ON THE IMPROVEMENT OF PERTURBATION THEORY FORMULAS

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The n-fold Compton effect and high-energy scattering of electrons and positrons by electrons are investigated, taking into account vacuum polarization diagrams composed of the simplest closed loops.

Bogoliubov and Shirkov[1] have shown that it is possible to improve the perturbation theory formulas, owing to the existence of the renormalization group. In the present paper the mentioned method is applied to the investigation of the Compton effect and high-energy scattering of electrons and positrons by electrons, taking into account vacuum polarization.

As known the renormalized Green's functions admit a group of finite transformations

$$G_1 \to G_2 = z_2 G_1; \quad \Gamma_1 \to \Gamma_2 = z^{-1}\Gamma_1; \quad z_1 = z_2; \ D_1 \to D_2 = z_3 D_1; \quad e_1 \to e_2 = z_3^{-1/2} e_1, \tag{1}$$

meaning that the same results are obtained when either $(G_1, D_1, \Gamma_1, e_1)$ or $(G_2, D_2, \Gamma_2, e_2)$ are used. The transformations (1) should be supplemented by the transformation of the generalized vertex part

$$\Gamma_1^{(n)} \to \Gamma_2^{(n)} = z_1^{-1}\Gamma_1^{(n)}, \text{ where } \Gamma^{(n)} = G^{-1}\frac{\delta^n G}{\delta(-eA)^n}\, G^{-1}, \tag{2}$$

and by the Green's function for two electrons

$$G_1^{(2)} \to G_2^{(2)} = z_2^2 G_1^2. \tag{3}$$

Using transformations (1) to (3) one obtains a functional equation for the Green's function, in a manner analogous to the one used by Bogoliubov and Shirkov.

We write the photon Green's function in the following form

$$D_{\mu\nu}(k) = -k^{-2}(g^{\mu\nu} - k_\mu k_\nu/k^2)\, d(k^2). \tag{4}$$

In the present paper we restrict ourselves to the following values of the function $d(k^2)$:

$$d(k^2) = \left[1 - \frac{e^2}{3\pi}\ln\frac{k^2}{m^2}\right]^{-1}. \tag{5}$$

This means that we are taking into account the simplest closed loops.

The generalized Green's function for the Compton effect in the high-energy region can be presented in the form

$$\Gamma_{\mu\nu} = \left(\gamma_\mu \frac{\hat{p}_1 + \hat{l}_1 + m}{(p_1 + l_1)^2 - m^2}\gamma_\nu + \gamma_\nu \frac{\hat{p}_1 - \hat{l}_2 + m}{(p_1 - l_2)^2 - m^2}\gamma_\mu\right)\Gamma((p_1 l_1), (p_2 l_2)), \tag{6}$$

where p_1 is the initial 4-momentum of the electron and ℓ_1 and ℓ_2 are the 4-momenta of the absorbed and emitted photons. Accordingly, the transition probability will be written in the form $d\sigma = d\sigma_0 W$ where $d\sigma_0$ is the effective cross section in the zeroth order perturbation theory.

Using the arguments analogous to the ones found in Ref. 1, we obtain the following equations for the function Γ:

$$\ln\frac{\Gamma(x, y, u, e^2)}{\Gamma(x_0, y, u, e^2)} = \int_{x_0}^{x}\frac{dt}{t}\left[\frac{\partial}{\partial\xi}\ln\Gamma\left(\xi, \frac{y}{t}, \frac{u}{t}, e^2 d(t, u, e^2)\right)\right]_{\xi=1} \tag{7}$$

$$\ln\frac{\Gamma(x_0, y, u, e^2)}{\Gamma(x_0, y_0, u, e^2)} = \int_{y_0}^{y}\frac{dt}{t}\left[\frac{\partial}{\partial\xi}\ln\Gamma\left(\frac{x_0}{t}, \xi, \frac{u}{t}, e^2 d(t, u, e^2)\right)\right]_{\xi=1}, \tag{8}$$

where
$$x = (p_1 l_1)/\lambda_2^2, \quad y = (p_1 l_2)/\lambda_2^2, \quad u = m^2/\lambda_2^2.$$

Equations, analogous to (7) and (8), hold also for the correction to the probability W.

The radiative corrections for the Compton effect are, at high energies, equal to[2]

$$W = 1 - \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right) \tag{9}$$

for $(p_1 l_1) \gg m^2$, $(p_1 l_1) \gg (p_1 l_2)$.

We will also assume that $\Delta\epsilon = m$. When a hard photon k is emitted whose 4-momentum satisfies the conditions

$$(p_1 k) \ll (p_1 p_2), \quad (p_2 k) \ll (p_1 p_2), \quad m^{-2}(p_1 p_2)(p_1 k) > (p_2 k) > (p_1 k)\, m^2/(p_1 p_2), \tag{10}$$

the expression for the probability, according to Abrikosov, should be multiplied by the factor

$$W_1 = \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right). \tag{11}$$

Using the perturbation-theory formulas (9) and (11) and Eqs. (7) and (8), we obtain the following expression for the n-fold Compton effect with emission of an arbitrary number of photons:

$$d\sigma = d\sigma_0 e^{-\overline{N}}(\overline{N})^n/n!; \tag{12}$$

$$\overline{N} = \frac{3}{2}\ln\frac{(p_1 l_1)}{m^2}\ln d\left(\frac{(p_1 l_2)}{m^2}\right) + \frac{3}{2}\ln\frac{(p_1 l_2)}{m^2}\ln d\left(\frac{(p_1 l_1)}{m^2}\right) + 3\ln\frac{(p_1 l_2)}{m^2} - \frac{9\pi}{e^2}\ln d\left(\frac{(p_1 l_2)}{m^2}\right) - 6\ln\frac{(p_1 l_1)}{m^2} + \frac{18\pi}{e^2}\ln d\left(\frac{(p_1 l_1)}{m^2}\right). \tag{13}$$

This expression holds for

$$e^2 d(m^{-2}(p_1 l_1)) \ll 1, \quad e^2 d(m^{-2}(p_1 l_2)) \ll 1. \tag{14}$$

If the conditions

$$\frac{e^2}{3\pi}\ln\frac{(p_1 l_1)}{m^2} \ll 1, \quad \frac{e^2}{3\pi}\ln\frac{(p_1 l_2)}{m^2} \ll 1, \tag{15}$$

are satisfied, we obtain $\overline{N} \to \overline{n}$, where, according to Ref. 3:

$$\overline{n} = \frac{e^2}{\pi}\left(\frac{3}{2}\ln^2\frac{(p_1 l_1)}{m^2} - \frac{1}{2}\ln^2\frac{(p_1 l_1)}{(p_1 l_2)}\right). \tag{16}$$

Making use of the perturbation-theory formula for the Compton-effect radiative corrections[2]

$$W_2 = 1 + \frac{e^2}{2\pi}\ln^2\frac{(p_1 l_1)}{m^2}, \quad (p_1 l_1) \approx (p_1 l_2) \gg m^2, \tag{17}$$

we obtain the following expression for the index of refraction at high frequencies:

$$1 - n^2 = \frac{4\pi N e^2}{m\omega^2}\exp\left\{-\frac{3}{2}\ln\frac{\omega}{m} + \frac{9\pi}{2e^2}\ln d\left(\frac{\omega}{m}\right)\right\}, \quad e^2 d\left(\frac{\omega}{m}\right) \ll 1, \tag{18}$$

where ω is the frequency of the primary photon in the laboratory system. For $(e^2/3\pi)\ln(\omega/m) \ll 1$, the known result of Abrikosov[3] is obtained.

Let us consider electron-electron scattering. For $p \gg m$ (p is the momentum of the electron in the center-of-mass system) and for scattering angles θ close either to zero or to π and satisfying respectively the conditions $p\sin(\theta/2) \sim 1$ or $p\cos(\theta/2) \sim 1$, the differential cross section, according to Akhizer and Polovin,[4] is given by

$$d\sigma = d\sigma_0\left(1 - 1.74\frac{e^2}{\pi}\ln\frac{(p_1 p_2)}{m^2}\right), \tag{19}$$

where p_1 and p_2 are the 4-momenta of the initial state of the electrons. Applying Eq. (7) to this expression, we obtain

$$d\sigma = d\sigma_0[d(m^{-2}(p_1 p_2))]^{-5.22} \quad \text{for } e^2 d(m^{-2}(p_1 p_2)) \ll 1. \tag{20}$$

In the case of high-energy positron-electron scattering at an angle $m/E \ll \pi - \theta \ll \pi/2$ in the center-of-mass system, we have[3]

$$d\sigma \Rightarrow d\sigma_0\left(1 + \frac{e^2}{2\pi}\ln^2\frac{(p_1q_1)}{(p_1q_2)}\right), \quad (p_1q_1) \gg (p_1q_2), \tag{21}$$

where p_1 and q_1 are the 4-momenta of the electron in the initial and final state, and q_2 is the 4-momentum of the positron in the final state. Applying Eqs. (7) and (8) to this expression we find

$$d\sigma = d\sigma_0 \exp\left\{-\frac{3}{2}\ln\frac{(p_1q_1)}{m^2}\ln d\left(\frac{(p_1q_2)}{m^2}\right) - \frac{3}{2}\ln\frac{(p_1q_2)}{m^2}\ln d\left(\frac{(p_1q_1)}{m^2}\right) - 3\ln\frac{(p_1q_1)}{m^2} + \frac{9\pi}{e^2}\ln d\left(\frac{(p_1q_1)}{m^2}\right)\right.$$
$$\left. - 3\ln\frac{(p_1q_2)}{m^2} + \frac{9\pi}{e^2}\ln d\left(\frac{(p_1q_2)}{m^2}\right)\right\}, \tag{22}$$

for $e^2 d(m^{-2}(p_1q_1)) \ll 1$, $e^2 d(m^{-2}(p_1p_2)) \ll 1$.

If we assume that

$$\frac{e^2}{3\pi}\ln\frac{(p_1q_1)}{m^2} \ll 1, \quad \frac{e^2}{3\pi}\ln\frac{(p_1q_2)}{m^2} \ll 1,$$

we obtain again the known result of Abrikosov.

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