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ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES

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Irreducible relativistic wave equations different from the Dirac equations are derived for spin ½ particles. The particles described by these equations may have one or more proper masses and the corresponding fields have a positive definite charge density. The existence of such equations is not incompatible with the well known proof of uniqueness of the Dirac equations.

ATTEMPTS to construct wave equations of the type

$$(\beta^k\partial_k - i\varkappa)\varphi = 0 \quad (1)$$

but different from the Dirac equations have been unsuccessful. The only exception is the set of equations due to Petras.[1] But these equations are not investigated in details in his paper, and it is therefore not clear whether they meet all of the physical requirements.

The known proofs by Wild[2] and by Gel'fand and Iaglom[3] on the uniqueness of the Dirac equations seemed to imply that anomalous equations for spin ½ particles do not exist, and that the equations of Petras[1] are physically unacceptable.

The proofs of the uniqueness of the Dirac equation are, however, not general. They rely on the assumption that the equation for spin ½ particles can be derived only with the representation of the total Lorentz group for the maximum value of spin ½.

In this paper we start from the representation of D_R for the maximum value of spin $^3\!/_2$ and we show that there exist anomalous equations for spin ½ particles, the simplest of them being the Petras equations.

1. RELATIVISTIC FORM OF ANOMALOUS WAVE EQUATIONS

If the equations (1) are covariant, the matrices β_k have to satisfy the known relationships

$$[\beta_k I_{lm}] = g_{kl}\beta_m - g_{km}\beta_l; \quad (2a)$$

$$[I_{kl}I_{mn}] = -g_{km}I_{ln} + g_{lm}I_{kn} + g_{kn}I_{lm} - g_{ln}I_{km}; \quad (2b)$$

$$\beta_j Z = Z\beta^j, \quad (2c)$$

where $I_{k\ell}$ represent infinitesimal rotations and Z is the matrix of space reflection.

If the representation of the total Lorentz group D_R is known, Eqs. (2) can be used to find the form of the matrix β_0. The remaining matrices β_ν ($\nu = 1, 2, 3$) can be determined from (2a).

It is known that there are three non-equivalent irreducible representations of the operators $I_{k\ell}$ and Z for the maximum value of spin $^3\!/_2$, with 12, 8 and 4 rows respectively, which we shall denote by τ_3, τ_2 and τ_1.

In order to construct the matrix β_0, we can use the general representation of the elements $I_{k\ell}$ and Z

$$I_{kl} = aI_{kl}^{(\tau_3)} + bI_{kl}^{(\tau_2)} + cI_{kl}^{(\tau_1)}; \quad Z = aZ^{(\tau_3)} + bZ^{(\tau_2)} + cZ^{(\tau_1)},$$

where a, b and c determine the number of irreducible representations of the same kind. However, in the following discussion, we will restrict ourselves to cases with $a = 1$, $b = 0$, $c = 1, 2, 3 \ldots$; these cases are the simplest because the existing field has then a positive definite charge density; in the general case the matrices β_k are either irreducible or not equal to the Dirac matrices.

We will denote the matrix β_0 constructed with these representations by

$$\beta_0 = \beta_0(\tau_3, c\tau_1).$$

It has been shown[4] that the matrix β_0 can be written as a direct product of two matrices

$$\beta_0 = \gamma_0 \times \alpha(\tau_3, c\tau_1), \tag{3}$$

where γ_0 is a Dirac matrix. The same reference shows the form of the matrix α ($\alpha = U\alpha_0 U^+$, α_0 is a matrix from Ref. 4) to be

$$\alpha \equiv \alpha(\tau_3, c\tau_1) = \begin{Vmatrix} 2l & & & \\ & 2l & & \\ & & -l & R \\ & & S & K \end{Vmatrix}$$

R is a row matrix with elements r_μ, $\mu = 1, 2, \ldots, c$; S is a column matrix with elements s_μ, $\mu = 1, 2, \ldots, c$.

Some of the coefficients of the matrix α can be assumed equal to zero.

If we set

$$l = 0; \ k_{\mu\nu} = 0, \ \mu \neq \nu; \ k_{\mu\mu} = k_\mu \neq 0; \ k_\mu \neq \pm k_\nu, \ \mu \neq \nu; \ r_\mu \neq 0; \ s_\mu \neq 0,$$

it is easy to verify that the matrices β_k remain irreducible and that the matrix α will have the simple form

$$\alpha \equiv \alpha(\tau_3, c\tau_1) = \begin{Vmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & r_1 & r_2 & \ldots & r_c \\ & & s_1 & k_1 & & & \\ & & s_2 & & k_2 & & \\ & & \vdots & & & \ddots & \\ & & s_c & & & & k_c \end{Vmatrix}$$

This matrix has in general c + 1 different eigenvalues λ_μ ($\mu = 1, 2, \ldots, c$) different from zero

For each eigenvalue λ_μ there are four independent solutions of the wave equation (1); these solutions correspond to the two signs of the energy $\pm E$ and to the two directions of spin $\pm 1/2$. The corresponding wave equations describe therefore a spin 1/2 particle which has, in the general case, c + 1 different mass values.

It follows that, in principle, there can exist spin 1/2 relativistic wave equations which differ from the Dirac equation. We shall call them anomalous because the spin 3/2 state never occurs although we start from the representation of D_R for the maximum value of spin 3/2. The spin 3/2 state is excluded by the condition $\ell = 0$.

It is easy to find the anomalous wave equations in explicit form. We have in the spin-tensor form

$$-\sum_{\mu=1}^{c} r_\mu (2\partial^k \chi^{(\mu)} + \gamma^l \gamma^k \partial_l \chi^{(\mu)}) = \varkappa B^k, \quad {}^1/_2 s_\mu \partial_k B^k - i k_\mu \gamma^k \partial_k \chi^{(\mu)} = \varkappa \chi^{(\mu)}, \tag{4}$$

where $\chi^{(\mu)}$ is the Dirac spinor, B^k is a spin-vector introduced by Rarita and Schwinger,[5] and γ^k are the usual Dirac matrices.

2. ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES WITH A SINGLE MASS EIGENVALUE

In the general case, i.e., for arbitrary values of the coefficients r_μ, s_μ and k_μ, the charge density of the field described by Eq. (4) is not positive definite. However, one can choose the relations between the coefficients of the matrix α such as to make the charge density positive definite.

The problem is treated best in the rest system of the particle. If Eq. (1) is solved in this system, letting

$$\partial_0 \to iE = i\varkappa/\lambda,$$

the following equation for the amplitude φ is obtained

$$(\beta^0 - \lambda)\varphi = 0. \tag{5}$$

The charge density is determined from the expression

$$\rho = \varphi^+ \Lambda \beta^0 \varphi, \tag{6}$$

where the matrix Λ has to satisfy the condition

$$\Lambda\beta_k = \beta_k^+ \Lambda.$$

The matrix Λ can be expressed as a direct product of the Dirac matrix γ^0 and the matrix $\eta = \mathrm{diag}\,\| 1, 1, 1, \epsilon_1, \epsilon_2, \ldots \epsilon_c \|$, $(\epsilon_\mu = \pm 1)$;

$$\Lambda = \gamma_0 \eta. \tag{7}$$

Substituting the direct products (3) and (7) into (5) and (6), we get

$$(\alpha - \lambda)\varphi_{1,3} = 0, \quad (\alpha + \lambda)\varphi_{2,4} = 0; \tag{5'}$$

$$\rho = \sum_{r=1}^{4} \varphi_r^+ \eta \alpha \varphi_r, \tag{6'}$$

where φ_r are the components of φ. The charge density will obviously be positive definite if we can show that one of the terms of the sum (6′), e.g., the term $\varphi_1^+ \eta \alpha \varphi_1$, is positive. The condition $\rho > 0$ can therefore be replaced by the condition

$$\varphi_1^+ \eta \alpha \varphi_1 = \lambda \varphi_1^+ \eta \varphi_1 > 0, \tag{8}$$

in which the amplitude φ_1 has five components φ_μ.

The case $\alpha = \alpha(\tau_3, \tau_1)$, i.e., $c = 1$, does not yield a relationship between the coefficients of the matrix α such that the relationship (8) be satisfied. We therefore turn to the case $\alpha = \alpha(\tau_3, 2\tau_1)$ i.e., $c = 2$.

If

$$r_1^2 = \varepsilon_1 k_1^2 k_2/(k_1 - k_2), \quad r_2^2 = \varepsilon_2 k_2^2(-k_1)/(k_1 - k_2), \quad k_1 > k_2, \quad s_1 = \varepsilon_1 r_1, \quad s_2 = \varepsilon_2 r_2, \tag{9}$$

the matrix α will have a single eigenvalue λ different from zero: $\lambda = k_1 + k_2$. This means that the particle will have a single mass $m = \kappa/|\lambda|$, or $m = \kappa$ if we choose $|\lambda| = 1$.

If all the dependent components are excluded from (8), we get

$$\lambda(\lambda^2/k_1 k_2)\psi_3^* \psi_3$$

which will obviously be positive definite if $\lambda > 0$, $k_2 > 0$ or $\lambda < 0$, $k_1 > 0 > k_2$. In the first case we have $\epsilon_1 = -\epsilon_2 = 1$; in the other case $\epsilon_1 = \epsilon_2 = -1$. In both cases the matrix β_0 is not Hermitian and it cannot be diagonalized because it has a multiple zero value. It is easy to show that a similarity transformation which would transform the case $\epsilon_1 = -\epsilon_2$ into the case $\epsilon_1 = \epsilon_2$ does not exist; it can be shown that all the other cases can be obtained through a similarity transformation.

The corresponding matrix β_0, which has 20 rows, satisfies the equation

$$\beta_0^2(\beta_0^2 - 1) = 0.$$

All the coefficients of α cannot be determined by the conditions (9) and the mass of the particle. One of the coefficients is independent and the wave function contains therefore a free parameter.

3. ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES WITH SEVERAL MASS EIGENVALUES

Let us consider the case $\alpha = \alpha(\tau_3, c\tau_1)$ with $c = 3$. We see that the coefficients of the matrix can be chosen in three independent ways such that the particle has two different masses $m_1 = \kappa/|\lambda_1|$, $m_2 = \kappa/|\lambda_2|$ and that the corresponding field has a positive definite charge density (i.e., the charge density has the same spin for both mass states, and is positive).

We will show the relationships which have to be satisfied by the coefficients of the matrix α in only one of the possible cases:

$$r_1^2 = k_1^2 (k_1 - \lambda_1)(k_1 - \lambda_2)/(k_3 - k_1)(k_1 - k_2), \quad r_2^2 = -k_2^2 (k_2 - \lambda_1)(k_2 - \lambda_2)/(k_1 - k_2)(k_2 - k_3),$$
$$r_3^2 = k_3^2 (k_3 - \lambda_1)(k_3 - \lambda_2)/(k_3 - k_1)(k_2 - k_3), \quad s_1 = r_1, \quad s_2 = -r_2, \quad s_3 = r_3, \tag{10}$$
$$\varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1, \quad \lambda_1 > k_1 > k_2 > k_3 > \lambda_2, \quad \lambda_2 < \lambda_1/2, \quad \lambda_1 + \lambda_2 = k_1 + k_2 + k_3.$$

This case is interesting because the masses of the particles are not independent—in the remaining two cases a similar restriction does not exist.

As it can be seen from Eq. (10), the wave equation will contain two free parameters.

The number of possible choices of the coefficients of the matrix increasing with the increase of c, there exist, for each matrix $\alpha = \alpha(\tau_3, c\tau_1)$, $c \geq 1$, physically acceptable anomalous equations. In general, they describe spin $\frac{1}{2}$ particles with $c - 1$ different masses.

CONCLUSION

The anomalous wave equations for spin $\frac{1}{2}$ particles can be generated from the Lagrange function because for each case there is a matrix Λ (7). The electromagnetic interaction can be introduced in the anomalous equations the usual way, i.e., by making the substitution

$$\partial_k \to \partial_k - ieA_k$$

without destroying their simultaneity. An obvious proof for the case $c = 2$ has been given in Ref. 1. Similarly, the introduction of other kinds of interactions does not present any difficulty; indeed the Lagrangian of the interaction contains, for a given kind of coupling (e.g., pseudoscalar), several independent invariants; we have therefore at our disposal several coupling constants and we can find some relationships between them such that the number of additional (initial) conditions contained in the system (4) is conserved when the interaction is introduced.

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