according to Landau, to use the Gallilean relativity principle. Then

$$\mathbf{v} \equiv \frac{\partial \varepsilon}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \mathrm{Sp}_{\sigma'} \int \hat{f}(\mathbf{p}, \mathbf{p}') \frac{\partial \hat{n}}{\partial \mathbf{p}'} d\mathbf{p}', \tag{16}$$

where m is the mass of the free electron. However, in a real metal m may differ from the mass of the free electron. Furthermore, this quantity, as well as the function $\hat{f}(\mathbf{p}, \mathbf{p}')$, can depend in principle on the direction. In addition, the region of large electromagnetic fields, $\hat{f}(\mathbf{p}, \mathbf{p}')$ may also depend on the electromagnetic field.

In conclusion I wish to thank V. L. Ginzburg for his interest in this work.

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APPLICATION OF MATRIX POLYNOMIALS TO DETERMINE SCATTERING PHASES

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A formalism of invariant matrix polynomials $L^{S',S}_{\ell',\ell;J}$ is developed for systems of particles of arbitrary spin. A general method for calculating $L^{S',S}_{\ell',\ell;J}$ is found, and the polynomials for total spin 0, ½ and 1 are written out explicitly. Equations (3.2) — (3.7) make the expansion of any invariant operator in polynomials $L^{S',S}_{\ell',\ell;J}$ a simple matter. It is shown that the coefficients in such an expansion of the scattering matrix are directly related to the phase-shifts. Formulae are derived for calculating the phase-shifts to any order of perturbation theory. In many specific examples this method simplifies the calculation of phase-shifts.

1. INTRODUCTION

To make comparisons of various theories with experiment, one must deal either with cross-sections or with phase-shifts. Whenever possible the phase-shifts are to be preferred, since they express the properties of the scattering with maximum conciseness. For example, the absolute sign of a phase is highly significant, as it indicates a qualitative difference (attractive or repulsive interaction) between two processes which may have equal cross-sections.

There is no existing theory which describes satisfactorily the strong interactions. But there are several theories[1-3] which give a reasonable qualitative picture of some particular processes. To compare these theories with one another, it is also convenient to study the behavior of the phase-shifts which they predict.

The interaction operator is specified in different ways in different theories. It may be given as a scattering matrix, as in perturbation theory,[4] or as a combination of various spin-dependent scalar potentials,[2,5] or as an integral operator, for example in the Tamm-Dancoff method.[6,7] To calculate the phases corresponding to given values of J, ℓ and s, one must expand the interaction operator in spin-dependent spherical harmonics $g_{J,M}^{\ell,s}(\mathbf{n})$, as for example in the papers of Tamm[6] and Zharkov.[8] But the lack of covariance of these harmonics makes the expansion very complicated. The problem is greatly simplified if one expands the interaction operator into the invariant matrix operators (2.6′) which are bilinear combinations of the $G_{J,M}^{\ell,s}(\mathbf{n})$. The superiority of the operators $L_{\ell',\ell;J}^{s',s}$ over the functions $G_{J,M}^{\ell,s}(\mathbf{n})$ is as great as that of the tensor notation $H_{\mu\nu}$ over the six components of the vectors E and H for writing down the Maxwell equations.

In addition, the use of these operators greatly simplifies the diagonalization of the S-matrix with respect to the spin s, and hence the specification of the scattering in terms of a minimum number of real phases. The situation is similar to that which arose in the work of Rohrlich and Eisenstein,[2] with the difference that we are here dealing with a known S-matrix and the problem of finding the phase-shifts is reduced to the solution of algebraic equations.

As a further generalization of the invariant polynomials (2.6′), one could introduce matrix polynomials depending on three or more unit vectors. This would simplify the description of inelastic collisions in which three or more particles are produced.

2. MATRIX POLYNOMIALS. EXPLICIT EXPRESSIONS

Let $\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle$ be an operator depending on two vectors **p** and **p′** and invariant under three-dimensional rotations and reflections of the coordinate-system. λ and λ' are spin variables describing the state of the system before and after collison. ξ and ξ' are sets of parameters describing the nature and the isotopic spin of incident and scattered particles. To eliminate the wave-functions of the initial and final states, we temporarily transform to an arbitrary representation g. Then the general principles of transformation theory[9] give

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{g'g}\langle\lambda', \xi'; \mathbf{p}'|g'\rangle\langle g'|R|g\rangle\langle g|\mathbf{p}; \xi, \lambda\rangle. \tag{2.1}$$

Separating the spin and angle dependence from the radial dependence of the wave-functions $\langle\lambda', \xi'; \mathbf{p}'|$ and $|\mathbf{p}; \xi, \lambda\rangle$, we have

$$\langle\lambda', \xi'; \mathbf{p}'| = \sum_{J'M'l's'}\langle\xi', p'; J', M', s', l'|\langle\lambda', \mathbf{n}'; J', M', s', l'| \tag{2.2}$$

$$|\mathbf{p}; \xi, \lambda\rangle = \sum_{JMls}|l, s, M, J; p, \xi\rangle|l, s, M, J; \mathbf{n}, \lambda\rangle, \tag{2.2'}$$

where

$$|l, s, M, J; \mathbf{n}, \lambda\rangle = (g_{J,M}^{l,s})^{+}(\mathbf{n}, \lambda), \quad \langle\lambda, \mathbf{n}; J, M, s, l| = g_{J,M}^{l,s}(\mathbf{n}, \lambda)$$

are eigenfunctions of the operators J^2, J_z, L^2 and S^2. Substituting Eq. (2.2) and (2.2′) into (2.1) and summing over g and g′, we obtain

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{J'M's'l'}\sum_{JMsl}\langle\xi', p'; J', M', s', l'|R|l, s, M, J; p, \xi\rangle \times\langle\lambda', \mathbf{n}'; J', M', s', l'|l, s, M, J; \mathbf{n}, \lambda\rangle. \tag{2.3}$$

Because of the rotation-invariance, J and M are conserved; the matrix elements of R which appear in the sum are diagonal in J and M and depend on M only through a factor $\delta_{M'M}$. Thus

$$\langle\xi', p'; J', M', s', l'|R|l, s, M, J; p, \xi\rangle = \delta_{J'J}\delta_{M'M}\langle\xi', p'; s', l'|R_J|l, s; p, \xi\rangle, \tag{2.4}$$

so that

$$\langle\lambda', \xi'; \mathbf{p}'|R|\mathbf{p}; \xi, \lambda\rangle = \sum_{J}\sum_{s's}\sum_{l'l}\langle\xi', p'; s', l'|R_J|l, s; p, \xi\rangle\sum_{M}\langle\lambda', \mathbf{n}'; J, M, s', l'|l, s, M, J; \mathbf{n}, \lambda\rangle. \tag{2.5}$$

The reflection-invariance of R restricts the summation over ℓ and ℓ' in Eq. (2.5) to values which conserve the parity π. Thus Eq. (2.5) is precisely the expansion of the operator R into generalized matrix polynomials[10] of the form

$$\langle\lambda',\ \mathbf{n}';\ J,\ s',\ l'\,|\,l,\ s,\ J;\ \mathbf{n},\ \lambda\rangle=\sum_M\langle\lambda',\mathbf{n}';J,M,s',l'\,|\,l,s,M,J;\mathbf{n},\lambda\rangle \tag{2.6}$$

or in a different notation*

$$L^{s',s}_{l';\,l;\,J}(\lambda',\ \mathbf{n}';\ \lambda,\ \mathbf{n})=4\pi\sum_M g^{l',s'}_{J,M}(\mathbf{n}',\ \lambda')\,(g^{l,s}_{J,M})^{+}(\mathbf{n},\lambda). \tag{2.6'}$$

Using the orthogonality relations of tensor harmonics, we can show that

$$\int L^{\alpha,\beta}_{lm;\,J}(\mathbf{n},\ \mathbf{n}')\,L^{\gamma,\delta}_{np;\,J_0}(\mathbf{n}',\ \mathbf{n}_0)\,d\mathbf{n}'=4\pi\delta_{JJ_0}\delta_{mn}\delta_{\beta\gamma}L^{\alpha,\delta}_{l,\,p;\,J}(\mathbf{n},\mathbf{n}_0). \tag{2.7}$$

We now consider the explicit calculation of the matrix polynomials (2.6′). First we express $g^{\ell,s}_{J,M}$ in terms of angle and spin functions $Y_{\ell,m}$ and $\chi_{s,\mu}$, according to the rules for transforming from the J, M, s, ℓ – representation to the ℓ, m, s, μ – representation. This gives

$$g^{l,s}_{J,M}(\mathbf{n},\ \lambda)=\sum_{m,\,\mu}\langle J,\ M,\ s,\ l\,|\,l,\ m;\ s,\ \mu\rangle\,Y_{l,\,m}(\mathbf{n})\,\chi_{s,\,\mu}(\lambda), \tag{2.8}$$

where $<J, M, s, \ell\,|\,\ell, m; s, \mu>$ is a Clebsch-Gordan coefficient.[11] In the representation with S^2 and S_z diagonal, $\chi_{s,\mu}(\lambda)=\delta_{\mu\lambda}$ and so

$$\begin{aligned}L^{s',\,s}_{l',\,l;J}(\mathbf{n}',\ \lambda';\ \mathbf{n},\ \lambda)=4\pi\sum_{\mu'\mu}\delta_{\mu'\lambda'}\delta_{\mu\lambda}\sum_M\langle J,\ M;\ s',\ l'\,|\,l';\ M-\mu';\ s',\ \mu'\rangle\\ \times\langle J,\ M;\ s,\ l\,|\,l,\ M-\mu;\ s,\ \mu\rangle\,Y_{l',M-\mu'}(\mathbf{n}')\,Y^{*}_{l,\,M-\mu}(\mathbf{n}).\end{aligned} \tag{2.9}$$

At first glance it would appear convenient to reduce the triple sum to a sum over M alone by using the factors $\delta_{\mu\lambda}$. But this is not the best way to proceed. It would give us the values of particular matrix elements with known λ and λ', but it would not allow us to express $L^{s',s}_{\ell',\ell;\,J}$ in closed form as a sum of vector spin operators[12] which reduce in the case s′ = s to ordinary spin matrices. So we keep the double sum over μ' and μ, and express $\delta_{\mu'\lambda'}\delta_{\mu\lambda}$ by means of $(2s'+1)(2s+1)$ combinations of the three components of the relevant vector operators. The summation over M can be performed by using recursive relations of the form†

$$\sqrt{(l\pm m+n_1)(l\pm(\mp m)+n_2)}\,Y_{l+i_1,\,m+i_2}(\mathbf{n})=D(l;\ n,\ i)\,Y_{lm}(\mathbf{n}), \tag{2.10}$$

where n, i are integers, and $D(\ell; n, i)$ is a differential operator independent of m and operating upon $Y_{\ell,m}(\mathbf{n})$. Having chosen the appropriate values of n, i, we avoid the appearance of Clebsch-Gordan coefficients involving m, and we can complete the summation by using the addition theorem for spherical harmonics

$$(2l+1)\,P_l(\mathbf{n}'\mathbf{n})\,/\,4\pi=\sum_m Y_{l,\,m}(\mathbf{n}')\,Y^{*}_{l,\,m}(\mathbf{n}). \tag{2.11}$$

Considering only processes in which the spin is not changed in the collision, we exhibit the explicit forms of the matrix polynomials for S = 0, ½, 1. These are the cases of greatest physical interest.

S = 0. The polynomials $L^{0,0}_{\ell,\ell;\,\ell}$ describe the scattering of two spinless particles or the singlet scattering of two particles with spin. They are identical up to a factor with Legendre polynomials

$$L^{0,0}_{l,\,l;\,l}(\mathbf{n}',\ \mathbf{n})=4\pi\sum_m Y_{l,\,m}(\mathbf{n}')\,Y^{*}_{l,\,m}(\mathbf{n})=(2l+1)\,P_l(\mathbf{n}'\mathbf{n}). \tag{2.12}$$

S = ½. The polynomials $L^{1/2,1/2}_{\ell',\ell;\,J}$ were first introduced by Tamm and collaborators[3] and have been applied extensively to the scattering of π-mesons by nucleons.[13,14] Transitions $\ell \rightarrow \ell' = \ell \pm 1$ are for-

*The factor 4π is put in so as to eliminate factors of $(1/4\pi)$ from the explicit forms of the polynomials.

†The sign $\pm(\mp m)$ means that all four combinations of sign are allowed for the two m's.

bidden by parity-conservation, so that only the two polynomials $L^{1/2,1/2}_{J-1/2,J-1/2;J}$ and $L^{1/2,1/2}_{J+1/2,J+1/2;J}$ are of interest. We derive the explicit form of the first of these from Eq. (2.9). Using the known values of the Clebsch-Gordan coefficients,[11] we find

$$L^{1/2,1/2}_{J-1/2,J-1/2,J}(\mathbf{n}',\lambda';\mathbf{n},\lambda)=\frac{4\pi}{2J}\Big\{\sum_M(J+M)Y_{J-1/2,M-1/2}(\mathbf{n}')Y^*_{J-1/2,M-1/2}(\mathbf{n})\delta_{1/2\lambda'}\delta_{1/2\lambda}$$

$$+\sum_M\sqrt{(J+M)(J-M)}\,Y_{J-1/2,M-1/2}(\mathbf{n}')Y^*_{J-1/2,M+1/2}(\mathbf{n})\delta_{1/2\lambda'}\delta_{-1/2\lambda} \tag{2.13}$$

$$+\sum_M\sqrt{(J-M)(J+M)}\,Y_{J-1/2,M+1/2}(\mathbf{n}')Y^*_{J-1/2,M-1/2}(\mathbf{n})\delta_{-1/2\lambda'}\delta_{1/2\lambda}+\sum_M(J-M)Y_{J-1/2,M+1/2}(\mathbf{n}')Y_{J-1/2,M+1/2}(\mathbf{n})\delta_{-1/2\lambda'}\delta_{-1/2\lambda}\Big\}$$

Transforming from J, M to $\ell = J - 1/2$, $m = M - 1/2$, and introducing spin matrices by

$$\delta_{1/2\lambda'}\delta_{1/2\lambda}=1/2(1+\sigma_z)_{\lambda'\lambda};\quad \delta_{1/2\lambda'}\delta_{-1/2\lambda}=1/2(\sigma_x+i\sigma_y)_{\lambda'\lambda};\quad \delta_{-1/2\lambda'}\delta_{1/2\lambda}=1/2(\sigma_x-i\sigma_y)_{\lambda'\lambda};\quad \delta_{-1/2\lambda'}\delta_{-1/2\lambda}=1/2(1-\sigma_z)_{\lambda'\lambda}; \tag{2.14}$$

we obtain

$$L^{1/2,1/2}_{J-1/2,J-1/2;J}(\mathbf{n}',\mathbf{n})=(2\pi/(2l+1))\Big\{\sum_m(l+m+1)Y_{l,m}(\mathbf{n}')Y^*_{l,m}(\mathbf{n})(1+\sigma_z)$$

$$+\sum_m\sqrt{(l+m)(l-m+1)}\,Y_{l,m-1}(\mathbf{n}')Y^*_{l,m}(\mathbf{n})(\sigma_x+i\sigma_y)+\sum_m\sqrt{(l-m)(l+m+1)}\,Y_{l,m+1}(\mathbf{n}')Y^*_{l,m}(\mathbf{n})(\sigma_x-i\sigma_y)$$

$$+\sum_m(l-m+1)Y_{l,m}(\mathbf{n}')Y^*_{l,m}(\mathbf{n})(1-\sigma_z)\Big\}. \tag{2.15}$$

The summation over m is done by using the recursion formulae

$$(l\pm m+1)Y_{l,m}=(l+1)Y_{l,m}\mp i\partial Y_{l,m}/\partial\varphi; \tag{2.16}$$

$$\sqrt{(l\pm m)(l\mp m+1)}\,Y_{l,m-1}=e^{\mp i\varphi}(i\cot\vartheta\,\partial Y_{l,m}/\partial\varphi\pm\sin\theta\,Y\partial_{l,m}/\partial\cos\theta). \tag{2.17}$$

We use also

$$\mathbf{n}'\mathbf{n}=\cos\theta'\cos\theta+\sin\theta'\sin\theta\cos(\varphi'-\varphi), \tag{2.18}$$

and obtain finally*

$$L^{1/2,1/2}_{J-1/2,J-1/2;J}(\mathbf{n}',\mathbf{n})=(l+1)P_l-i\boldsymbol{\sigma}[\mathbf{n}'\mathbf{n}]P_l',\ l=J-1/2,\ J=1,2,3,\ldots \tag{2.19}$$

A similar calculation gives

$$L^{1/2,1/2}_{J+1/2,J+1/2,J}(\mathbf{n}',\mathbf{n})=lP_l+i\boldsymbol{\sigma}[\mathbf{n}'\mathbf{n}]P_l',\ l=J+1/2,\ J=0,1,2,\ldots \tag{2.20}$$

S = 1. The matrix polynomials $L^{1,1}_{\ell',\ell;J}$ greatly facilitate the analysis of the interaction of two fermions in the triplet state. Becasue of parity-conservation, the four polynomials $L^{1,1}_{J+1,J;J}$ $L^{1,1}_{J,J+1;J}$ do not appear in the expansion of R. Out of nine possible polynomials, only five have physical interest, three diagonal in ℓ

$$L^{1,1}_{J-1,J-1;J}(\mathbf{n}',\mathbf{n})\equiv L_{11,J}(\mathbf{n}',\mathbf{n}),\quad L^{1,1}_{J,J;J}(\mathbf{n}',\mathbf{n})\equiv L_{22,J}(\mathbf{n}',\mathbf{n}),\quad L^{1,1}_{J+1,J+1;J}(\mathbf{n}',\mathbf{n})\equiv L_{33,J}(\mathbf{n}',\mathbf{n})$$

and three non-diagonal

$$L^{11}_{J-1,J+1;J}(\mathbf{n}',\mathbf{n})\equiv L_{13\,J}(\mathbf{n}',\mathbf{n});\quad L^{11}_{J+1,J-1;J}(\mathbf{n}',\mathbf{n})\equiv L_{31,J}(\mathbf{n}',\mathbf{n}),$$

Also $L_{31,J}(\mathbf{n}',\mathbf{n}) = [L_{13,J}(\mathbf{n},\mathbf{n}')]^+$. As in the case $S = 1/2$, we start from Eq. (2.9). Since λ' and λ now take three values, we must introduce instead of σ_x, σ_y, σ_z, the spin operators for S = 1,

$$S_x=\frac{1}{\sqrt2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix};\quad S_y=\frac{1}{\sqrt2}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix};\quad S_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

*In Ref. 3 [Eq. (4.6)], Ref. 14 [Eq. (5.16)], and Ref. 13 [Eq. (5.12)], there are misprints in the expressions for $L_\ell^\pm$. Either $\sigma[\mathbf{n}'\mathbf{n}]$ should be divided by sin θ, or $P_\ell^1(\cos\theta)$ should be changed to $P_\ell'(\cos\theta)$.

Then

$$\delta_{1\lambda'}\delta_{1\lambda} = \frac{1}{2}S_z(S_z+1);\ \delta_{1\lambda'}\delta_{0\lambda} = \frac{1}{\sqrt{2}}S_z(S_x+iS_y);\ \delta_{1\lambda'}\delta_{-1\lambda} = \frac{1}{2}(S_x+iS_y)^2;$$
$$\delta_{0\lambda'}\delta_{1\lambda} = \frac{1}{\sqrt{2}}(S_x-iS_y)S_z;\ \delta_{0\lambda'}\delta_{0\lambda} = 1-S_z^2;\ \delta_{0\lambda'}\delta_{-1\lambda} = -\frac{1}{\sqrt{2}}(S_x+iS_y)S_z; \tag{2.21}$$
$$\delta_{-1\lambda'}\delta_{1\lambda} = \frac{1}{2}(S_x-iS_y)^2;\ \delta_{-1\lambda'}\delta_{0\lambda} = -\frac{1}{\sqrt{2}}S_z(S_x-iS_y);\ \delta_{-1\lambda'}\delta_{-1\lambda} = \frac{1}{2}S_z(S_z-1).$$

In addition to Eq. (2.16), (2.17), the summation over m requires four more recursion formulae,

$$\sqrt{(l\pm m)(l\pm m-1)}\,Y_{l-1,\,m\mp 1} = \sqrt{\frac{2l-1}{2l+1}}\,e^{\mp i\varphi}\left(\pm l\sin\theta\, Y_{l,m} + \frac{i}{\sin\theta}\frac{\partial Y_{l,m}}{\partial\varphi} + \sin\theta\cos\theta\frac{\partial Y_{l,m}}{\partial\cos\theta}\right); \tag{2.22}$$

$$\sqrt{(l\mp m+1)(l\mp m+2)}\,Y_{l+1,\,m\mp 1} = \sqrt{\frac{2l+3}{2l+1}}\,e^{\mp i\varphi}\left(\pm(l+1)\sin\theta\, Y_{l,m} + \frac{i}{\sin\theta}\frac{\partial Y_{l,m}}{\partial\varphi} \pm \sin\theta\cos\theta\frac{\partial Y_{l,m}}{\partial\cos\theta}\right). \tag{2.23}$$

Omitting the lengthy intermediate steps, we exhibit the final expressions for all five polynomials,

$$L_{11,J}(\mathbf{n}',\mathbf{n}) = \frac{1}{l+1}\{(l+1)P_l + [2(\mathbf{n}'\mathbf{n}) + (l+1)(\mathbf{Sn})(\mathbf{Sn}') - (l+2)(\mathbf{Sn}')(\mathbf{Sn})]P_l' - (\mathbf{S}[\mathbf{n}'\mathbf{n}])^2 P_l''\}; \tag{2.24}$$

$$L_{22,J}(\mathbf{n}',\mathbf{n}) = \frac{2l+1}{l(l+1)}\{l(l+1)P_l + [-2(\mathbf{n}'\mathbf{n}) + (\mathbf{Sn}')(\mathbf{Sn})]P_l' + (\mathbf{S}[\mathbf{n}'\mathbf{n}])^2 P_l''\}; \tag{2.25}$$

$$L_{33,J}(\mathbf{n}',\mathbf{n}) = \frac{1}{l}\{-lP_l + [2(\mathbf{n}'\mathbf{n}) - l(\mathbf{Sn})(\mathbf{Sn}') + (l-1)(\mathbf{Sn}')(\mathbf{Sn})]P_l' - (\mathbf{S}[\mathbf{n}'\mathbf{n}])^2 P_l''\}; \tag{2.26}$$

$$L_{13,J}(\mathbf{n}',\mathbf{n}) = \frac{1}{\sqrt{l(l-1)}}\{l[-2l + (2l-1)(\mathbf{Sn}')^2]P_l + [2(\mathbf{n}'\mathbf{n})$$
$$-(2l-1)(\mathbf{n}'\mathbf{n})(\mathbf{Sn}')^2 + (l-1)((\mathbf{Sn})(\mathbf{Sn}') + (\mathbf{Sn}')(\mathbf{Sn}))]P_l' - (\mathbf{S}[\mathbf{n}'\mathbf{n}])^2 P_l''\}; \tag{2.27}$$

$$L_{31,J}(\mathbf{n}',\mathbf{n}) = \frac{1}{\sqrt{(l+1)(l+2)}}\{(l+1)[-2(l+1) + (2l+3)(\mathbf{Sn}')^2]P_l + [2(\mathbf{n}'\mathbf{n})$$
$$+(2l+3)(\mathbf{n}'\mathbf{n})(\mathbf{Sn}')^2 - (l+2)((\mathbf{Sn})(\mathbf{Sn}') + (\mathbf{Sn}')(\mathbf{Sn}))]P_l' - (\mathbf{S}[\mathbf{n}'\mathbf{n}])^2 P_l''\}. \tag{2.28}$$

Similar polynomials for S = 1 were obtained by Ritus[15] using a somewhat different method.

3. EXPANSION OF INVARIANT OPERATORS INTO MATRIX POLYNOMIALS

The preceding formulae show that the $L^{1,1}_{l',l;\,J}$ have a complicated form, and it is a laborious task to expand an operator in these polynomials by ordinary methods. However, the requirements of rotation and reflection invariance restrict the possible form of an operator. The most general invariant operator R is in fact*

$$R = V_1 + (\mathbf{Sn})^2 V_2 + (\mathbf{Sn}')^2 V_3 + (\mathbf{Sn}')(\mathbf{Sn})V_4 + (\mathbf{Sn})(\mathbf{Sn}')V_5, \tag{3.1}$$

where $V_i = V_i(\mathbf{n}'\mathbf{n})$ are arbitrary functions of $\mathbf{n}'\mathbf{n}$. The problem of finding the expansion coefficients is greatly simplified by using the following identities which follow from Eq. (2.24) — (2.28),

$$(2J+1)P_J = L_{11,\,J+1} + L_{22,\,J} + L_{33,\,J-1}; \tag{3.2}$$

$$(2J+1)(\mathbf{n}'\mathbf{n})P_J = \frac{J+1}{2J+3}L_{11,\,J+2} + \frac{J}{2J-1}L_{11,\,J} + \frac{J+1}{2J+3}L_{22,\,J+1} + \frac{J}{2J-1}L_{22,\,J-1} + \frac{J+1}{2J+3}L_{33,\,J} + \frac{J}{2J-1}L_{33,\,J-2}; \tag{3.3}$$

$$(2J+1)(\mathbf{Sn})^2 P_J = \frac{J+2}{2J+3}L_{11,\,J+1} + L_{22,\,J} + \frac{J-1}{2J-1}L_{33,\,J-1} + \frac{\sqrt{(J+1)(J+2)}}{2J+3}L_{13,\,J+1} + \frac{\sqrt{J(J-1)}}{2J-1}L_{31,\,J-1}; \tag{3.4}$$

$$(2J+1)(\mathbf{Sn}')^2 P_J = \frac{J+2}{2J+3}L_{11,\,J+1} + L_{22,\,J} + \frac{J-1}{2J-1}L_{33,\,J-1} + \frac{\sqrt{J(J-1)}}{2J-1}L_{13,\,J-1} + \frac{\sqrt{(J+1)(J+2)}}{2J+3}L_{31,\,J+1}; \tag{3.5}$$

$$(2J+1)(\mathbf{Sn}')(\mathbf{Sn})P_J = \frac{J+1}{2J+1}L_{11,\,J} + \frac{J-1}{2J-1}L_{22,\,J-1} + \frac{J+2}{2J+3}L_{22,\,J+1}$$
$$+\frac{J}{2J+1}L_{33,\,J} + \frac{\sqrt{J(J+1)}}{2J+1}L_{13,\,J} + \frac{\sqrt{J(J+1)}}{2J+1}L_{31,\,J}; \tag{3.6}$$

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*The invariant combination $i\mathbf{S}[\mathbf{n}'\mathbf{n}]$ is contained in the fourth and fifth terms by $i\mathbf{S}[\mathbf{n}'\mathbf{n}] = (\mathbf{Sn}')(\mathbf{Sn}) - (\mathbf{Sn})(\mathbf{Sn}')$, and $(\mathbf{S}'[\mathbf{n}'\mathbf{n}])^2$ is contained in all five by $(\mathbf{S}[\mathbf{n}'\mathbf{n}])2 = 2[1-(\mathbf{n}'\mathbf{n})^2] - (\mathbf{Sn})^2 - (\mathbf{Sn}')^2 + (\mathbf{n}'\mathbf{n})[(\mathbf{Sn})(\mathbf{Sn}') + (\mathbf{Sn}')(\mathbf{Sn})]$.

$$(2J+1)(\mathbf{Sn})(\mathbf{Sn}')P_J = \frac{J+1}{2J+3}L_{11,\,J+2} + \frac{2J}{(2J+1)(2J-1)}L_{11,\,J} + \frac{J}{2J-1}L_{22,\,J-1}$$

$$+ \frac{J+1}{2J+3}L_{22,\,J+1} - \frac{2(J+1)}{(2J+1)(2J+3)}L_{33,\,J} + \frac{J}{2J-1}L_{33,\,J-2} + \frac{\sqrt{J(J+1)}}{2J+1}L_{13,\,J} + \frac{\sqrt{J(J+1)}}{2J+1}L_{31,\,J}. \quad (3.7)$$

Sometimes we encounter operators which contain not only the P_ℓ but also their derivatives P'_ℓ and P''_ℓ. Terms containing P'_ℓ and P''_ℓ can be expressed as simple combinations of matrix polynomials $L^{1,1}_{\ell',\ell;\,J}$ similar to Eq. (3.2) —(3.7), but for reasons of space we do not here go into the details.

4. DETERMINATION OF PHASE SHIFTS FROM THE SCATTERING MATRIX

The S-matrix is as a rule calculated using quantum field theory, in which for a variety of reasons it is convenient to work in the p-representation. Thus the scattering matrix is first obtained as an operator depending on the initial and final moment $\mathbf{p}$ and $\mathbf{p}'$ and on the parameters λ, ξ, λ', ξ'. To find the phases we transform to the J, M, s, ℓ, p, ξ – representation,

$$\langle\lambda', \xi'; \mathbf{p}'|S|\mathbf{p}; \xi, \lambda\rangle = \sum_{J_2M_2s_2l_2}\sum_{J_1M_1s_1l_1}\int\langle\lambda', \xi'; \mathbf{p}'|J_2, M_2, s_2, l_2, p_2, \xi_2\rangle$$
$$\times p_2^2dp_2\langle\xi_2, p_2; l_2, s_2, M_2, J_2|S|J_1, M_1, s_1, l_1; p_1, \xi_1\rangle p_1^2dp_1\langle\xi_1, p_1; l_1, s_1, M_1, J_1|\mathbf{p}; \xi, \lambda\rangle, \quad (4.1)$$

with the transformation-functions

$$\langle\lambda, \xi; \mathbf{n}, p|J, M, s, l, p', \xi'\rangle = \langle\lambda, \mathbf{n}|J, M, s, l\rangle\langle\xi, p|p', \xi'\rangle, \quad (4.2)$$

$$\langle\lambda, \mathbf{n}|J, M, s, l\rangle = g^{l,s}_{J,M}(\mathbf{n}, \lambda), \quad (4.3)$$

$$\langle\xi, p|p', \xi'\rangle = (1/p)\,\delta(p-p')\,\delta_{\xi'\xi} \quad (4.4)$$

normalized by the conditions

$$\sum_{JMsl}\langle\lambda', \mathbf{n}'|J, M, s, l\rangle\langle J, M, s, l|\mathbf{n},\lambda\rangle = \delta_{\lambda'\lambda}\delta(\mathbf{n}'-\mathbf{n}), \quad (4.5)$$

$$\sum_{\lambda}\int\langle l', s', M', J'|\mathbf{n}, \lambda\rangle\, d\mathbf{n}\,\langle\lambda, \mathbf{n}|J, M, s, l\rangle = \delta_{JJ'}\delta_{MM'}\delta_{ss'}\delta_{ll'}, \quad (4.6)$$

$$\sum_{\xi_1}\int\langle\xi', p'|p_1, \xi_1\rangle p_1^2dp_1\langle\xi_1, p_1|p, \xi\rangle = \delta_{\xi'\xi}\,\delta(p'-p). \quad (4.7)$$

If we compare the expansion of the S-matrix in matrix polynomials $L^{s',s}_{\ell',\ell;\,J}(\mathbf{n}', \lambda'; \mathbf{n}, \lambda)$

$$\langle\lambda', \xi'; \mathbf{p}'|S|\mathbf{p}; \xi, \lambda\rangle = \sum_{s's}\sum_{l'l}\alpha^{s's}_{l',l;\,J}(p', \xi'; p, \xi)\,L^{s's}_{l',l;\,J}(\mathbf{n}', \lambda'; \mathbf{n}, \lambda) \quad (4.8)$$

with the expansion (4.1), we see that the expansion coefficients $\alpha^{s',s}_{\ell',\ell;\,J}$ are related to the J, M, s, ℓ, p, ξ —representation of the S-matrix by

$$4\pi\delta_{J'J}\delta_{M'M}p'p\alpha^{s's}_{l',l;\,J}(p', \xi'; p, \xi) = \langle\xi', p'; J', M', s', l'|S|l, s, M, J; p, \xi\rangle \quad (4.9)$$

or

$$4\pi p'p\alpha^{s',s}_{l',l;\,J}(p', \xi'; p, \xi) = \langle\xi', p'; s', l'|S|l, s; p, \xi\rangle. \quad (4.10)$$

To introduce the phase-shifts, we use two fundamental properties of the S-matrix; it must be unitary and symmetric. The first property follows from the normalization to unity of the flux of incident and scattered particles, or from the conservation of the number of particles. The second property follows from the time-reversibility of the theory, which is connected with the principle of detailed balance. Since the S-matrix is unitary, it can always be written in the form

$$S = \exp(2iQ), \quad (4.11)$$

where Q is a hermitian phase-matrix. Then, since Q is hermitian, it can be reduced by a unitary transformation U to a diagonal matrix Q_0 with N real diagonal elements*

$$Q = U^{-1}Q_0U. \quad (4.12)$$

*Here N is the number of different sets of s, ℓ, ξ, corresponding to a given value of J.

But S is symmetric, i.e. $S^+ = S^*$, and therefore Q and U are real. The orthogonality of U imposes $\frac{1}{2}N(N+1)$ additional conditions, so that only $\frac{1}{2}N(N+1)$ out of the original $2N^2$ real parameters are independent. Of these, $\frac{1}{2}N(N-1)$ are still to some extent arbitrary, since they are connected with the matrix U and depend on the choice of representation. Only the N diagonal elements, which are the actual phase shifts, express essential properties of the interaction. Thus the problem of calculating phase-shifts reduces to the diagonalization of the matrix α, and the elements of the diagonalized matrix α' are related to the phases by

$$4\pi p' p \alpha_0^{(k)}(p', p) = e^{2i\delta_k}, \quad k = 1, 2, \ldots, N; \tag{4.13}$$

or by

$$\tan 2\delta_k = \operatorname{Im}\alpha_0^{(k)} / \operatorname{Re}\alpha_0^{(k)}. \tag{4.14}$$

We often deal not with the S-matrix itself but with R = S − 1. If we denote by $a_{\ell',\ell;\,J}^{s',s}$ the coefficients in the expansion of R into polynomials $L_{\ell',\ell;\,J}^{s',s}$, then we have instead of Eq. (4.13), (4.14),

$$-2\pi i p' p a_0^{(k)}(p', p) = e^{i\delta_k}\sin\delta_k \tag{4.15}$$

$$\tan\delta_k = -\operatorname{Re} a_0^{(k)} / \operatorname{Im} a_0^{(k)}. \tag{4.16}$$

Eq. (4.14) and (4.16) do not quite solve the problem, because they are obtained by assuming the exact unitarity of the S-matrix, which ensures the compatibility of the real equations which follow from the complex equations (4.13) and (4.15). But in practice the S-matrix is often given as a finite sum which is unitary only up to the order of the last term included in the sum. Then it can happen, for example, that the equations

$$-2\pi p' p \operatorname{Re} a_{(k)} = \sin^2\delta_k; \tag{4.17}$$

$$2\pi p' p \operatorname{Im} a_{(k)} = \sin\delta_k \cos\delta_k \tag{4.18}$$

are incompatible, and Eqs. (4.14) and (4.16) may then introduce errors which of lower order than the terms neglected in the S-matrix. We now derive a general expression for the phase-shift which preserves the order of accuracy with which the S-matrix is given. The S-matrix can be written in the form

$$S = 1 + \sum_{n=1}^{\infty} g^n C_n; \tag{4.19}$$

$$C_n = \frac{(-i)^n}{g^n n!} \int_{-\infty}^{\infty} dx_n \ldots \int_{-\infty}^{\infty} dx_1 P(H(x_n) \ldots H(x_1)). \tag{4.20}$$

The coefficients in the expansion of R into polynomials $L_{\ell',\ell;\,J}^{s',s}$ are then

$$a_{l',l;\,J}^{s',s} = \sum_{n=1}^{\infty} g^n b_{l',l;\,J}^{s',s}(n). \tag{4.21}$$

After diagonalization with respect to s and ℓ, Eq. (4.17) and (4.18) give*

$$2\pi p^2 \sum_{n=1}^{\infty} g^n \operatorname{Re} b(n) = -\sin^2\delta; \tag{4.22}$$

$$2\pi p^2 \sum_{n=1}^{\infty} g^n \operatorname{Im} b(n) = \sin\delta\cos\delta, \tag{4.23}$$

hence

$$|\sin\delta| = 2\pi p^2 \left[\left(\sum_{n=1}^{\infty} g^n \operatorname{Re} b(n)\right)^2 + \left(\sum_{n=1}^{\infty} g^n \operatorname{Im} b(n)\right)^2\right]^{1/2} = 2\pi p^2 \left[\sum_{n=2}^{\infty} g^n f_n\right]^{1/2}, \tag{4.24}$$

where

$$f_{2m} = 2\sum_{k=1}^{m-1} (\operatorname{Re} b(k) \operatorname{Re} b(2m-k) + \operatorname{Im} b(k) \operatorname{Im} b(2m-k)) + [\operatorname{Re} b(m)]^2 + [\operatorname{Im} b(m)]^2,$$

* Without loss of generality we put p′ = p.

$$f_{2m+1} = 2\sum_{k=1}^{m} (\mathrm{Re}\, b(k)\, \mathrm{Re}\, b(2m+1-k) + \mathrm{Im}\, b(k)\, \mathrm{Im}\, b(2m+1-k)).$$

The sign of δ, as we see from Eq. (4.23), is given by the imaginary part of the first non-vanishing term $b(n_0)$ in the series (4.21).

5. CALCULATION OF PHASE-SHIFTS FOR NUCLEON-ANTINUCLEON SCATTERING

In Sec. 4 we showed how to deduce the phase-shifts from the scattering matrix for systems of particles of arbitrary spin. As an example we consider nucleon-antinucleon scattering.

We take the S-matrix given by the covariant formulation of perturbation theory,[16]

$$S = \sum_{n=0}^{\infty} S^{(n)}, S^{(n)} = \frac{(-i)^{(n)}}{n!} \int_{-\infty}^{\infty} dx_n \ldots \int_{-\infty}^{\infty} dx_1 P(H(x_n)\ldots H(x_2)H(x_1)) \tag{5.1}$$

The units are chosen so that $h = c = 1$. For pseudoscalar meson theory with pseudoscalar charge-symmetric interaction,

$$H(x) = gN(\bar{\psi}(x)\gamma^5\tau_i\varphi_i(x)\psi(x)). \tag{5.2}$$

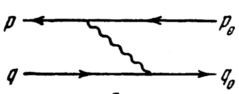
Here $\psi(x)$ and $\varphi_i(x)$, $(i = 1, 2, 3)$ are operators of the free nucleon and meson fields, τ_i are nucleon isotopic spin operators, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, $(\gamma^5)^+ = -\gamma^5$, and N denotes a normal product. Transition matrix elements can be written down by the usual rules.[4] In the lowest non-vanishing approximation we have

$$S^{(2)}_{1f\leftarrow i} = -\frac{ig^2}{(2\pi)^2} \frac{(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu}M^2}{(E_pE_qE_{p_0}E_{q_0})^{1/2}} \frac{\bar{u}_\sigma(\mathbf{p})\gamma^5 u_{\sigma_0}(\mathbf{p}_0)\cdot\bar{v}_{\rho_0}(\mathbf{q}_0)\gamma^5 v_\rho(\mathbf{q})}{(\mathbf{p}-\mathbf{p}_0)^2-(E_p-E_{p_0})^2+\mu^2}\,\delta(\mathbf{p}+\mathbf{q}-\mathbf{p}_0-\mathbf{q}_0)\,\delta(E_p+E_q-E_{p_0}-E_{q_0}); \tag{5.3}$$

$$S^{(2)}_{2f\leftarrow i} = \frac{ig^2}{(2\pi)^2} \frac{(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0}M^2}{(E_pE_qE_{p_0}E_{q_0})^{1/2}} \frac{\bar{u}_\sigma(\mathbf{p})\gamma^5 v_\rho(\mathbf{q})\cdot\bar{v}_{\rho_0}(\mathbf{q}_0)\gamma^5 u_{\sigma_0}(\mathbf{p}_0)}{(\mathbf{p}_0+\mathbf{q}_0)^2-(E_{p_0}+E_{q_0})^2+\mu^2}\,\delta(\mathbf{p}+\mathbf{q}-\mathbf{p}_0-\mathbf{q}_0)\,\delta(E_p+E_q-E_{p_0}-E_{q_0}), \tag{5.4}$$

where $\mathbf{p}_0(\mathbf{q}_0)$ and $\mathbf{p}(\mathbf{q})$ are the initial and final momenta of the nucleon (antinucleon), $\sigma_0(\rho_0)$ and $\sigma(\rho)$ are the spin indices of nucleon (antinucleon) in initial and final states, $\mu_0(\nu_0)$ and $\mu(\nu)$ are the isotopic spin indices of nucleon (antinucleon), M and μ are the nucleon and meson masses, and u and v are bispinors of the form

$$u(\mathbf{p}) = (2M(E_p+M))^{-1/2}\begin{pmatrix} E_p+M \\ \mathbf{p}\boldsymbol{\sigma} \end{pmatrix}; \qquad v(\mathbf{q}) = (2M(E_q+M))^{-1/2}\begin{pmatrix} \boldsymbol{\sigma}\mathbf{q} \\ E_q+M \end{pmatrix}.$$



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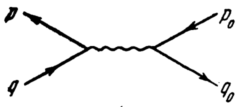

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The Feynman diagrams corresponding to $S_{1f\leftarrow i}$ and $S_{2f\leftarrow i}$ are shown in Figs. a and b.

Passing to the center-of-mass system, we have*

$$\delta(\mathbf{p}+\mathbf{q}-\mathbf{p}_0-\mathbf{q}_0) = \delta(\mathbf{p}+\mathbf{q})\,\delta(\mathbf{p}_0+\mathbf{q}_0);$$

$$\delta(E_p+E_q-E_{p_0}-E_{q_0}) = \delta(2E_p-2E_{p_0}) = (E_0/2p_0)\,\delta(p-p_0),$$

Expanding the bilinear combinations of u and v with $\mathbf{q}_0 = -\mathbf{p}_0 = -\mathbf{n}_0 p_0$ and $\mathbf{q} = -\mathbf{p} = -\mathbf{n}p$, we obtain

$$S^{(2)}_{1,2f\leftarrow i} = R_{1,2}\,\delta(\mathbf{p}+\mathbf{q})\,\delta(\mathbf{p}_0+\mathbf{q}_0)\,\delta(p-p_0); \tag{5.5}$$

$$R_1 = -\frac{ig^2}{32\pi^2}(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu}\frac{p_0}{E_0}\frac{((\mathbf{n}-\mathbf{n}_0)\boldsymbol{\sigma})_{\sigma\sigma_0}((\mathbf{n}-\mathbf{n}_0)\boldsymbol{\sigma})_{\rho_0\rho}}{p_0^2(\mathbf{n}-\mathbf{n}_0)^2+\mu^2}; \tag{5.6}$$

$$R_2 = (ig^2/8\pi^2)(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0}(E_0/p_0)\,\delta_{\sigma\rho}\delta_{\sigma_0\rho_0}/(-4E_0^2+\mu^2). \tag{5.7}$$

It is convenient to express these quantities in terms of the outer products $\tau_i \times \tau_k = \tau_{1i}\tau_{2k}$ and $\sigma_i \times \sigma_k = \sigma_{1i}\sigma_{2k}$. This is done by means of

$$(\tau_i)_{\mu\mu_0}(\tau_i)_{\nu_0\nu} = \tau_1\tau_2 - 2\tau_{1y}\tau_{2y}; \tag{5.8}$$

$$(\tau_i)_{\mu\nu}(\tau_i)_{\nu_0\mu_0} = 1/2\,(3 - \tau_1\tau_2 + 2\tau_{1y}\tau_{2y}); \tag{5.9}$$

$$(\sigma\mathbf{a})_{\sigma\sigma_0}(\sigma\mathbf{a})_{\rho_0\rho} = (\sigma, \mathbf{a})(\sigma_2\mathbf{a} - 2\sigma_{2y}a_y); \tag{5.10}$$

*Hereinafter we write E_0 for E_{p_0}.

$$\delta_{\sigma\rho}\,\delta_{\sigma_0\rho_0} = {}^1/_2\,(1 + \sigma_1\sigma_2 - 2\sigma_{1y}\,\sigma_{2y}). \tag{5.11}$$

The matrices R_1 and R_2 in this representation are formally not invariant* under rotations either in ordinary space or in isotopic space. To obtain R_1 and R_2 in manifestly invariant form, we must transform to a new representation $R^1 = URU^+$ with $U = \tau_{2y}\,\sigma_{2y}$. Dropping the primes, we obtain in the new representation

$$R_1 = -\frac{ig^2}{32\pi^2}(\tau_1\tau_2)\frac{p_0}{E_{p_0}}\,\frac{(\sigma_1(\mathrm{n}-\mathrm{n}_0))(\sigma_2(\mathrm{n}-\mathrm{n}_0))}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}\,; \tag{5.12}$$

$$R_2 = \frac{ig^2}{32\pi^2}(3+\tau_1\tau_2)\,\frac{E_0}{p_0}\,\frac{1-\sigma_1\sigma_2}{(-4E_0^2+\mu^2)}\,. \tag{5.13}$$

The transition to a representation using the total spin $S = {}^1/_2(\sigma_1 + \sigma_2)$ and $T = {}^1/_2(\tau_1 + \tau_2)$ is simplified by using the relations

$$(\sigma_1 a)(\sigma_2 a) = 2(Sa)^2 - a^2, \quad \sigma_1\sigma_2 = 2S^2 - 3, \quad \tau_1\tau_2 = 2T^2 - 3. \tag{5.14}$$

The result is then

$$R_1 = \frac{ig^2}{16\pi^2}(2T^2-3)\,\frac{p_0}{E_0}\,\frac{1-(\mathrm{nn}_0)-(\mathrm{S}(\mathrm{n}-\mathrm{n}_0))^2}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}, \quad R_2 = \frac{ig^2}{8\pi^2}\,\frac{T^2(2-S^2)}{-4E_0^2+\mu^2}\,\frac{E_0}{p_0}\,. \tag{5.15}$$

Equation (5.15) implies that, for $S = 1$ or for $T = 0$, the annihilation term vanishes in this approximation, and the scattering is given by the exchange term alone.

We consider the case of triplet scattering ($S = 1$) and expand the matrix R_1 in polynomials $L^{1,1}_{\ell',\ell;\,J}(\mathbf{n},\mathbf{n}_0)$. First we must expand $1/[\mathbf{n}-\mathbf{n}_0)^2 + c^2]$ in Legendre polynomials

$$\frac{p_0}{E_0}\,\frac{1}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2} = \sum_{j=0}^{\infty} A_j(p_0)\,P_j(\mathbf{nn}_0), \tag{5.16}$$

Next, using Eqs. (3.2) — (3.7) of Ref. 1, we obtain

$$R_1 = \frac{ig^2}{16\pi^2}(2T^2-3)\sum_j \frac{A_j(p_0)}{2j+1}\Big\{-\frac{1}{2j+3}L_{11,\,j+1} + \frac{1}{2j+1}L_{11,\,j} + \frac{j+2}{2j+3}L_{22,\,j+1} - L_{22,\,j} + \frac{j-1}{2j-1}L_{22,\,j-1} - \frac{1}{2j+1}L_{33,\,j}$$
$$+\frac{1}{2j-1}L_{33,\,j-1} - \frac{\sqrt{(j+1)(j+2)}}{2j+3}(L_{13,\,j+1}+L_{31,\,j+1}) + 2\frac{\sqrt{j(j+1)}}{2j+1}(L_{13,\,j}+L_{31,\,j}) - \frac{\sqrt{j(j-1)}}{2j-1}(L_{13,\,j-1}+L_{31,\,j-1})\Big\}. \tag{5.17}$$

With a change in the summation variable, this expansion takes the form

$$R_1 = \frac{ig^2}{16\pi^2}(2T^2-3)\sum_{J=0}^{\infty}\Big\{a_J^{(1)}L_{11,\,J} + a_J^{(2)}L_{22,\,J} + a_J^{(3)}L_{33,\,J} + a_J(L_{13,\,J}+L_{13,\,J})\Big\},$$

$$(2J+1)\,a_J^{(1)} = \frac{1}{2J+1}A_J(p_0) - \frac{1}{2J-1}A_{J-1}(p_0), \quad (2J+1)\,a_J^{(2)} = \frac{J}{2J+3}A_{J+1}(p_0) - A_J(p_0) + \frac{J+1}{2J-1}A_{J-1}(p_0), \tag{5.18}$$

$$(2J+1)\,a_J^{(3)} = \frac{1}{2J+3}A_{J+1}(p_0) - \frac{1}{2J+1}A_J(p_0), \quad (2J+1)\,a_J = \sqrt{J(J+1)}\Big(-\frac{1}{2J+3}A_{J+1}(p_0) + \frac{2}{2J+1}A_J(p_0) - \frac{1}{2J-1}A_{J-1}(p_0)\Big).$$

In the case of singlet scattering ($S = 0$) the matrices R_1 and R_2 become

$$R_1 = \frac{ig^2}{16\pi^2}(2T^2-3)\,\frac{p_0}{E_0}\,\frac{1-(\mathrm{nn}_0)}{p_0^2(\mathrm{n}-\mathrm{n}_0)^2+\mu^2}, \quad R_2 = \frac{ig^2}{4\pi^2}\,\frac{T^2}{-4E_0^2+\mu^2}\,\frac{E_0}{p_0}\,. \tag{5.19}$$

Expanding R_1 and R_2 into polynomials $L^{0,0}_{\ell',\ell;\,\ell}(\mathbf{n},\mathbf{n}_0)$ which are simply multiples of Legendre polynomials[1]

$$L^{0,0}_{l',\,l;\,l} \equiv L_l^0(\mathbf{n},\mathbf{n}_0) = (2l+1)\,P_l(\mathbf{nn}_0), \tag{5.20}$$

we find

$$R_1 = \frac{ig^2}{16\pi^2}(2T^2-3)\sum_{l=0}^{\infty} b_l(p_0)\,L_l^0(\mathbf{n},\mathbf{n}_0), \tag{5.21}$$

---

*This does not mean that there is any failure of spin-conservation, since in this representation $S_x = {}^1/_2(\sigma_{1x} - \sigma_{2x})$, $S_y = {}^1/_2(\sigma_{1y} + \sigma_{2y})$, $S_z = {}^1/_2(\sigma_{1z} - \sigma_{2z})$, and similarly $T_x = {}^1/_2(\tau_{1x} - \tau_{2x})$, $T_y = {}^1/_2(\tau_{1y} + \tau_{2y})$, $T_z = {}^1/_2(\tau_{1z} - \tau_{2z})$.

$$R_2 = \frac{ig^2}{4\pi^2} \frac{T^2}{-4E_0^2 + \mu^2} \frac{E_0}{p_0} L_0^0 (\mathbf{n}, \mathbf{n}_0), \tag{5.22}$$

where $(2\ell + 1) b_\ell (p_0) = B_\ell (p_0)$ is a coefficient in the expansion of

$$p_0 (1 - (\mathbf{n}\mathbf{n}_0)) / E_0 [p_0^2 (\mathbf{n} - \mathbf{n}_0)^2 + \mu^2]$$

in Legendre polynomials.

We saw in Sec. 4 that the phase shifts are in general to be found by diagonalizing the matrix $a^{s's}_{\ell'\ell, J}$ which is formed from the coefficients in the expansion of the operator $R = S - 1$ in matrix polynomials $L^{s's}_{\ell'\ell, J}$. The elements a_0^k of the diagonalized matrix are related to the phase-shifts δ_k by

$$-2\pi i p_0^2 a_0^k (p_0) = e^{i\delta_k} \sin \delta_k.$$

Consider the case of triplet scattering. When $J = 0$, only one polynomial $L^1_{33, 0}$ is different from zero. Thus, except for

$$a_{33} \equiv a_0^{(3)} = \left(-A_0 (p_0) + \frac{1}{3} A_1 (p_0)\right) \frac{ig^2}{16\pi^2} (2T^2 - 3), \quad A_0 (p_0) = \frac{1}{4p_0 E_0} \ln \left(1 + \frac{4p_0^2}{\mu^2}\right), \tag{5.23}$$

$$A_1 (p_0) = \frac{3}{2p_0 E_0} \left\{\frac{1}{2}\left(1 + \frac{\mu^2}{2p_0^2}\right) \ln \left(1 + \frac{4p_0^2}{\mu^2}\right) - 1\right\}, \tag{5.24}$$

all elements of the matrix a_{ik} are zero. The single phase-shift, corresponding to the transition $^3P_0 \rightarrow {}^3P_0$, is given by

$$\sin \delta = (g^2 / 8\pi) p_0^2 (2T^2 - 3) (-A_0 (p_0) + {}^1/_3 A_1 (p_0)). \tag{5.25}$$

Compare Eq. (4.24) of Ref. 1. The phases for transitions with $J \geq 1$ are found similarly.

In conclusion I express my deep gratitude to Academician I. E. Tamm for his valuable criticisms.

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