BOUND STATES IN POSITRONIUM

IU. A. TARASOV

Moscow State University

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It is shown that the bound states of an electron and positron may be studied by examining the poles of the photon propagator.

WE consider the bound states of an electron and a positron. Starting from the Bethe-Salpeter equation in the ladder approximation, and taking Fourier transforms of the wave-function $\psi_{\alpha\beta}(1, 2)$ and of the functions S_F, D_F, we obtain the momentum-space equation

$$A_{n_1n_2}(\mathbf{p}, p_0) = -\frac{ie^2}{(2\pi)^4}\int\sum_{n_1'n_2'}\left\{\frac{[\bar{u}^{n_1'}(\mathbf{p}')\gamma_\mu u^{n_1}(\mathbf{p})][\bar{u}^{n_2}(\mathbf{p})\gamma_\mu u^{n_2'}(\mathbf{p}')]D_F(p-p')}{(\delta_{n_1}E_p+W/2-p_0)(\delta_{n_2}E_p-W/2-p_0)}\right.$$
$$\left.-\frac{[\bar{u}^{n_2}(\mathbf{p})\gamma_\mu u^{n_1}(\mathbf{p})][\bar{u}^{n_1'}(\mathbf{p}')\gamma_\mu u^{n_2'}(\mathbf{p}')]D_F(W)]}{(\delta_{n_1}E_p+W/2-p_0)(\delta_{n_2}E_p-W/2-p_0)}\right\}A_{n_1'n_2'}(\mathbf{p}', p_0'). \tag{1}$$

Here n = 1, 2 denote solutions of the Dirac equation with positive energy, n = 3, 4 those with negative energy, and $\delta_1 = \delta_2 = 1$, $\delta_3 = \delta_4 = -1$. The second term in Eq. (1) is the exchange term. We define the three-dimensional amplitude by

$$a_{n_1n_2}(\mathbf{p}) = \int A_{n_1n_2}(p)\,dp_0.$$

We obtain the adiabatic approximation[1] from Eq. (1) if we replace D_F by $\int D_F(p)\,\delta(p_0)\,dp_0$. The first non-adiabatic approximation is obtained by substituting the adiabatic expression for $A_{n_1n_2}(p)$ on the right of Eq. (1) and integrating with respect to p_0 and p_0'. Neglecting the minus particles, we have the old Tamm-Dancoff equation for positronium. We are interested in the contribution of the exchange term. After dropping tensor terms and integrating over angles, we obtain the equations for the triplet s-state ($\sigma_1\sigma_2 = 1$),

$$a^\varepsilon(p) = \lambda\int\left\{\frac{K_1(p, p', \varepsilon W)\,a^\varepsilon(p')}{2E_p-\varepsilon W} + \frac{K_2(p, p', \varepsilon W)\,a^{-\varepsilon}(p')}{2E_p-\varepsilon W} - \frac{4[a^\varepsilon(p')+a^{-\varepsilon}(p')]}{W^2(2E_p-\varepsilon W)}\right\}p'^2\,dp', \tag{2}$$

Here $\epsilon = 1$ for the plus-component (a^{++}), and $\epsilon = -1$ for the minus-component (a^{--}); $\lambda = e^2/4\pi^2 = 1/137\pi$.

In the adiabatic approximation

$$K_1(p, p', \varepsilon W) = K_2 = \frac{1}{pp'}\ln\frac{p+p'}{|p-p'|}.$$

and in the first non-adiabatic approximation

$$K_1 = \frac{1}{pp'}\ln\frac{p+p'+E_p+E_{p'}-\varepsilon W}{|p-p'|+E_p+E_{p'}-\varepsilon W},\quad K_2 = \frac{1}{pp'}\ln\frac{p+p'+E_p+E_{p'}}{|p-p'|+E_p+E_{p'}}.$$

We introduce the notation

$$\chi = \int 4[a^\varepsilon(p')+a^{-\varepsilon}(p')]\frac{p'^2dp'}{W^2}.$$

and look for a solution of Eq. (2) of the form

$$a^3(p) = \lambda\Gamma^\varepsilon(p, W)\,\chi/(2F_0-\varepsilon W) + g^\varepsilon(p).$$

Substituting this ansatz into Eq. (2), we find that Γ^ϵ satisfies

$$\Gamma^{\varepsilon}(p, W) = 1 + \lambda \int \left\{ \frac{K_1 \Gamma^{\varepsilon}(p', W)}{2E_{p'} - \varepsilon W} + \frac{K_2 \Gamma^{-\varepsilon}(p', W)}{2E_{p'} + \varepsilon W} \right\} p'^2 dp', \tag{3}$$

while g^{ϵ} satisfies the homogeneous equation with kernels K_1 and K_2. The homogeneous system of equations for a^{ϵ} has energy eigenvalues W differing from the eigenvalues of the equation for g^{ϵ}. Therefore, if W is one of the eigenvalues for a^{ϵ}, the equation for g^{ϵ} has no non-zero solution. Substituting a^{ϵ} into the expression for χ, we find

$$\chi\,[W^2 - \Pi(W^2)] = 0, \text{ with } \Pi(W^2) = -4\lambda \sum_{\varepsilon} \int \frac{\Gamma^{\varepsilon}(p, W)\, p^2 dp}{2E_p - \varepsilon W}.$$

Since the equation for Γ^{ϵ} is the equation for a vertex function, $\Pi(W^2)$ is the photon polarization operator. Eq. (21) will have a nontrivial solution only for values of W at which the photon propagator $D_F' = 1/(W^2 - \Pi(W^2))$ has a pole. To find the poles we must carry out a renormalization. We separate a divergent factor from the vertex function

$$\Gamma^{\varepsilon}(p, W) = Z\Gamma_c^{\varepsilon}(p, W).$$

The renormalized function Γ_c^{ϵ} is defined[2] by the condition $\Gamma_c^{\epsilon}(0, m) = 1$. Substituting $\Gamma^{\epsilon}(p, W) = Z\,\Gamma_c^{\epsilon}(p, W)$ into Eq. (3) and separating $\Gamma_c^{\epsilon}(p, W)$ into two parts

$$\Gamma_c^{\varepsilon}(p, W) = \Gamma_c^{\varepsilon}(p, m) + \Phi^{\varepsilon}(p, W),$$

we obtain the equations

$$\Gamma_c^{\varepsilon}(p, m) = 1 + \lambda \int_0^{\infty} \left\{ \left[\frac{K_1(p, p', \varepsilon m) - K_1(0, p', \varepsilon m)}{2E_{p'} - \varepsilon m} \right] \Gamma_c^{\varepsilon}(p', m) + \left[\frac{K_2(p, p') - K_2(0, p')}{2E_{p'} + \varepsilon m} \right] \Gamma_c^{-\varepsilon}(p', m) \right\} p'^2 dp';$$

$$\Phi^{\varepsilon}(p, W) = \lambda \int_0^{\infty} \left\{ \frac{K_1(p, p', \varepsilon W)\, \Phi^{\varepsilon}(p', W)}{2E_{p'} - \varepsilon W} + \frac{K_2(p, p')\, \Phi^{-\varepsilon}(p', W)}{2E_{p'} + \varepsilon W} \right\} p'^2 dp' + P^{\varepsilon}(p, W),$$

where

$$P^{\varepsilon}(p, W) = \lambda \int_0^{\infty} \left\{ \left[\frac{K_1(p, p', \varepsilon W)}{2E_{p'} - \varepsilon W} - \frac{K_1(p, p', \varepsilon m)}{2E_{p'} - \varepsilon m} \right] \Gamma_c^{\varepsilon}(p', m) + \left[\frac{K_2(p, p')}{2E_{p'} + \varepsilon W} - \frac{K_2(p, p')}{2E_{p'} + \varepsilon m} \right] \Gamma_c^{-\varepsilon}(p', m) \right\} p'^2 dp'.$$

After removing the overlapping divergences, the finite part of the polarization operator is separated,*

$$\Pi(W^2) = \Pi(0) + \Pi'(0)\, W^2 + \Pi_c(W^2).$$

A calculation up to terms of order λ gives the result $\Pi'(0) = -1/4$ in the adiabatic approximation and $\Pi'(0) = -1/2$ in the first non-adiabatic approximation (after dividing by Z^2). The charge-renormalization is thus finite. The quantity $\Pi(W^2) - \Pi(0)$ is obtained as an integral involving $\Gamma_c^{\epsilon}(p, m)$ and $\Phi^{\epsilon}(p, W)$. The function $\Gamma_c^{\epsilon}(p, m)$ is calculated by successive approximation. The asymptotic form of $\Gamma_c^{\epsilon}(p, m)$ is $p^{-2\lambda}$ in the adiabatic and $p^{-\lambda}$ in the first non-adiabatic approximation. In the adiabatic approximation $\Phi^{+}(p, W) = \Phi^{-}(p, W)$ and so the two equations reduce to one, giving the result

$$Z^2 \Pi_c(W^2) = \Pi(W^2) - \Pi(0) - \Pi'(0)\, W^2 = Z^2 \left\{ -4\lambda \int_0^{\infty} \frac{\Phi^{+}(p, W)\, \Gamma_c^{+}(p, m)\, 4E_p (W^2 - m^2)\, p^2 dp}{(4E_p^2 - W^2)(4p^2 + 3m^2)} + O(\lambda) \right\}.$$

$\Phi^{+}(p, W)$ satisfies a non-homogeneous equation. A variational calculation shows that up to terms of order λ we may replace the kernel of the homogeneous integral equation corresponding to the given non-homogeneous equation by the kernel which describes the motion of an electron in a Coulomb field. We make the change of variables

$$p/m = t\sqrt{\beta}, \quad \beta = |E|/m, \quad W = 2m - |E|$$

and replace the unknown function by

$$\varphi(t) = t\Phi^{+}(t)/\sqrt{t^2 + 1},$$

*In the old Tamm-Dancoff method it is impossible to separate Π_c covariantly, since Π depends on W and not only on W^2.

Then the equation for Φ^+ becomes symmetric, and the binding energy appears only in the coefficient of the integral,

$$\varphi(t) = \gamma \int [(t^2+1)(t'^2+1)]^{-1/2} \ln \frac{t+t'}{|t-t'|} \varphi(t')\, dt' + f(t), \quad \gamma = \frac{\lambda}{\sqrt{\beta}}. \tag{4}$$

The solution of a symmetric equation may be written

$$\varphi(t) = f(t) + \gamma \sum_{n=1}^{\infty} \frac{f_n \varphi_n(t)}{\gamma_n - \gamma}, \quad f_m = (f, \varphi_m),$$

where γ_n are the eigenvalues and φ_n the eigenfunctions of Eq. (4), namely

$$\varphi_n(t) = \frac{2}{\sqrt{\pi}} (1+t^2)^{-n-1/2} \sum_{m=0}^{n-1} (-1)^m \frac{2n(2n-1)\dots(2n-2m)}{(2m+1)!} t^{2m+1}.$$

Since Π_c contains a factor λ, the quantity $W^2 - \Pi_c(W^2)$ can vanish only near to an eigenvalue of the equation for Φ^+. For example, suppose γ is near to γ_1; then we substitute $\varphi(t)$ into $\Pi_c(W^2)$ and obtain the result

$$W^2 - \Pi_c(W^2) = 4m^2 - 4m^2(\lambda^2\pi^2\beta_1/\Delta\beta_1) + O(\lambda) = 0,$$

where $\beta_1 = E_1/m$, $E_1 = me^4/4h^2$, and $\Delta\beta_1$ is the ground-state level-shift. Therefore $W^2 - \Pi_c(W^2)$ vanishes for $\Delta\beta_1 = \beta_1\alpha^2$ or $\Delta E_1 = E_1\alpha^2$, with $\alpha = 1/137$. This result agrees with the perturbation theory calculation of the level-shift in the ground-state of positronium.[3] For the other states, taking γ near to γ_n, we find

$$W^2 - \Pi_c(W^2) = 4m^2 - 4m^2(\beta_n\lambda^2\pi^2/\Delta\beta_n \cdot n) + O(\lambda) = 0,$$

with $\beta_n = \beta_1/n^2$. Therefore $W^2 - \Pi_c(W^2)$ vanishes when

$$\Delta\beta_n = \alpha^2\beta_1/n^3.$$

We conclude that the study of the poles of a propagator can give information about the bound states of a fermion and an anti-fermion. The results are consistent with Lehmann's theorem.[4] We are currently applying the method to a study of the bound states of nucleon and antinucleon.

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[1] M. Lévy, Phys. Rev. 88, 72 (1952).

[2] R. H. Dalitz and F. J. Dyson, Phys. Rev. 99, 301 (1955).

[3] A. I. Akhiezer and V. B. Berestetskii, Квантовая электродинамика (Quantum Electrodynamics) (Gostekhizdat, Moscow, 1953), § 38.

[4] H. Lehmann, Nuovo cimento 11, 342 (1954).

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