If we try a solution of Eq. (1) in the form

$$G = A(\rho)\sin(\rho + \delta(\rho)) \tag{4}$$

imposing the additional condition

$$dG/d\rho = A(\rho)\cos(\rho + \delta(\rho)), \tag{5}$$

on the functions $A(\rho)$ and $\delta(\rho)$, we obtain from Eq. (1), in view of Eqs. (4) and (5), the following expressions for $A(\rho)$ and $\delta(\rho)$:

$$dA/d\rho = A\left[l(l+1)\rho^{-2} + U(\rho)\right]\sin(\rho+\delta)\cos(\rho+\delta), \tag{6}$$

$$d\delta/d\rho = -\left[l(l+1)\rho^{-2} + U(\rho)\right]\sin^2(\rho+\delta). \tag{7}$$

We may note that since $d\ln G/d\rho = \cot[\rho + \delta(\rho)]$, in view of Eq. (2), when $\rho \to 0$

$$\sin(\rho + \delta(\rho)) = \rho/(l+1). \tag{8}$$

Using the last expression and Eq. (7) we find

$$\delta(\rho) = -\frac{l}{l+1}\rho - \frac{1}{(l+1)^2}\int_0^\rho \rho\gamma(\rho)\,d\rho.$$

Using Eq. (8) we find the following expression from Eq. (6):

$$A(\rho) = A_0\rho^l\left(1 + \gamma_0\rho/(l+1) + \ldots\right),$$

which applies for values of ρ close to zero.

If the limitations imposed on the function $U(\rho)$ are satisfied it is obvious that

$$\delta(\rho) \to -\pi l/2 + \delta_l \quad \text{for} \quad \rho \to \infty.$$

It should be noted that in the case $\gamma(\rho) = \gamma_0$, in view of the fact that $\sin^2(\rho + \delta)$ is bounded, it follows from Eq. (7) that at large values of ρ

$$\delta(\rho) = -\pi l/2 + \delta_l - \alpha\ln\rho,$$

where α is a constant.

Because of the monotonic variation of $\delta(\rho)$ at large values of ρ it is possible to integrate Eqs. (6) and (7) numerically with high accuracy and to determine the quantity δ_l — the phase of the scattered wave. It is obvious that this form of the method of variation of arbitrary constants can be easily extended to the case in which the function $U(\rho)$ is complex.

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45

NUCLEAR SUBSHELLS AND DEFORMATION IN THE REGION PAST LEAD

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NUCLEAR deformation causes a change in the shape of the nuclear potential well. As a result the usual sequence for filling levels, described by Mayer,[1] becomes inaccurate and the scheme proposed by Nilsson must be employed.[2] An interesting feature is the fact that nuclear deformation is not a gradual effect; rather, it is found that the onset of deformation occurs suddenly at certain critical nucleon numbers. There is a sharp increase in deformation in the region of the lanthanides after N = 88. There is evidence that the collective properties of the nuclei become important at Z = 64.[3] The reverse trend, that from highly-deformed nuclei to spherically-symmetric nuclei, is not so sharp (this change probably takes place at N = 112 and Z = 74[3]). The sudden depression of the lowest levels, which are of rotational character, after Z = 86 indicates that the abrupt increase in deformation in the region of the heavy nuclei sets in after Z = 86.[4]

At the present time the scheme by which the levels are filled in the heavy-nucleus region has not been determined. The energy classification of α-decay indicates the presence of irregularities at N = 152[5] and, to a lesser extent, at Z = 96 but it is still not clear whether or not these numbers are to be associated with a nuclear deformation effect. We have set ourselves the problem of determining: a) with which N and Z numbers nuclear deformation effects are to be associated, b) whether or not any of the subshells are filled in the heavy-nucleus region, c) the energy associated with filling of the subshells and the effect of deformation.

We consider this problem from the point of view of energetics, assuming at the outset, that any irregularity effects are to be associated only with an even number of protons or neutrons. For this purpose we compare the energy increments required to add neutrons and protons. We are interested not in the absolute values of the energy increments for adding neutrons but in their difference. Provisionally we assume that no energy is required for adding neutrons in the nuclei Pu^{238} and Pu^{239} and assume that the difference is 0.2 Mev in Pu^{240} and Pu^{241}. Using the energetics of $\alpha - n$ and $\beta - n$ chains[6] we have calculated the energy involved in adding neutrons to almost all the known heavy nuclei.[7] As has already been indicated by one of us,[8] the energy $E_n(A^*, N)$ required for adding a neutron to a fictitious nucleus (A^*, N), lying on the β-stability curve Z^*, is in general different from the energy $E_n(A, N)$ required to add a neutron to another nucleus with the same number of neutrons N (but naturally with a different number of protons Z and which does not lie on the Z^* curve) by an amount which is proportional to $Z - Z^*(A)$:

$$E_n(A^*,\ N) = E_n(A,\ Z) - \alpha\{Z - Z^*(A)\}, \tag{1}$$

where $\alpha = 0.425$ Mev and $Z^*(A)$ is the value of the fictitious isobar which can have the smallest mass for a given A. In the region of heavy nuclei, which is being considered here, the dependence of Z^* on A is given approximately by the empirical formula[8]

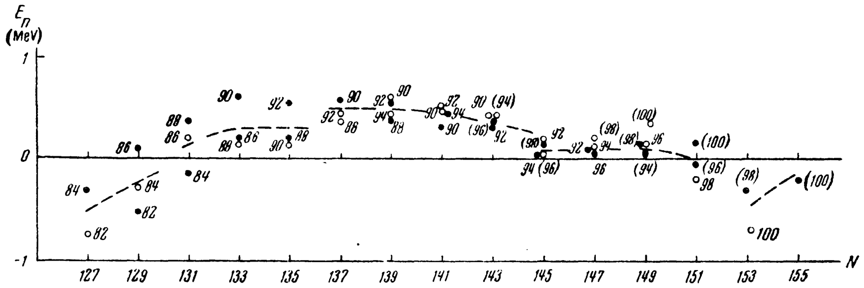
$$Z^* = 0.356\,A + 9.1, \tag{2}$$

not taking account of the small fluctuations that odd-even effects and periodic effects produce in the empirical curve $Z^*(A)$.

According to Eqs. (1) and (2), the quantities $E_n(A^*, N)$ computed from the empirical values of $E_n(A, Z)$ for different nuclei with the same parity and with the same N should be approximately the same. Actually, as is apparent from the figure (in which we have plotted $E_n(A^*, N)$ vs. N), points pertaining to the same N are almost coincident. In order to show with greatest clarity the discontinuity points in the curve $E_n(A^*, N) = f(N)$, this curve is drawn through the mean (for each N) value. It is apparent from the figure that the curve of "reduced" incremental neutron energies goes through a number of discontinuities (which are found in the same places for nuclei with all four types of parity) which indicate singularities in the energies at N = 130, 136, 144, and 152.

A similar investigation of the energies required for adding protons leads to singularities at Z = 86, 92 and 96.

To evaluate the significance of these numbers we have examined the positions of the lowest rotational levels; it has been found that a sharp drop occurs at N = 130, particularly after Z = 86; this effect extends, in a less pronounced manner, to N = 136 and Z = 92, after which the levels become stabilized. It may be assumed in any case that the numbers Z = 86 and N = 130 are to be associated with a sharp change in deformation. The forbidenness factor in α-decay in even-even nuclei in which the ground state of the initial nucleus is the ground state for the final nucleus, in particular, in transitions to the second level of the final nucleus, is reduced sharply after Z = 86 and N = 130 and then increases following Z = 92;[9] subsequently it is reduced somewhat after Z = 96. In nuclei with an odd number of neutrons there is a sharp increase in the forbiddenness factor for α-decay with neutron number following the numbers N = 136, 144 and 152; Z = 92 and 96. Comparing these results with data on the position of the levels we may conclude that the numbers Z = 86 and N = 130 are to be associated with a sharp increase in deformation. The decrease in the forbiddenness factor for α-decay following N = 130 and Z = 86 is due to the



The "reduced" energies for adding neutrons E_n^* as a function of the number of neutrons N for nuclei with Z-even, N-odd; ●—for nuclei with, A = 4n + 3; ○—for nuclei with A = 4n + 1. The numbers around the points indicate the number of protons Z; the values of Z are enclosed in brackets when interpolated values of E_α have been used for calculating E_n^*.

reduction of the potential barrier in the direction of nuclear elongation. The particular numbers N = 144 and N = 156, Z = 96 are, however, not connected exclusively with deformation. To what has been said it should be added that: (a) the probability of spontaneous fission increases sharply following the number N = 152[10] and the numbers Z = 92 and Z = 96 are somewhat fissionable, (b) the total cross sections for slow-neutron capture increase sharply after N = 152[11] and become weaker after N = 144, and (c) if one of the nuclei between the ground states of which β-decay occurs has N = 145, N = 137, or Z = 95, it is found that log $f\tau$ is larger than in other neighboring nuclei.

It is interesting to note that all the sub-magic numbers which have been found N = 136, 144, 152 and Z = 92 and 96 are included in the usual Mayer scheme if the order of neutron levels is taken as (Ref. 5):

$$\ldots |i_{11/2}\,|_{126}\; 2\,g_{9/2}\,\vdots_{(136)}\; 3\,d_{5/2}\,\vdots_{(142)}\; 4\,s_{1/2}\,\vdots_{(144)}\; g_{7/2}\,\vdots_{(152)} \ldots,$$

and the proton order is taken as:

$$\ldots h_{11/2}\,|_{82}\; h_{9/2}\,\vdots_{92}\; p_{3/2}\,\vdots_{96} \ldots .$$

In reality, however, both situations are much more complicated; for example, in some cases the known spins of the heavy nuclei are not even remotely in agreement with the described schemes.

The total effect of nuclear deformation, as it is apparent from the figure, is about 0.7 Mev for neutrons and for protons. The subshell effect is only about 0.2 Mev except for N = 152 in which the effect is of the order of 0.4 Mev.

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46