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SCATTERING OF ELECTROMAGNETIC WAVES IN A PLASMA

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Combination scattering by plasma density oscillations may occur when electromagnetic waves are propagated in a plasma. The intensity of combination scattering of electromagnetic waves in a plasma is determined in the absence and in the presence of a constant uniform magnetic field.

I. It is well known that there can exist in a plasma weakly damped electromagnetic oscillations which are associated with oscillations of plasma density whose frequency (without taking dispersion into account) is given by[1] $\Omega = \sqrt{4\pi n_0 e^2/m}$. The existence of these oscillations leads to a periodic variation of the dielectric constant in the plasma. Because of this, combination scattering of electromagnetic waves propagated in the plasma becomes possible, i.e., if a wave of frequency $\omega_0 > \Omega$ is propagated in the plasma then at the same time waves with frequencies $\omega = \omega_0 \pm n\Omega$, where n is an integer, will also be propagated. The object of this paper is to determine the intensity of these waves.

Let us first determine the dielectric constant of the plasma. We start with the equation

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\frac{d\mathbf{E}}{dt} + \frac{4\pi}{c}\mathbf{j}_{\text{free}}, \tag{1}$$

where $\mathbf{j}_{\text{free}}$ is the current density associated with the motion of the plasma electrons and is equal to $\mathbf{j}_{\text{free}} = en\mathbf{v}$, n is the electron density, $\mathbf{v}$ is the electron velocity which is related to the electric field by the equation $\dot{\mathbf{v}} = e\mathbf{E}/m$. From these equations it follows that

$$\mathbf{v}(\mathbf{r}, t) = \frac{e}{m}\int^{t} \mathbf{E}(\mathbf{r}, t')\,dt',$$

$$\mathbf{j}_{\text{free}}(\mathbf{r}, t) = \frac{e^2}{m} n(\mathbf{r}, t)\int^{t} \mathbf{E}(\mathbf{r}, t')\,dt'.$$

Substituting this expression for $\mathbf{j}_{\text{free}}$ into Eq. (1), we find

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi e^2}{m} n(\mathbf{r}, t)\int^{t} \mathbf{E}(\mathbf{r}, t')\,dt'.$$

We now introduce the dielectric constant operator by $\mathbf{D} = \hat{\varepsilon}\mathbf{E}$. Then

$$\hat{\varepsilon}(\mathbf{r}, t)\,\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + \frac{4\pi e^2}{m}\int^{t} dt' n(\mathbf{r}, t')\int^{t'} dt''\mathbf{E}(\mathbf{r}, t''). \tag{2}$$

Let the electric field vary with time in accordance with the harmonic law $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t}$. The result of applying the dielectric constant operator to $\mathbf{E}(\mathbf{r}, t)$ reduces in this case to multiplication by

$$\varepsilon_\omega(\mathbf{r}, t) = 1 - \frac{4\pi e^2}{im\omega} e^{i\omega t} \int^t dt' n(\mathbf{r}, t') e^{-i\omega t'}. \tag{3}$$

We now note that the plasma electron density $n(\mathbf{r}, t)$ contains the constant term n_0 plus a small additional term $\delta n(\mathbf{r}, t) = \delta n(\mathbf{r}) \cos \Omega t$, $\delta n \ll n_0$. Therefore

$$\varepsilon_\omega(\mathbf{r}, t) = \varepsilon(\omega) + \delta\varepsilon_\omega(\mathbf{r}, t), \tag{4}$$

where

$$\varepsilon(\omega) = 1 - \frac{\Omega^2}{\omega^2}, \quad \delta\varepsilon_\omega(\mathbf{r}, t) = -\frac{2\pi e^2 \delta n(\mathbf{r})}{m\omega}\left(\frac{e^{i\Omega t}}{\omega - \Omega} + \frac{e^{-i\Omega t}}{\omega + \Omega}\right).$$

If we introduce the operator for the fluctuation in the dielectric constant

$$\delta\hat{\varepsilon} = (\hat{\varepsilon} - 1)^{1/2} \frac{\delta n(\mathbf{r}, t)}{n_0} (\hat{\varepsilon} - 1)^{1/2}, \tag{5}$$

where $\hat{\epsilon}$ is the operator for the dielectric constant of a plasma of density n_0 which is determined by means of the formula

$$\hat{\varepsilon} e^{-i\omega t} = \varepsilon(\omega) e^{-i\omega t},$$

then

$$\delta\hat{\varepsilon} e^{-i\omega t} = \delta\varepsilon_\omega(\mathbf{r}, t) e^{-i\omega t}.$$

2. Let the plane monochromatic wave $\mathbf{E}_0(\mathbf{r}, t) = \mathbf{e}_0 E_0 \exp(i\mathbf{k}_0 \cdot \mathbf{r} - i\omega_0 t)$ be propagated in the plasma. Then the electric and magnetic fields of the scattered wave may be determined by means of Maxwell's equations in which the current and charge densities have in the first approximation the form

$$\mathbf{j} = \frac{1}{4\pi} \frac{\partial}{\partial t} (\delta\hat{\varepsilon} \mathbf{E}_0), \quad \rho = -\frac{1}{4\pi} \operatorname{div}(\delta\hat{\varepsilon} \mathbf{E}_0). \tag{6}$$

Since $\delta\epsilon$ has a time dependence of the form $e^{\mp i\Omega t}$ the current $\mathbf{j}$ will excite a wave of frequency $\omega_0 \pm \Omega$. In the second approximation we would obtain the current

$$\mathbf{j}' = \frac{1}{4\pi} \frac{\partial}{\partial t} (\delta\hat{\varepsilon} \mathbf{E}),$$

where $\mathbf{E}$ is the electric field of the scattered wave. This current obviously excites a wave of frequency $\omega_0 \pm 2\Omega$, etc. Thus in a plasma when an electromagnetic wave of frequency ω_0 is being propagated combination frequencies $\omega_0 \pm n\Omega$ arise where n is an integer. In future we shall restrict ourselves to an investigation only of the combination frequencies $\omega_0 \pm \Omega$.

Introducing instead of the field instensities $\mathbf{E}$ and $\mathbf{H}$ of the scattered wave the vector and the scalar potentials $\mathbf{A}$ and φ and assuming that $\operatorname{div} \mathbf{A} = 0$ we shall start with the following basic equations:

$$\Delta \mathbf{A} - \frac{\hat{\varepsilon}}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{\hat{\varepsilon}}{c} \frac{\partial}{\partial t} \operatorname{grad} \varphi = -\frac{4\pi}{c} \mathbf{j}, \tag{7}$$

$$\hat{\varepsilon} \Delta \varphi = -4\pi\rho. \tag{8}$$

The increase per unit time in the energy of the electromagnetic field is given by the expression

$$\frac{\partial W}{\partial t} = \frac{1}{8\pi} \operatorname{Re} \int \left\{ \mathbf{E} \frac{\partial \mathbf{D}^*}{\partial t} + \mathbf{H} \frac{\partial \mathbf{H}^*}{\partial t} \right\} dv = \frac{1}{2} \operatorname{Re} \frac{1}{c} \int \mathbf{j} \dot{\mathbf{A}}^* dv + \frac{1}{2} \operatorname{Re} \int \rho \dot{\varphi}^* dv,$$

where $\mathbf{D}$ is the electric displacement vector. Here the first term determines the increase in the energy of the transverse oscillations of the electromagnetic field, i.e., the intensity of scattering of electromagnetic waves

$$I = \frac{1}{2c} \operatorname{Re} \int \mathbf{j} \dot{\mathbf{A}}^* dv, \tag{9}$$

while the second term determines the increase in the energy of the longitudinal oscillations of the plasma

$$I' = \frac{1}{2}\operatorname{Re}\int \rho\dot{\varphi}^* dv. \tag{10}$$

3. We seek a solution of Eq. (7) of the form

$$\mathbf{A}(\mathbf{r},\ t) = \sum_\lambda q_\lambda(t)\,\mathbf{A}_\lambda(\mathbf{r}), \tag{11}$$

where

$$\Delta\mathbf{A}_\lambda + k_\lambda^2\mathbf{A}_\lambda = 0,\ \ \mathbf{A}_\lambda(\mathbf{r}) = \mathbf{e}^\lambda\,(4\pi c^2/\varepsilon(\omega_\lambda))^{1/2} e^{i\mathbf{k}_\lambda\mathbf{r}}. \tag{12}$$

$\mathbf{e}^\lambda$ is the polarization unit vector ($\mathbf{e}^\lambda\cdot\mathbf{k}_\lambda = 0$); $\mathbf{k}_\lambda$ and ω_λ are related by the dispersion relation $\omega_\lambda^2 = k_\lambda^2c^2/\epsilon(\omega_\lambda)$, while the constant has been chosen so that the following normalization condition is satisfied

$$\int \mathbf{A}_\lambda\mathbf{A}_{\lambda'}^* dv = (4\pi c^2/\varepsilon(\omega_\lambda))\,\delta_{\lambda\lambda'}$$

(the index λ determines both $\mathbf{k}$ and the polarization).

Using (7) and noting that

$$\int \varepsilon\,\operatorname{grad}\dot{\varphi}\mathbf{A}_\lambda^* dv = 0,$$

we obtain the following equation for the determination of q_λ:

$$\ddot{q}_\lambda + \omega_\lambda^2 q_\lambda = f_\lambda(t),\ \ f_\lambda(t) = \frac{1}{c}\int \mathbf{j}\mathbf{A}_\lambda^* dv. \tag{13}$$

This equation has the form of the equation for the forced vibrations of an oscillator acted on by the force $f_\lambda(t)$.

Setting

$$f_\lambda(t) = \sum_\omega b_\lambda^\omega e^{-i\omega t},$$

we shall write the solution of (13) in the form[2,3]

$$q_\lambda = \frac{1}{2\omega_\lambda}\sum_\omega \left\{\frac{1}{\omega_\lambda - \omega} + i\pi\delta(\omega_\lambda - \omega)\right\} b_\lambda^\omega e^{-i\omega t}. \tag{14}$$

We substitute the expansion (11) into (9). Making use of (14) we obtain the following formula for the intensity of scattering:

$$I = \frac{\pi}{4}\sum_\omega \sum_{\lambda=1,2} \int |b_\lambda^\omega|^2 \delta(\omega_\lambda - \omega)\frac{d\mathbf{k}_\lambda}{(2\pi)^3}, \tag{15}$$

where the summation is taken over the two polarization directions of the scattered wave.

Making use of the relation between the fluctuation in the dielectric constant and the deviation of the electron density from its equilibrium value we shall obtain the current that appears in (7)

$$\mathbf{j} = -\frac{i}{8\pi}\sum_{\omega=\omega_0\ \ \Omega} [1-\varepsilon(\omega)]^{1/2}[1-\varepsilon(\omega_0)]^{1/2}\frac{\delta n(\mathbf{r})}{n_0}\,\omega E_0\mathbf{e}_0 e^{i\mathbf{k}_0\mathbf{r} - i\omega t}. \tag{16}$$

Substituting this expression into (13) we obtain

$$b_\lambda^\omega = -\frac{iE_0\omega}{4\sqrt{\pi}\,n_0\sqrt{\varepsilon(\omega)}}[1-\varepsilon(\omega)]^{1/2}[1-\varepsilon(\omega_0)]^{1/2}(\mathbf{e}_0\mathbf{e}^\lambda)\,\delta n_{\mathbf{k}-\mathbf{k}_0},\qquad \delta n_{\mathbf{k}-\mathbf{k}_0} = \int \delta n(\mathbf{r})\,e^{-i(\mathbf{k}-\mathbf{k}_0)\mathbf{r}}d\mathbf{r}.$$

From this, in accordance with (15), we obtain the following expression for the intensity of scattering of electromagnetic waves:

$$I = \sum_{\omega=\omega_0+\Omega} \int I_\omega(\vartheta,\ \varphi)\,do, \tag{17}$$

$$I_\omega(\vartheta, \varphi) = \frac{2E_0^2 \sqrt{\varepsilon(\omega)}}{(8\pi)^3 n_0^2 c^3} \omega^4 [1 - \varepsilon(\omega)] [1 - \varepsilon(\omega_0)] \overline{|\delta n_{\mathbf{k}-\mathbf{k}_0}|^2} \{\cos^2 \varphi + \sin^2 \varphi \cos^2 \vartheta\}, \qquad (18)$$

ϑ is the angle between the propagation vector of the scattered wave **k** and the wave vector of the incident wave $\mathbf{k}_0$, while φ is the angle between the direction of polarization of the incident wave $\mathbf{e}_0$ and the normal to the plane $(\mathbf{k}_0, \mathbf{k})$. The average value of the square of the absolute value of the Fourier component of the fluctuation in the electron density which appears in (18) may be expressed in terms of the Fourier component of the correlation function

$$\overline{|\delta n_{\mathbf{k}}|^2} = n_0 \{1 + \nu_{\mathbf{k}}\}, \quad \nu_{\mathbf{k}} = \int \nu(\mathbf{r})\, e^{-i\mathbf{k}\mathbf{r}} dv.$$

(The correlation function is defined, as is well known, by means of the relation

$$n_0 \nu(\mathbf{r}_2 - \mathbf{r}_1) = \overline{\delta n(\mathbf{r}_1)\, \delta n(\mathbf{r}_2)} - n_0 \delta(\mathbf{r}_2 - \mathbf{r}_1).)$$

Thus

$$I_\omega(\vartheta, \varphi) = \frac{2E_0^2 \sqrt{\varepsilon(\omega)}\, \omega^4}{(8\pi)^3 n_0 c^3} [1 - \varepsilon(\omega)] [1 - \varepsilon(\omega_0)] (1 + \nu_{\mathbf{k}-\mathbf{k}_0}) (\cos^2 \varphi + \sin^2 \varphi \cos^2 \vartheta). \qquad (18')$$

On expanding $\delta n(\mathbf{r}, t)$ into plane waves

$$\delta n(\mathbf{r}, t) = \sum_{\mathbf{k}} \delta n_{\mathbf{k}}(t)\, e^{i\mathbf{k}\mathbf{r}},$$

one may obtain the total energy of the plasma in the form

$$\int \left\{ \frac{n_0 m (\delta v)^2}{2} + \frac{m s^2}{2n_0} (\delta n)^2 + \frac{(\delta E)^2}{8\pi} \right\} dv = \sum_{\mathbf{k}}{}' \frac{m}{2n_0 k^2} \{ |\delta \dot{n}_{\mathbf{k}}|^2 + \omega_{\mathbf{k}}^2 |\delta n_{\mathbf{k}}|^2 \},$$

where

$$\omega_k^2 = \Omega^2 + k^2 s^2, \quad s^2 = (\partial p / \partial \rho)_T$$

(p is the pressure, ρ is the density).

If one assumes that the equipartition law holds, then

$$\overline{(m / 2n_0 k^2) \{ |\delta \dot{n}_{\mathbf{k}}|^2 + \omega_k^2 |\delta n_{\mathbf{k}}|^2 \}} = T;$$

T is the plasma temperature in ergs. Assuming that $s^2 = T/m$ we obtain[4]

$$\overline{|\delta n_{\mathbf{k}}|^2} = n_0 k^2 / (4\pi n_0 e^2 / T + k^2).$$

For the sake of definiteness we shall henceforth use this formula for the average value of the square of the absolute value of the Fourier component of the density fluctuations.

The quantity $I_\omega(\vartheta, \varphi)$ represents the intensity of radiation of the combination frequency $\omega = \omega_0 \pm \Omega$ per unit solid angle. In the case of an unpolarized incident wave, averaging $I_\omega(\vartheta, \varphi)$ over the angle φ gives

$$I_\omega(\vartheta) = \frac{E_0^2}{(8\pi)^3 n_0^2 c^3} (1 - \varepsilon)(1 - \varepsilon_0) \sqrt{\varepsilon}\, \omega^4 \frac{n_0 (k^2 + k_0^2 - 2kk_0 \cos\vartheta)}{4\pi n_0 e^2 / T + k^2 + k_0^2 - 2kk_0 \cos\vartheta} (1 + \cos^2 \vartheta), \qquad (19)$$

$$\varepsilon_0 = \varepsilon(\omega_0), \quad \varepsilon \equiv \varepsilon(\omega), \quad k_0^2 = \omega_0^2 \varepsilon_0 / c^2, \quad k^2 = \omega^2 \varepsilon / c^2.$$

We note that if $\vartheta = \pi/2$ then the scattered radiation will be polarized in the plane $(\mathbf{k}_0, \mathbf{k})$ irrespectively of the character of polarization of the incident wave.

We consider two limiting cases: $(\mathbf{k} - \mathbf{k}_0)^2 \ll 4\pi n_0 e^2/T$ and $(\mathbf{k} - \mathbf{k}_0)^2 \gg 4\pi n_0 e^2/T$. In the first case the collective oscillations of the plasma play an essential role, and the total intensity of radiation of frequency ω is equal to

$$I_\omega = \frac{e^2 E_0^2 T}{24\pi m^2 c^5} \frac{\omega^2}{\omega_0^2} \sqrt{\varepsilon} (\omega^2 \varepsilon + \omega_0^2 \varepsilon_0), \quad \omega = \omega_0 \pm \Omega. \qquad (20)$$

On dividing the total intensity of scattering $I = I_{\omega_0+\Omega} + I_{\omega_0-\Omega}$ by the energy flux density of the incident wave we obtain the scattering coefficient

$$\Sigma = \frac{e^2T}{3m^2c^6} \sum_{\omega=\omega_0\pm\Omega} \frac{\omega^2}{\omega_0^2} \sqrt{\frac{\varepsilon}{\varepsilon_0}} (\omega^2\varepsilon + \omega_0^2\varepsilon_0). \tag{21}$$

In the second limiting case the average quadratic fluctuation in the electron density is equal to n_0 and the intensity of scattering is determined by Rayleigh's formula

$$I = \frac{1}{3\cdot 16\pi^2} E_0^2 \frac{(\varepsilon_0 - 1)^2}{n_0c^3} \varepsilon_0^{1/2}\omega_0^4. \tag{22}$$

In this case the scattering coefficient is

$$\Sigma = (8\pi/3)(e^2/mc^2)^2 n_0.$$

4. Let us now investigate the scattering of electromagnetic waves in a plasma in the presence of an external magnetic field $\mathbf{H}_0$. In this case the plasma is an optically active anisotropic medium whose properties may be described by the dielectric constant tensor[5]

$$\varepsilon_{ik} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix},$$

where

$$\varepsilon_1 = 1 - \Omega^2/(\omega^2 - \omega_H^2), \quad \varepsilon_2 = \omega_H\Omega^2/\omega(\omega^2 - \omega_H^2), \quad \varepsilon_3 = 1 - \Omega^2/\omega^2, \quad \omega_H = eH/mc.$$

The electric and the magnetic fields of the scattered wave should be determined from the following expressions for the current density and the charge density:

$$\mathbf{j}_i = \frac{1}{4\pi}\frac{\partial}{\partial t}(\delta\hat{\varepsilon}_{ik}E_k^0), \quad \rho = -\frac{1}{4\pi}\frac{\partial}{\partial x_i}(\delta\hat{\varepsilon}_{ik}E_k^0), \tag{23}$$

where $\delta\hat{\varepsilon}_{ik}$ is the operator for the fluctuation in the dielectric constant of the plasma in the presence of a magnetic field. Repeating the calculations which led to formula (5) one may show that this operator has the following form:*

$$\delta\hat{\varepsilon}_{ik} = (\hat{\varepsilon} - 1)^{1/2}\frac{\delta n(\mathbf{r}, t)}{n_0}(\hat{\varepsilon}_{ik} - \delta_{ik})(\hat{\varepsilon} - 1)^{-1/2}. \tag{24}$$

Instead of equations (7) and (8) we must now use

$$\Delta A_i - \frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_k} - \frac{\varepsilon_{ik}}{c^2}\frac{\partial^2 A_k}{\partial t^2} - \frac{\varepsilon_{ik}}{c}\frac{\partial}{\partial x_k}\frac{\partial\varphi}{\partial t} = -\frac{4\pi}{c}j_i, \quad \varepsilon_{ik}\partial^2\varphi/\partial x_i\partial x_k = -4\pi\rho, \tag{25}$$

and the condition[6]

$$\partial(\varepsilon_{ik}A_k)/\partial x_i = 0.$$

Choosing for the normal mode waves

$$\mathbf{A}^\lambda(\mathbf{r}) = (4\pi c^2/\varepsilon_{ik}e_i^\lambda e_k^{\lambda *})^{1/2}\mathbf{e}^\lambda e^{i\mathbf{k}_\lambda\mathbf{r}}, \tag{26}$$

normalized in accordance with the condition

$$\int A_i^\lambda \varepsilon_{ik} A_k^{\lambda' *} dv = 4\pi c^2 \delta_{\lambda\lambda'},$$

where $\mathbf{e}^\lambda$ is the complex polarization vector

$$\mathbf{e}^\lambda = \left\{\frac{\varepsilon_2}{n_\lambda^2 - \varepsilon_1}; \; i; \; i\frac{n_\lambda^2 \sin\vartheta\cos\vartheta}{n_\lambda^2\sin^2\vartheta - \varepsilon_3}\right\}, \quad \lambda = 1, 2$$

*In deriving this formula it should be kept in mind that the equation of motion has the form $\dot{\mathbf{v}} = e\mathbf{E}/m + (e/mc)[\mathbf{v}\times\mathbf{H}_0]$.

and n_λ is the index of refraction

$$n_{1,2}^2 = \frac{(\varepsilon_1^2 - \varepsilon_2^2)\sin^2\vartheta + \varepsilon_1\varepsilon_3(1+\cos^2\vartheta) \pm \sqrt{(\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_1\varepsilon_3)^2 \sin^4\vartheta + 4\varepsilon_2^2\varepsilon_3^2\cos^2\vartheta}}{2(\varepsilon_1\sin^2\vartheta + \varepsilon_3\cos^2\vartheta)},$$

we obtain in this case the following expression for the driving force:

$$f_\lambda(t) = \sum_{\omega=\omega_0\pm\Omega} B_\lambda^\omega e^{-i\omega t}, \quad B_\lambda^\omega = -\frac{i}{4\sqrt{\pi}}(\varepsilon_{ik}e_i^{\lambda *}e_k^{\lambda})^{-1/2} E_0 (\delta n_{\mathbf{k}_\lambda - \mathbf{k}_\mu}/n_0)\{\delta_{ik} - \varepsilon_{ik}(\omega_0)\} e_k^{\mu} e_i^{\lambda *}\omega_0,$$

where $\mathbf{k}_\mu$ and $\mathbf{e}^\mu$ are the propagation vector and the polarization vector of the incident wave.

It can be shown that the intensity of scattering at the combination frequency ω is given by the following general formula:

$$I_\omega(\vartheta, \varphi) = \frac{E_0^2\omega_0^2\omega^2}{(8\pi)^3 n_0^2 c^3} \sum_{\lambda=1,2} \frac{|e_i^\mu e_i^{\lambda *} - \varepsilon_{ik}(\omega_0) e_k^\mu e_i^{\lambda *}|^2}{\varepsilon_{ik}(\omega) e_i^{\lambda *} e_k^\lambda} n_\lambda^3(\omega) \frac{2n_0(\mathbf{k}_\lambda - \mathbf{k}_\mu)^2}{4\pi n_0 e^2/T + (\mathbf{k}_\lambda - \mathbf{k}_\mu)^2}. \tag{27}$$

If the incident wave is travelling along the magnetic field then the intensity is given by the following expression

$$I_\omega^{\pm}(\vartheta) = \frac{E_0^2\omega_0^2\omega^2}{(8\pi)^3 n_0^2 c^3}\{1 - \varepsilon_1(\omega_0) \pm \varepsilon_2(\omega_0)\}^2 \sum_{\lambda=1,2} S_\lambda(\omega, \vartheta) n_\lambda^3(\omega) \frac{2n_0(\mathbf{k}_\lambda - \mathbf{k}_\mu)^2}{4\pi n_0 e^2/T + (\mathbf{k}_\lambda - \mathbf{k}_\mu)^2}, \tag{28}$$

$$S_\lambda(\omega, \vartheta) = \left(1 \pm \frac{\varepsilon_2}{n_\lambda^2 - \varepsilon_1}\right)^2 \Big/ \left[\varepsilon_1 \frac{\varepsilon_2^2}{(n_\lambda^2 - \varepsilon_1)^2} + \varepsilon_1 + \varepsilon_3 \frac{n_\lambda^4 \sin^2\vartheta\cos^2\vartheta}{(n_\lambda^2\sin^2\vartheta - \varepsilon_3)^2} + \frac{2\varepsilon_2^2}{n_\lambda^2 - \varepsilon_1}\right],$$

where the signs + and − refer respectively to right and left polarization of the incident wave.

If $(\mathbf{k}_\lambda - \mathbf{k}_\mu)^2 \ll 4\pi n_0 e^2/T$ then the scattering is due mainly to the collective oscillations of plasma density and the angular distribution of the scattering is given by the following expression

$$I_\omega^{\pm}(\vartheta) = \frac{2E_0^2 T\Omega^2\omega^4}{(8\pi)^3 n_0 m c^5 (\omega_0 \pm \omega_H)^2} \sum_{\lambda=1,2} S_\lambda(\omega, \vartheta) n_\lambda^5(\omega, \vartheta)\left\{1 + \frac{\omega_0^2}{\omega^2}\frac{\varepsilon_1(\omega_0) \pm \varepsilon_2(\omega_0)}{n_\lambda^2(\omega, \vartheta)} - 2\frac{\omega_0}{\omega}\frac{\sqrt{\varepsilon_1(\omega_0) \pm \varepsilon_2(\omega_0)}}{n_\lambda(\omega, \vartheta)}\cos\vartheta\right\}. \tag{29}$$

The presence of the magnetic field leads to a considerable change in the angular distribution of the scattering of electromagnetic waves.

In the limiting case of a weak magnetic field ($\epsilon_1 = \epsilon_3$, $\epsilon_2 \ll 1$) expression (24) becomes simplified

$$I_\omega^{\pm}(\vartheta) = \frac{2}{(8\pi)^3}\frac{E_0^2 T\omega^4\omega_0^2}{n_0 m\Omega^2 c^5}[\varepsilon_1(\omega_0) - 1 \pm \varepsilon_2(\omega_0)]^2 \left\{\varepsilon_1^{3/2}\left(1 + \frac{\omega_0^2\varepsilon_{10}}{\omega^2\varepsilon_1} - \right.\right.$$

$$\left. - 2\frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_{10}}{\varepsilon_1}}\cos\vartheta\right)(1 + \cos^2\vartheta) \pm \varepsilon_{20}\sqrt{\varepsilon_1}\left(\frac{\omega_0^2}{\omega^2} - \frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_1}{\varepsilon_{10}}}\cos\vartheta\right)(1 + \cos^2\vartheta)$$

$$\left.\left. \mp 2\varepsilon_2\sqrt{\varepsilon_1}\left(1 - \frac{5}{2}\cos^2\vartheta + \frac{\omega_0^2}{\omega^2}\frac{\varepsilon_{10}}{\varepsilon_1} - 2\frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_{10}}{\varepsilon_1}}\cos\vartheta - \frac{3\omega_0^2}{2\omega^2}\frac{\varepsilon_{10}}{\varepsilon_1}\cos^2\vartheta + 4\frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_{10}}{\varepsilon_1}}\cos^2\vartheta\right)\right\},$$

$$n_{1,2}^2 = \varepsilon_1 \pm \varepsilon_2|\cos\vartheta|, \quad \varepsilon_{10} \equiv \varepsilon_1(\omega_0), \quad \varepsilon_{20} \equiv \varepsilon_2(\omega_0). \tag{30}$$

In the case of a strong magnetic field ($\epsilon_2 = 0$, $\epsilon_1 \neq \epsilon_3$) we have

$$I_\omega^{\pm}(\vartheta) = \frac{2E_0^2 T\omega^4\omega_0^2}{(8\pi)^3 n_0 m\Omega^2 c^5}(\varepsilon_{10} - 1)^2\left\{\varepsilon'_1\left(1 + \frac{\omega_0^2}{\omega^2}\frac{\varepsilon_{10}}{\varepsilon_1} - 2\frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_{10}}{\varepsilon_1}}\cos\vartheta\right) + \frac{n_2^7}{\varepsilon_1^2}\left(1 + \frac{\omega_0^2}{\omega^2}\frac{\varepsilon_{10}}{n_2^2} - 2\frac{\omega_0}{\omega}\sqrt{\frac{\varepsilon_{10}}{n_2^2}}\cos\vartheta\right)\cos^2\vartheta\right\},$$

$$n_1^2 = \varepsilon_1, \quad n_2^2 = \varepsilon_1\varepsilon_3/(\varepsilon_1\sin^2\vartheta + \varepsilon_3\cos^2\vartheta). \tag{31}$$

If $(\mathbf{k}_\lambda - \mathbf{k}_\mu)^2 \gg 4\pi n_0 e^2/T$ the plasma oscillations may be neglected. In this case the scattering of the electromagnetic wave is determined by the random thermal motion of the individual electrons of the plasma. The angular distribution of the scattering is given by the following formula

$$I^{\pm}(\vartheta) = \frac{2}{(8\pi)^3}\frac{E_0^2}{n_0c^3}\{\varepsilon_1(\omega_0) - 1 \pm \varepsilon_2(\omega_0)\}^2\omega_0^4 \sum_{\lambda=1,2} S_\lambda(\omega_0, \vartheta)\, n_\lambda^3(\omega_0, \vartheta). \tag{32}$$

This expression is evidently a generalization of Rayleigh's formula for the intensity of scattering to the case of an optically active anisotropic medium.

5. Let us also determine the absorption of electromagnetic waves due to the excitation of plasma oscillations in which the energy of the incident electromagnetic wave is converted into the energy of longitudinal plasma oscillations.

The longitudinal field of the oscillations is characterized by the scalar potential φ satisfying the equation (8). We seek the solution of this equation in the form

$$\varphi(\mathbf{r}, t) = \sum_\sigma q_\sigma(t)\,\Phi_\sigma(\mathbf{r}), \quad \Phi_\sigma(\mathbf{r}) = \sqrt{4\pi c^2}\, e^{i\mathbf{k}_\sigma\mathbf{r}}. \tag{33}$$

The charge density ρ may be written, making use of (6), in the form

$$\rho = \frac{1}{8\pi}\sum_{\omega=\omega_0\pm\Omega}\frac{\Omega^2E_0}{n_0\omega_0\omega}(\mathbf{e}_0 \operatorname{grad}\delta n(\mathbf{r}))\, e^{i\mathbf{k}_0\mathbf{r}-i\omega t}. \tag{34}$$

Substitution of (33) and (34) into equation (8) leads to the following expression for q_σ:

$$q_\sigma = \frac{i}{4\sqrt{\pi}}\sum_{\omega=\omega_0\pm}\frac{E_0\Omega^2}{c\omega_0\omega n_0k_\sigma^2}(\mathbf{e}_0\mathbf{k}_\sigma)\,\delta n_{\mathbf{k}_\sigma-\mathbf{k}_0}\frac{1}{\varepsilon(\omega)}e^{-i\omega t}. \tag{35}$$

For $\epsilon(\omega)$ we shall take the following expression for the dielectric constant of the plasma:

$$\varepsilon(\omega) = 1 - \Omega^2/\omega(\omega + i\gamma),$$

where γ is the collision frequency of electrons in the plasma. (It is necessary to take collisions into account for otherwise at $\omega_0 = 2\Omega$ the absorption coefficient becomes infinite.)

The amount of energy absorbed per unit time per unit volume is given by formula (10) which gives after taking into account (33), (34), and (35)

$$I' = \frac{1}{2}\operatorname{Re}\sum_\sigma \dot{q}_\sigma^*\int\rho\Phi_\sigma^* dv = \frac{1}{16\pi}\sum_{\omega=\omega_0\pm}\frac{E_0^2\Omega^4}{n_0^2\omega_0^2}\int\frac{(\mathbf{e}_0\mathbf{k})^2}{k^2}\frac{\Omega^2\gamma}{(\omega^2-\Omega^2)^2+\gamma^2\omega^2}\overline{|\delta n_{\mathbf{k}-\mathbf{k}_0}|^2}\frac{d\mathbf{k}}{(2\pi)^3}. \tag{36}$$

Assuming that $(\mathbf{k} - \mathbf{k}_0)^2 \ll 4\pi n_0e^2/T$ and carrying out in (36) integration over the angles and over $\kappa = |\mathbf{k} - \mathbf{k}_0|$ from 0 to $\kappa_m = 1/a$, where $a = \sqrt{T/4\pi n_0e^2}$ is the Debye radius, we obtain the following expression for the absorption coefficient:

$$\sum{}' \approx \frac{1}{60\pi^2}\frac{\Omega^4}{n_0a^3c\varepsilon_0^{1/2}\omega_0^2}\frac{\Omega^2\gamma}{(\omega^2-\Omega^2)^2+\gamma^2\omega^2}, \quad \omega = \omega_0 \pm \Omega. \tag{37}$$

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