Thus, for given values of energy and angle, the results of double and triple scattering experiments can yield 11 independent relations among the parameters of the transition matrix in the case of elastic scattering of neutrons against deuterons. One additional relation can be obtained from experiments on the scattering of protons against deuterons, since due to charge invariance, all the above results also apply to this case (if one only includes the electromagnetic interaction).

As shown in the work of Smorodinskii and others,[5] the transition matrix is determined by as many real functions as there are variables in its most general formulation. Thus, if one carries out triple scattering experiments and obtains the above mentioned experimental data, it should be possible to carry out a phase shift analysis.

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NUCLEAR FORCES AND THE SCATTERING OF π MESONS

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The interaction of two fixed nucleons in pseudoscalar meson theory with pseudovector coupling is considered. The principal part of the functional of the two-nucleon state is represented in the form of a product of single-nucleon functionals. Consideration is given only to the states without real mesons and with one real meson. A procedure is developed for reducing two-nucleon renormalized matrix elements to single nucleon elements, which are then calculated by the method of Chew, Low, and Wick. The potential of order e^{-2R} is calculated. It consists of two parts: one part is proportional to f^4 (f is the interaction constant), and the other is a function of the phases of π meson scattering on nucleons.

RECENTLY, Chew and Low[1] and Wick[2] considered the one-nucleon problem from a new point of view. Characteristic of their approach is the attempt to solve the problem without perturbation theory, and thus to deal only with renormalized quantities.

In considering the two-nucleon problem in the region of nonrelativistic energies, it may be assumed that the meson clouds of the interacting nucleons conserve their individuality.

Therefore, we may feel confident that in this energy region, quantities referring to two interacting nucleons will be expressed by single-nucleon quantities, so that the method of Chew, Low, and Wick may also be used in the two-nucleon problem. In the present work, we will approach the theory of nucleon interaction from this point of view.*

1. STATEMENT OF THE PROBLEM

We consider a field of π mesons, which interacts with two nucleons, fixed at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. The energy operator for such a system is

$$H = H_0 + U_1 + U_2, \tag{1}$$

where H_0 is the energy operator of the meson field

$$H_0 = \sum_q q_0 a_q^+ a_q, \tag{2}$$

and U_A is the interaction operator of nucleon A with the meson field in symmetrical theory

$$U_A = \sum_q \{V_{Aq}^0 a_q e^{i\mathbf{qr}_A} + V_{Aq}^0 a_q^+ e^{-i\mathbf{qr}_A}\}; \tag{3}$$

$$V_{Aq}^0 = i f_0 [(\sigma_A \mathbf{q}) / \sqrt{2q_0}] \tau_{Aq} v(q). \tag{4}$$

Here, $q_0 = (1 + q^2)^{1/2}$, the spin operator σ_A and the isotopic spin operator τ_A refer to nucleon A; a_q and a_q^+ are the annihilation and creation operators of a meson in the state q; for simplicity, one symbol q is used to denote all the quantum numbers of the meson — the momentum and the third projection of the isotopic spin. We use a system of units in which $h = c = \mu = 1$ (μ is the meson mass), f_0 is the non-renormalized coupling constant (in a rationalized system), and v(q) is the source function.

The eigenfunctions Ψ^n of the Hamiltonian H refer to states with two nucleons and different numbers of real mesons: without mesons (n = 0), with one meson (n = 1), etc. We are interested in Ψ^0 — the state with two nucleons, fixed at a distance $R = |\mathbf{r}_2 - \mathbf{r}_1|$, and surrounded by a cloud of virtual mesons. In the representation in which the creation operator a_q^+ is a multiplicative operator on an auxiliary function $\bar{a}_q$, the Schroedinger equation for Ψ^0 is written in the form

$$H\Psi_\sigma^0(1, 2, \bar{a}) = \{2E_0 + E_\sigma(\mathbf{R})\}\Psi_\sigma^0(1, 2, \bar{a}). \tag{5}$$

The energy eigenvalue consists of two parts — the self-energy of both nucleons $2E_0$ and the static interaction energy of the nucleons $E_\sigma(R)$; $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. The index $\sigma \equiv (S', I', S_3, I_3)$ characterizes the eigenvalues of the total spin, total isotopic spin, and their third projections. The state vector $\Psi^0(1, 2, \bar{a})$ is a function of the parameters $\mathbf{r}_1$, $\mathbf{r}_2$ and the spin-charge variables of nucleons 1 and 2. Moreover, $\Psi_\sigma^0(1, 2, \bar{a})$ is a function of the variables of the virtual mesons, and Ψ_σ^0 is a functional of $\bar{a}_q$. It is sometimes convenient to use the representation

$$\Psi_\sigma^0(1, 2, \bar{a}) = \Psi_\sigma^0(1, 2, a^+)\Lambda_0, \tag{6}$$

where Λ_0 is the vacuum state of the meson field.

Our problem is the calculation of Ψ_σ^0 and $E_\sigma(\mathbf{R})$. We will attempt to reduce the two-nucleon problem to a one-nucleon problem. For this purpose it is necessary to study products of one-nucleon state vectors $F_\alpha(1, \bar{a})$ and $F_\beta(2, \bar{a})$ (α is the spin-charge index, taking four values). The quantity $F(1, \bar{a})$ is a solution of the Schrödinger equation

$$H_1 F(1, \bar{a}) \equiv (H_0 + U_1) F(1, \bar{a}) = E_0 F(1, \bar{a}). \tag{7}$$

Similar to $\Psi^0(1, 2, \bar{a})$, the one-nucleon state $F(1, \bar{a})$ is also a function of the variables of the meson field. We will use the formula

$$F(1, \bar{a}) = F(1, a^+)\Lambda_0. \tag{8}$$

Different spin-charge states are orthogonal

$$(F_\alpha(1, \bar{a}), F_\beta(1, \bar{a})) = \delta_{\alpha\beta}. \tag{9}$$

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*A brief communication of part of the results was published in J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1262 (1957) [Soviet Phys. JETP 5, 1030 (1957)].

The dependence of $F(1, \bar{a})$ on the coordinate $\mathbf{r}_1$ has the form

$$F(1, \bar{a}) = e^{-i\mathbf{u}\mathbf{r}_1} F^0(1, \bar{a}) = F(1, \bar{a}_q e^{-i\mathbf{q}\mathbf{r}_1}), \tag{10}$$

where $\mathbf{u}$ is the momentum operator of the meson field, and $F^0(1, \bar{a})$ is independent of $\mathbf{r}_1$.

We will consider the asymptotic behavior of $\Psi^0(1, 2, \bar{a})$ at great distances R, at which the nucleons do not interact. In this case the meson cloud in each nucleon will have the same character as in the absence of the other nucleon. Therefore, as $R \to \infty$, the functional of the two-nucleon state $\Psi^0(1, 2, \bar{a})$ must be a linear combination of products of the one-nucleon functionals $F(1, \bar{a})F(2, \bar{a})$: as $R \to \infty$

$$\Psi^0_\sigma(1, 2, \bar{a}) \to \Phi^\sigma(1, 2, \bar{a}) = \sum c_{\alpha\beta} F_\alpha(1, \bar{a}) F_\beta(2, \bar{a}), \tag{11}$$

i.e., the functional $\Phi^\sigma(1, 2, \bar{a})$ must be a solution of the Schrödinger equation

$$(H - 2E_0)\Phi^\sigma(1, 2, \bar{a}) = 0, \quad R \to \infty. \tag{12}$$

This was shown by Ekstein[3] in a nonstationary treatment. For the stationary treatment adopted by us, the asymptotic behavior of Ψ^0 and the state vector with one real meson Ψ^q are considered in the Appendix.

The description of noninteracting nucleons by the use of the state vector (11) differs from the usual description in that the meson clouds of both nucleons are taken into account from the very beginning. The products $\Phi_{\alpha\beta}(1, 2, \bar{a}) = F_\alpha(1, \bar{a})F_\beta(2, \bar{a})$ are orthogonal only as $R \to \infty$

$$(\Phi_{\alpha\beta}(1, 2, \bar{a}), \Phi_{\alpha'\beta'}(1, 2, \bar{a})) = \delta_{\alpha\alpha'}\delta_{\beta\beta'}, \quad R \to \infty. \tag{13}$$

For finite R, a term depending on R and not containing $\delta_{\alpha\alpha'}\delta_{\beta\beta'}$ is added to the right hand side of Eq. (13). The nonorthogonality of $\Phi_{\alpha\beta}(1, 2, \bar{a})$ is a result of the fact that the kinematic interaction, which is associated with the identity and statistics of the mesons and is similar to the interaction by means of Pauli's principle for a Fermi field, is already taken into account in $\Phi_{\alpha\beta}$. The region of nonorthogonality may be identified with the region in which the meson clouds overlap, and one and the same meson may belong to both clouds.

We will seek a solution of the Schrödinger equation (5) for finite R in the form:

$$\Psi^0_\sigma(1, 2, \bar{a}) = \Phi^\sigma(1, 2, \bar{a}) + \chi_\sigma(1, 2, \bar{a}), \tag{14}$$

where $\Phi^\sigma(1, 2, \bar{a})$, defined by Eq. (11), coincides with $\Psi^0_\sigma(1, 2, \bar{a})$ as $R \to \infty$. Only the states Ψ^0_μ and Ψ^0_μ (with one real meson) are considered in the expansion of χ_σ in eigenfunctions of the operator H. We are interested only in distances at which potentials of the order e^{-R} and e^{-2R} are of fundamental importance. As we will see below, in Sec. 3, the first term of Φ^σ in Eq. (14) already yields the potential e^{-R} and the major part of the potential e^{-2R}. We therefore consider χ_σ to be a small quantity. The terms neglected here yield terms in the potential of the order e^{-3R} and higher. We then obtain

$$\Psi^0_\sigma(1, 2, \bar{a}) = \frac{1}{(\Phi^\sigma, \Phi^\sigma)}\Big\{\Phi^\sigma - \sum_{\mu \neq \sigma} (\Phi^\mu, \Phi^\sigma)\Phi^\mu - \sum_{\mu q} \frac{1}{q_0}(\Phi^{q\mu}, H\Phi^\sigma)\Phi^{q\mu}\Big\}; \tag{15}$$

$$E_\sigma(R) = \frac{1}{(\Phi^\sigma, \Phi^\sigma)}\Big\{(\Phi^\sigma, H\Phi^\sigma) - \sum_{\mu \neq \sigma} (\Phi^\mu, \Phi^\sigma)(\Phi^\sigma, H\Phi^\mu)$$
$$- \sum_{\mu q} \frac{1}{q_0}(\Phi^{q\mu}, H\Phi^\sigma)(\Phi^\sigma, H\Phi^{q\mu})\Big\}. \tag{16}$$

Here and in the future, we will assume $E_0 = 0$, since the self-energy enters everywhere only as a difference. We replaced Ψ^0_σ by Φ^σ in the denominators of Eqs. (15) and (16), which corresponds to taking account of terms of the order e^{-R}. Let us note that the smallness of χ_σ is not associated with the smallness of the coupling constant. $\Phi^{q\mu}$ is defined by Eq. (A.6).

2. INTRODUCTION OF SEPARATE COORDINATES FOR MESON CLOUDS

The matrix elements in Eqs. (15) and (16) have the form

$$(\alpha\beta | L | \alpha'\beta') = (F_\alpha(1, \bar{a}) F_\beta(2, \bar{a}), L(a, a^+) F_{\alpha'}(1, \bar{a}) F_{\beta'}(2, \bar{a})), \tag{17}$$

where L contains the normal product of the creation and annihilation operators, and is a function of the operators σ_A and τ_A. Such expressions may be calculated if the one-nucleon functionals $F(1, \bar{a})$ and $F(2, \bar{a})$ (which may be found, for example, by the use of the intermediate coupling method) are known. However, the use of an explicit form of the one-nucleon functionals is undesirable because it would lead to the consideration of virtual mesons belonging to the clouds, and the effect of the self-energy would be mixed with the effects of the interaction. Therefore, we will attempt to calculate matrix elements of the type (17) by expressing them through matrix elements between states of the physical nucleon. This will allow us to use only the general properties of the functionals $F(1, \bar{a})$ and $F(2, \bar{a})$, and not their concrete form. With this approach, all the matrix elements encountered will be renormalized.

In the coordinate representation, the roles of annihilation and creation operators are fulfilled by $\varphi^{(+)}(\mathbf{r})$ and $\varphi^{(-)}(\mathbf{r})$ — the annihilating and creating parts of the meson operator $\varphi(\mathbf{r})$. If $\varphi^{(-)}(\mathbf{r})$ is a multiplicative operator, then the operator $\varphi^{(+)}(\mathbf{r})$ has the form:

$$\varphi^{(+)}(\mathbf{r}) = \int \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \frac{\delta}{\delta\varphi^{(-)}(\mathbf{r}')}, \quad \Delta^{(+)}(\mathbf{r}-\mathbf{r}') = [\varphi^{(+)}(\mathbf{r}), \varphi^{(-)}(\mathbf{r}')]. \tag{18}$$

Let B_A be the region occupied by the meson cloud of nucleon A. Then $F(A, a^+)$ depends only on those variables of $\varphi^{(-)}(\mathbf{r})$ for which $\mathbf{r}$ lies in the region $B_A: F(A, a^+) = F(A, \varphi_{B_A}^{(-)})$.

We will introduce separate variables for the meson clouds of nucleons 1 and 2. We will denote the field variables in the functional $F(1, \varphi_{B_1}^{(-)})$ by $\varphi_1^{(-)}(\mathbf{r})$, and the field variables in the functional $F(2, \varphi_{B_2}^{(-)})$ by $\varphi_2^{(-)}(\mathbf{r})$. We will introduce the annihilation operators $\varphi_1^{(+)}(\mathbf{r})$ and $\varphi_2^{(+)}(\mathbf{r})$ and with the commutators

$$[\varphi_1^{(+)}(\mathbf{r}), \varphi_1^{(-)}(\mathbf{r}')] = [\varphi_2^{(+)}(\mathbf{r}), \varphi_2^{(-)}(\mathbf{r}')] = \Delta^{(+)}(\mathbf{r}-\mathbf{r}'), \quad [\varphi_1^{(+)}(\mathbf{r}), \varphi_2^{(-)}(\mathbf{r}')] = 0. \tag{19}$$

The operator $\varphi_A^{(+)}(\mathbf{r})$ has the meaning of an annihilation operator of a meson of the cloud of nucleon A. With regard to the product $F(1, \varphi_1^{(-)})\, F(2, \varphi_2^{(-)})$, the operator $\varphi^{(+)}(\mathbf{r})$ will be equivalent to the operator

$$\varphi^{(+)}(\mathbf{r}) = \varphi_1^{(+)}(\mathbf{r}) + \varphi_2^{(+)}(\mathbf{r}) = \int \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \left\{\frac{\delta}{\delta\varphi_1^{(-)}(\mathbf{r}')} + \frac{\delta}{\delta\varphi_2^{(-)}(\mathbf{r}')}\right\}. \tag{20}$$

For $L = 1$, the matrix element (17) is now written in the form:

$$(\Lambda_0, F^*(1, \varphi^{(+)})\, F^*(2, \varphi^{(+)})\, F(1, \varphi_1^{(-)})\, F(2, \varphi_2^{(-)})\, \Lambda_0). \tag{21}$$

[No assumptions are made in going from Eq. (17) to Eq. (21)]. It is clear from Eqs. (20) and (21) that the introduction of separate variables for the meson clouds will be convenient in the case in which $F^*(A, \varphi^{(+)}) = F^*(A, \varphi_1^{(+)} + \varphi_2^{(+)})$ may be expressed by $F^*(A, \varphi_A^{(+)})$.

We will assume that the commutator $[\varphi_1^{(+)}(\mathbf{r}_2), F(1, \varphi_1^{(-)})]$ is considerably smaller than the commutator $[\varphi_2^{(+)}(\mathbf{r}_2), F(2, \varphi_2^{(-)})]$. This corresponds to the assumption that in the common meson cloud of two interacting nucleons, the meson clouds of the individual mesons may be distinguished. Then in the calculation of (21), we may consider the operator $\varphi_2^{(+)}$ in $F^*(1, \varphi_1^{(+)})$ to be small compared with $\varphi_1^{(+)}$, and the operator $\varphi_1^{(+)}$ in $F^*(2, \varphi_1^{(+)} + \varphi_2^{(+)})$ to be small compared with $\varphi_2^{(+)}$. For small $\varphi_2^{(+)}$

$$F^*(1, \varphi_1^{(+)} + \varphi_2^{(+)}) \approx F^*(1, \varphi_1^{(+)}) + \int \frac{\delta F^*(1, \varphi_1^{(+)})}{\delta\varphi_1^{(+)}(\mathbf{r})} d^3r\, \Delta^{(+)}(\mathbf{r}-\mathbf{r}')\, d^3r' \frac{\delta}{\delta\varphi_2^{(-)}(\mathbf{r}')} + \cdots \tag{22}$$

and similarly for $F^*(2, \varphi_1^{(+)} + \varphi_2^{(+)})$. Making the transition to the creation and annihilation operators a_{Aq}^+ and a_{Aq} of a meson in the A^{th} cloud in momentum space, we obtain

$$(\alpha\beta \mid \alpha'\beta') = (F_\alpha(1, \bar{a}_1)\, F_\beta(2, \bar{a}_2),\ (1 + \hat{N})\, F_{\alpha'}(1, \bar{a}_1)\, F_{\beta'}(2, \bar{a}_2)), \tag{23}$$

$$\hat{N} = \sum_q [a_{1q}^+ a_{2q} + a_{2q}^+ a_{1q}]. \tag{24}$$

In the general case, in the calculation of (17), we must first apply all the operators a_q and a_q^+ to the functionals $F(1, \bar{a})$, $F(2, \bar{a})$, and then use expansions of the type (22).

The first term in (22) is the functional of nucleon 1, not interacting with nucleon 2. The second term in (22), which decreases with the distance R as e^{-R}, is associated with the exchange of mesons between clouds 1 and 2. The following term in expansion (22) decreases as e^{-2R}. Retaining only the first term in expansions of the type (22), we obtain a potential of the order of e^{-R}. If we also include the second term, we find a potential of the order of e^{-2R}.

As a result, we arrive at matrix elements of products of the operators a_{Aq} and $\overset{+}{a}_{Aq}$ between states of the noninteracting physical nucleons $F(1, \bar{a}_1)\, F(2, \bar{a}_2)$. These matrix elements in turn reduce to products of one-nucleon matrix elements.

From the products $\Phi_{\alpha\beta}(1, 2; \bar{a}_1, \bar{a}_2) = F_\alpha(1, \bar{a}_1)\, F(2, \bar{a}_2)$ we may already construct in the usual way the eigenfunctions $\Phi^\sigma(1, 2; \bar{a}_1, \bar{a}_2)$, $\sigma \equiv (S', I', S_3', I_3')$. The spin and isotopic spin operators of two non-interacting nucleons are defined as $S = S_1(a_1) + S_2(a_2)$; $I = I_1(a_1) + I_2(a_2)$, where S_A and I_A refer to an isolated nucleon with a meson cloud.

3. CALCULATION OF THE TWO-NUCLEON POTENTIAL

In the future we will write $|\alpha\beta>$ instead of the state $F_\alpha(1, \bar{a}_1)\, F_\beta(2, \bar{a}_2)$ and $|\alpha>$ instead of the one-nucleon state $F_\alpha(1, \bar{a}_1)$. In the calculations we will frequently use the following formulas from the works of Chew and Low[1] and of Wick[2]:

$$a_{1q}|\alpha\rangle = -\frac{1}{H_1 + q_0} V_{1q}^0 e^{-i\mathbf{q}\mathbf{r}_1}|\alpha\rangle, \tag{25}$$

$$\langle\alpha|V_q^0|\alpha'\rangle = (u_\alpha V_q u_{\alpha'}), \tag{26}$$

where $H_1 = H_0(a_1) + U_1(a_1)$; u_α is the spin-charge function of the bare nucleon, and V_q contains the renormalized charge f.

We find from Eqs. (23) and (24) and the normalization condition (9),

$$\begin{aligned}(\alpha\beta|\alpha'\beta') &= \langle\alpha\beta|1+\hat{N}|\alpha'\beta'\rangle = \delta_{\alpha\alpha'}\delta_{\beta\beta'} + \sum_q\{\langle\alpha|a_{1q}^+|\alpha'\rangle\langle\beta|a_{2q}|\beta'\rangle + \langle\alpha|a_{1q}|\alpha'\rangle\langle\beta|a_{2q}^+|\beta'\rangle\}\\ &= \delta_{\alpha\alpha'}\delta_{\beta\beta'} + (u_\alpha(1)\, u_\beta(2), N u_{\alpha'}(1)\, u_{\beta'}(2)),\end{aligned} \tag{27}$$

where N depends only on the spin and charge operators:

$$N = \frac{f^2}{(2\pi)^3}\int\frac{d^3q}{q_0^3} e^{i\mathbf{q}\mathbf{R}} (\sigma_1\mathbf{q})(\sigma_2\mathbf{q})(\tau_1\tau_2)\, v^2(q). \tag{28}$$

It follows from the form of the operator N that the functionals $\Phi^\sigma(1, 2, \bar{a})$ are orthogonal:

$$(\Phi^\sigma(1,2,\bar{a}),\ \Phi^{\sigma'}(1,2,\bar{a})) = \delta_{\sigma\sigma'}(1 + N_\sigma),\quad \sigma \equiv (S', I', S'_3, I_3'), \tag{29}$$

where N_σ is the mean value of N in the state σ.

The potential W(R) between the nucleons is defined as an operator whose mean value in the state σ (in a phenomenological treatment) is equal to the interaction energy E_σ, calculated by Eq. (16). It follows from Eq. (29) that the second term in Eq. (16) is zero. We will consider the principal term in the interaction energy (26), which is equal to

$$(\Phi^\sigma,\ H\Phi^\sigma)/(1 + N_\sigma). \tag{30}$$

The numerator in Eq. (30) is a linear combination of matrix elements $(\alpha\beta|H|\alpha'\beta')$, which by the use of Eqs (A.3) and (22) may be reduced to the form

$$(\alpha\beta|H|\alpha'\beta') = \langle\alpha\beta|(1+\hat{N})[U_1^+(a_2) + U_2^+(a_1)]|\alpha'\beta'\rangle. \tag{31}$$

Here, $U_1^+(a_2)$ is the annihilating part of the operator U_1 with annihilation operators a_{2q}. If we neglect the weighting operator $\hat{N}$ in Eq. (31), i.e., we neglect the distortion of the meson clouds, then by using (25) and (26) we obtain the usual pseudovector interaction in the lowest approximation of perturbation theory (with renormalized charge f):

$$\langle \alpha\beta \,|\, U_1^+(a_2) + U_2^+(a_1) \,|\, \alpha'\beta' \rangle = (u_\alpha(1)\, u_\beta(2),\; W_0 u_{\alpha'}(1)\, u_{\beta'}(2)), \tag{32}$$

where W_0 is a pseudovector potential

$$W_0(R) = -\frac{f^2}{(2\pi)^3} \int \frac{d^3q}{q_0^2}\, e^{i\mathbf{q}\mathbf{R}}\, (\sigma_1\mathbf{q})(\sigma_2\mathbf{q})(\tau_1\tau_2)\, v^2(q). \tag{33}$$

The second part $(\alpha\beta|\,H\,|\,\alpha'\beta')$, which depends on the operator $\hat{N}$, leads to an interaction energy of the order e^{-2R}. This part contains products of matrix elements of the first nucleon

$$\langle \alpha \,|\, a_{1q}^+ V_{1q'}^0 \,|\, \alpha' \rangle, \quad \langle \alpha \,|\, a_{1q} a_{1q'} \,|\, \alpha' \rangle, \quad \langle \alpha \,|\, a_{1q}^+ a_{1q'} \,|\, \alpha' \rangle$$

times analogous matrix elements of the second nucleon. Each of these matrix elements may be calculated by the methods of Chew and Low[1] and of Wick.[2] For example

$$\langle \alpha \,|\, a_{1q}^+ V_{1q'}^0 \,|\, \alpha' \rangle = -\sum_{n_1} \frac{\langle \alpha \,|\, V_{1q}^0 \,|\, n_1 \rangle \langle n_1 \,|\, V_{1q'}^0 \,|\, \alpha' \rangle}{E_{n_1} + q_0}, \tag{34}$$

where $|n_1\rangle$ is a complete set of eigenfunctionals of the one-nucleon energy operator H_1. In the product of the sum (34) times the analogous sum for nucleon 2 (with summation index n_2), we must leave only the terms $n_1 = n_2 = 0$ (no mesons; functional $|\mu\rangle$ and $|\nu\rangle$), and also $n_1 = 1$, $n_2 = 0$ and $n_1 = 0$, $n_2 = 1$ (one meson; functionals $|q\mu\rangle$ and $|q\nu\rangle$), because we are considering only states without mesons and with one real meson.

We will first single out the term $n_1 = n_2 = 0$ from $\langle \alpha\beta|\,\hat{N}\,[U_1^+(a_2) + U_2^+(a_1)]\,|\,\alpha'\beta'\rangle$. After some transformations, it has the form:

$$-\sum_{qq'} e^{i(\mathbf{q}+\mathbf{q}')\mathbf{R}} \Big(u_\alpha(1)\, u_\beta(2),\, 2V_{1q}V_{1q'}V_{2q}V_{2q'}\, \frac{q_0 + q_0'}{q_0^2 q_0'^2}\, u_{\alpha'}(1)\, u_{\beta'}(2) \Big) + (u_\alpha(1)\, u_\beta(2),\; W_1(\mathbf{R})\, u_{\alpha'}(1)\, u_{\beta'}(2)). \tag{35}$$

In the calculation of $(\Phi^\sigma, H\Phi^\sigma)$, the first term in (35) gives $N_\sigma W_{0\sigma}$, where $W_{0\sigma}$ is the mean value of the potential (33) in the state σ. The second term in (35) contributes $W_{1\sigma}/(1 + N_\sigma) \approx W_{1\sigma}$ to the energy (30), if the interaction of order e^{-3R} is neglected. Thus, the without-mesons part of the matrix element $(\alpha\beta|\,H\,|\,\alpha'\beta')$ leads to the potential $W_0 + W_1$, where

$$W_1(\mathbf{R}) = -\frac{f^4}{(2\pi)^6} \int d^3q\, d^3q'\, e^{i(\mathbf{q}+\mathbf{q}')\mathbf{R}}\, \frac{(2q_0 + q_0')\, v^2(q)\, v^2(q')}{q_0^2 q_0'^3 (q_0 + q_0')} \{2(\tau_1\tau_2)(\mathbf{q}\mathbf{q}')^2 + 3(\sigma_1[\mathbf{q}\mathbf{q}'])(\sigma_2[\mathbf{q}\mathbf{q}'])\}. \tag{36}$$

Let us consider the one-meson part of $(\alpha\beta|\,H\,|\,\alpha'\beta')$. The expression

$$\begin{aligned} -\sum_{kk'q\mu} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}}\, & \frac{\langle \alpha \,|\, V_{1k}^0 \,|\, \mu q \rangle \langle \mu q \,|\, V_{1k'}^0 \,|\, \alpha \rangle}{k_0 k_0' (q_0 + k_0)(q_0 + k_0')(k_0 + k_0')} (u_\beta(2),\, [V_{2k}V_{2k'}(k_0 q_0 + k_0' q_0 + 2k_0 k_0') \\ & + 2V_{2k'}V_{2k}(k_0 q_0 + k_0' q_0 + k_0^2 + k_0'^2 + k_0 k_0')]\, u_\beta(2)). \end{aligned} \tag{37}$$

corresponds to the case $n_1 = 1$, $n_2 = 0$.

The quantity $\langle \mu q|\,V_{1k}\,|\,\alpha\rangle$ may be expressed through the phases δ_{ij} of the scattering of a meson on a nucleon.[2] Since according to experiment the phase δ_{33} is dominant over a broad energy region, we will neglect all phases except δ_{33}. Then,

$$\langle \mu q \,|\, V_k^0 \,|\, \alpha \rangle \approx -\frac{2\pi}{(k_0 q_0)^{1/2} q^3} \frac{v(k)}{v(q)}\, e^{i\delta_{33}(q)} \sin\delta_{33}(q)\, (u_\mu,\; P_{33}(qk)\, u_\alpha), \tag{38}$$

where P_{33} is the projection operator onto state 33: $P_{33}(qk) = \left(\delta_{kq} - \tfrac{1}{3}\tau_q\tau_k\right)(3(\mathbf{q}\mathbf{k}) - (\sigma\mathbf{q})(\sigma\mathbf{k}))$. After substituting (38) into (37) we see the cases $n_1 = 0$, $n_2 = 1$ and $n_1 = 1$, $n_2 = 0$ give the same results. If we neglect terms in e^{-3R} in the interaction energy, the one-meson part of $(\alpha\beta|\,H\,|\,\alpha'\beta')$ leads to the potential

$$\begin{aligned} W_2' = -\frac{4f^2}{(2\pi)^6} \int d^3k\, d^3k'\, dq\, & \frac{v^2(k)\, v^2(k')\sin^2\delta_{33}(q)\, e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}}}{(q_0 + k_0)(q_0 + k_0')(k_0 + k_0')\, v^2(q)\, q^2 q_0 k_0^2 k_0'^2} \Big\{ \frac{1}{3} [3 + (\tau_1\tau_2)]\, [2(\mathbf{k}\mathbf{k}')^2 (4q_0 k_0 + 3k_0 k_0' + 2k_0'^2) \\ & + (\sigma_1[\mathbf{k}\mathbf{k}'])(\sigma_2[\mathbf{k}\mathbf{k}'])\, k_0 (k_0' - 2k_0)] - [(\tau_1\tau_2) - 1]\, [2(\mathbf{k}\mathbf{k}')^2 - (\sigma_1[\mathbf{k}\mathbf{k}'])(\sigma_2[\mathbf{k}\mathbf{k}'])(2k_0 q_0 + 2k_0^2 + k_0 k_0')] \Big\}. \end{aligned} \tag{39}$$

Thus, the principal term (30) in the energy E_σ is connected with the potential $W_0 + W_1 + W_2'$.

We will now consider the last term in the energy E_σ [Eq. (16)]. In order to calculate it, we must find the matrix element

$$(\Phi^q_{\mu\nu}, H\Phi_{\alpha'\beta'}), \tag{40}$$

where Φ^q is given by Eq. (A.6). The calculation of (40) is made easier by the fact that we are only interested in potentials of the order e^{-2R}. Therefore, we need retain in (40) only those terms decreasing no more rapidly than e^{-R}. We have

$$(H - q_0 - i\varepsilon)^{-1} V_{1q} F(1,\bar{a}) F(2,\bar{a}) \approx F(2,\bar{a}) (H_0 + U_1 - q_0 - i\varepsilon)^{-1} V_{1q} F(1,\bar{a}) + \ldots, \tag{41}$$

where the unwritten terms lead to potentials of higher order. For the same reason, the first term is sufficient in an expansion of the type (22). As a result of the calculations,

$$(\Phi^q_{\mu\nu}, H\Phi_{\alpha'\beta'}) \approx (F_\mu(1,\bar{a}_1) F^q_\nu(2,\bar{a}_2) + F^q_\mu(1,\bar{a}_1) F_\nu(2,\bar{a}_2),\ [U_1^+(a_2) + U_2^+(a_1)] F_{\alpha'}(1,\bar{a}_1) F_{\beta'}(2,\bar{a}_2)), \tag{42}$$

where $F^q_\mu(1, \bar{a}_1)$ is the one-nucleon functional of a state with one meson. Using Eqs. (25), (26), (38), and (42), we find that the last term in (16) is

$$-\sum_{q\mu\nu} \frac{1}{q_0} \left| (\Phi^q_{\mu\nu}, H\Phi^\sigma) \right|^2 = W''_{2\sigma}, \tag{43}$$

where $W''_{2\sigma}$ is the mean value of the potential W''_2:

$$\begin{aligned} W''_2 = -\frac{4}{3}\frac{f^2}{(2\pi)^6} \int d^3k d^3k' dq e^{i(\mathbf{k}+\mathbf{k}')\mathbf{R}} \frac{(2k_0 + q_0)(2k'_0 + q_0) v^2(k) v^2(k') \sin^2\delta_{33}(q)}{k_0^2 k_0'^2 q_0^2 q^2 v^2(q)(q_0 + k_0)(q_0 + k'_0)} \\ \times [3 + (\tau_1\tau_2)] \{2(\mathbf{kk}')^2 + (\sigma_1[\mathbf{kk}'])(\sigma_2[\mathbf{kk}']) - {}^7/_9 (\sigma_1\mathbf{k}')(\sigma_2\mathbf{k})(\mathbf{kk}') \\ + {}^4/_9 k^2 (\sigma_1\mathbf{k}')(\sigma_2\mathbf{k}') + {}^1/_9 k^2 k'^2 - {}^1/_9 (\mathbf{kk}')^2 (1 + (\sigma_1\sigma_2))\}. \end{aligned} \tag{44}$$

The total potential W is the sum of expressions (33), (36), (39), and (44):

$$W = W_0 + W_1 + W'_2 + W''_2. \tag{45}$$

The following assumptions were made in the derivation of Eq. (45): (a) states with two or more (real) mesons were neglected, (b) it was implied that the various terms in the potential could be characterized by the way in which they decrease with distance (the potential of 2n-th order decreases as e^{-nR}), the computed potential W contains all terms of the second and fourth orders, (c) the phase δ_{33} is considered to be dominant and the remaining phases are neglected. It is not difficult, however, to take account of the remaining phases. Moreover, the two-nucleon potential was defined by us as a potential between nucleons with meson clouds, which led to the appearance of the denominator $1 + N_\sigma$ in the expression for the energy [Eqs. (16) and 30].

The basic difference between potential (45) and other known potentials[4-6] is contained in the terms W_2' and W_2'', which depend on the phase of the scattering of π mesons on nucleons. These terms have not been previously obtained. W_2'' differs from zero in the triplet charge state. The magnitude of W_2' is considerably greater in the triplet charge state than in the singlet state. It may therefore be expected that the terms W_2' and W_2'' will be imprtant in the triplet states.

The potential of Taketani et al.[4] represents the without-meson part of the term (30), i.e., $W_{0\sigma} + W_{1\sigma}/(1 + N_\sigma)$. In the transition to Eq. (36), we discarded the term N_σ in the denominator. The denominator in the potential of Taketani et al.[4] contains in addition the renormalized term $3\Delta_2'$, which should not be there according to the point of view of the present work.

The potential of fourth order V_4 of Brueckner and Watson[5] and of Gartenhaus[6] corresponds rather to the potential between nucleons without clouds. This potential may be obtained from the without-meson part of (30) by discarding N_σ in the denominator. In our notation $V_4 = W_1 + NW_0$.

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APPENDIX

THE FUNCTIONALS Ψ^0 AND Ψ^q AS $R \to \infty$

The operator a_q is an operator of differentiation with respect to $\bar{a}_q$. Since $F(1, \bar{a})$ commutes with $\partial F(2, \bar{a})/\partial \bar{a}_q$, we have

$$H_0F(1,\bar{a})\,F(2,\bar{a}) = F(1,\bar{a})\,H_0F(2,\bar{a}) + F(2,\bar{a})\,H_0F(1,\bar{a}); \tag{A.1}$$

$$U_1^+F(1,\bar{a})\,F(2,\bar{a}) = F(2,\bar{a})\,U_1^+F(1,\bar{a}) + F(1,\bar{a})\,U_1^+F(2,\bar{a}), \tag{A.2}$$

where U_1^+ is the annihilating part of the operator U_1. Hence, in view of Eq. (7),

$$(H - 2E_0)\,F(1,\bar{a})\,F(2,\bar{a}) = F(1,\bar{a})\,U_1^+F(2,\bar{a}) + F(2,\bar{a})\,U_2^+F(1,\bar{a}). \tag{A.3}$$

The dependence of the right hand side of (A.3) on R may be evaluated. Using Eq. (25), we have

$$U_2^+F(1,\bar{a}) = -\sum_q V_{2q}^0\,\frac{e^{-i\mathbf{qR}}}{H_0 + U_1 + q_0 - E_0}\,V_{1q}^{0+}F(1,\bar{a}). \tag{A.4}$$

On the right hand side of Eq. (A.4), we have the integral

$$\int \frac{(\sigma_1\mathbf{q})\,(\sigma_2\mathbf{q})}{(\varepsilon_n + q_0)\,q_0}\,e^{-i\mathbf{qR}}\,d^3q, \quad \varepsilon_n > 0, \tag{A.5}$$

which decreases exponentially as $R \to \infty$. Hence, Eq. (12) follows. For large R, the one-meson state Ψ^q is

$$\Psi^q(1,2,\bar{a}) \to \Phi^q(1,2,\bar{a}) = \{a_q^+ - (H - q_0 - 2E_0 \pm i\varepsilon)^{-1}\,(V_{1q}^0 e^{i\mathbf{qr}_1} + V_{2q}^0 e^{i\mathbf{qr}_2})\}\,\Phi(1,2,\bar{a}), \tag{A.6}$$

which can be verified also by using Eq. (A.3).

Note added in proof (September 16, 1957). After this work had been sent to the publisher, the author obtained Phys. Rev. 104, No. 6, 1956, with articles by Miyazawa and by Klein and McCormick, in which the two-nucleon problem is also considered. The approach to the problem and the method of our article is completely different from that developed in the articles of Miyazawa and of Klein and McCormick.

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