = \tan\varphi\left\{A_K - \frac{1}{2}\frac{1 + j^2\frac{\partial V}{\partial p}}{p - p_0}\frac{\Delta_F a}{\pi(a^2+b^2)}\ln x\right\} \quad \text{for } x > 0,$$

$$\Delta_K = \frac{\Delta v_{0y}}{v_{0y}} + \frac{V/V_0 - \partial V/\partial V_0}{V_0 - V}\Delta_{V_0} + \frac{1 - j^2\partial V/\partial p_0}{2(p - p_0)}\Delta_{p_0} + \frac{1 + j^2\partial V/\partial p}{2(p - p_0)}\frac{\Delta_F b}{a^2+b^2}.$$

Thus the curvature of the shock wave in the subsonic case also has a logarithmic singularity at the point of intersection with a weak discontinuity.

There now remains only the connecting of the quantities Δ_F, Δ_{p_0}, etc. with the intensity of the incident weak discontinuity. Let us first consider the case of the incidence of the usual weak discontinuity of the acoustic type. Inasmuch as the weak discontinuity can fall along either of two characteristics k_{+0} and k_{-0}, we must distinguish two cases (Figs. 2 and 3). The connection of the quantity Δ_{p_0} with the intensity of the incident weak discontinuity follows from elementary geometric considerations:

$$\Delta_{p_0} = \frac{k_{\pm 0}}{\sqrt{1 + k_{\pm 0}^2}}\Delta_{\pm} = \frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2 - 1}}{c_0\left[v_{0x}\sqrt{M_0^2 - 1} \pm v_{0y}\right]}\Delta_{\pm}. \tag{3.11}$$

Therefore, with the help of Eqs. (2.25), we get

$$\Delta_{V_0} = -V_0^2\,\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0^3\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm},$$

$$\Delta_{v_{0x}} = \frac{V_0}{M_0^2}\,\frac{[\mp v_{0y}\sqrt{M_0^2-1} - v_{0x}]\,[v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}]}{c_0^3\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm}, \qquad (3.12)$$

$$\Delta_{v_{0y}} = \frac{V_0}{M_0^2}\,\frac{[\pm v_{0x}\sqrt{M_0^2-1} - v_{0y}]\,[v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}]}{c_0^3\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm}.$$

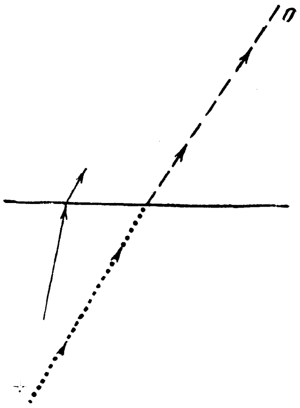

FIG. 2

Further, we find:

$$\Delta_F = \Big\{\frac{2j^2V_0}{M_0^2c_0^2}\,[(V_0 - V)(V_0\cot^2\varphi - V) \mp (V^2 + V_0^2\cot^2\varphi)\cot\varphi\sqrt{M_0^2-1}]$$
$$- \Big[V_0\cot^2\varphi\Big(1 + j^2\frac{\partial V}{\partial p_0}\Big) - V\Big(1 - j^2\frac{\partial V}{\partial p_0}\Big)\Big] - \frac{j^2V_0^2}{c_0^2}\Big[\Big(1 - \frac{\partial V}{\partial V_0}\Big)(V + V_0\cot^2\varphi)$$
$$-2\frac{V}{V_0}(V_0 - V)\Big]\Big\}\,\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm};$$

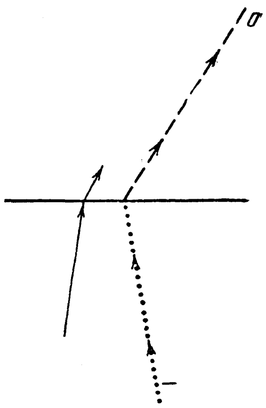

FIG. 3

$$\Delta_V = -\Big(\frac{V^2}{c^2} + \frac{\partial V}{\partial p}\Big)\frac{b}{a^2+b^2}\,\Delta_F + \Big(\frac{\partial V}{\partial p_0} - \frac{V_0^2}{c_0^2}\frac{\partial V}{\partial V_0}\Big)\,\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm};$$

$$\Delta_{J_0} = \frac{1}{2}(V_0 - V)\Big(1 + j^2\frac{\partial V}{\partial p}\Big)\frac{b}{a^2+b^2}\,\Delta_F - \frac{1}{2}(V_0 - V)\Big[1 + j^2\frac{\partial V}{\partial p_0}$$
$$+ \frac{j^2V_0^2}{c_0^2}\Big(2\frac{V}{V_0} - 1 + \frac{\partial V}{\partial V_0}\Big)\Big]\,\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm};$$

$$\Delta_K = \frac{1 + j^2\dfrac{\partial V}{\partial p}}{2(p - p_0)}\,\frac{b}{a^2+b^2}\,\Delta_F + \Big[\frac{1 - j^2\partial V/\partial p_0}{2(p - p_0)} - \frac{V/V_0 - \partial V/\partial V_0}{V_0 - V}\,\frac{V_0^2}{c_0^2} \qquad (3.13)$$
$$+ \frac{\pm\cot\varphi\sqrt{M_0^2-1} - 1}{M_0^2c_0^2}\Big]\,\frac{v_{0x}v_{0y} \pm c_0^2\sqrt{M_0^2-1}}{c_0\,[v_{0x}\sqrt{M_0^2-1} \pm v_{0y}]}\,\Delta_{\pm}.$$

The upper and lower signs in these equations refer to the cases a and b, respectively.

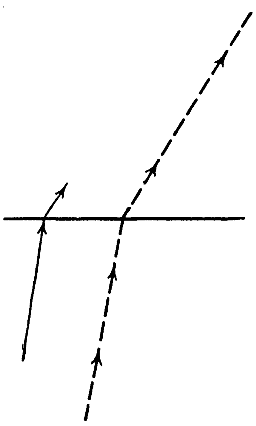

FIG. 4

Now let us consider the case of the interaction of a weak tangential discontinuity with the shock wave (Fig. 4). In correspondence with Eqs. (2.26), we have

$$\Delta_{p_0} = 0; \quad \Delta_{V_0} = \frac{v_{0y}}{M_0c_0}\Big(\frac{\partial V}{\partial s}\Big)_{p_0}\Delta_1; \quad \Delta_{v_{0x}} = \frac{v_{0x}v_{0y}}{M_0^3c_0^3}\,\Delta_2; \quad \Delta_{v_{0y}} = \frac{v_{0y}^2}{M_0^3c_0^3}\,\Delta_2. \qquad (3.14)$$

Therefore, with the help of elementary calculations, we obtain

$$\Delta_F = 2j^3(V_0 - V)V_0(V - V_0\cot^2\varphi)\frac{\Delta_2}{M_0^3c_0^3} + \frac{j^3}{M_0c_0}\Big[\Big(1 - \frac{\partial V}{\partial V_0}\Big)(V + V_0\cot^2\varphi)$$
$$-2\frac{V}{V_0}(V_0 - V)\Big]V_0\Big(\frac{\partial V}{\partial s}\Big)_{p_0}\Delta_1; \quad \Delta_V = -\Big(\frac{V^2}{c^2} + \frac{\partial V}{\partial p}\Big)\frac{b}{a^2+b^2}\,\Delta_F + \frac{jV_0}{M_0c_0}\Big(\frac{\partial V}{\partial s}\Big)_{p_0}\frac{\partial V}{\partial V_0}\Delta_1;$$
$$\Delta_{J_0} = \frac{1}{2}(V_0 - V)\Big(1 + j^2\frac{\partial V}{\partial p}\Big)\frac{b}{a^2+b^2}\,\Delta_F + \frac{jV_0}{M_0c_0}\,\Delta_2 + \frac{1}{2}j^3(V_0 - V)\Big(2\frac{V}{V_0} - 1$$
$$+ \frac{\partial V}{\partial V_0}\Big)\frac{V_0}{M_0c_0}\Big(\frac{\partial V}{\partial s}\Big)_{p_0}\Delta_1; \quad \Delta_K = \frac{1 + j^2\,\partial V/\partial p}{2(p - p_0)}\,\frac{b}{a^2+b^2}\,\Delta_F + \frac{jV_0}{M_0^3c_0^3}\Delta_2 + \frac{V/V_0 - \partial V/\partial V_0}{V_0 - V}\,\frac{jV_0}{M_0c_0}\Big(\frac{\partial V}{\partial s}\Big)_{p_0}\Delta_1.$$

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Translated by R. T. Beyer.