[10] B. Rossi, High Energy Particles, New York, 1952 (Russ. Transl.).
[11] Hirokawa, Komori, and Gawa, Nuovo cimento 4, 736 (1956).
[12] H. Carmichael and J. F. Steljes, Phys. Rev. 99, 1542 (1955).
[13] G. N. Fowler, Proc. Phys. Soc. A68, 482 (1955).

Translated by H. Kasha
9

SOVIET PHYSICS JETP VOLUME 6, NUMBER 1 JANUARY, 1958

NONLINEAR MESON FIELD EQUATIONS

A. A. BORGARDT

Dnepropetrovsk State University

Submitted to JETP editor October 27, 1956

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 60-66 (July, 1957)

All possible variants of pseudoscalar mesodynamics with third-order self-action are considered. Asymptotic expressions are obtained for the nonrelativistic potentials of point nucleons. Studies are also made of the solutions of the resulting nonlinear equations that correspond to plane waves, and of the question of the definite character of the energy density.

THE introduction of nonrenormalizable nonlinear equations into the quantum field theory seems to be the only promising way of overcoming a number of difficulties in the present mesodynamics and meson theory of nuclear forces. The different values of the coupling constant $g^2/\hbar c$ obtained from various experiments on the interaction of nucleons give a quite definite indication that the interaction of nucleons is a nonlinear one, and that the value of the coupling constant is always an effective value depending on the nature of the experiment. Also it is hardly possible to give a convincing explanation of the fact of the saturation of nuclear forces outside the framework of a nonlinear theory.

In the present paper we consider nonlinearities in the equations of the meson field in the form of terms of third order with respect to φ. Out of all possible forms of such self-action, that of the type $\lambda\varphi^3$, and the associated Schiff equation[1–3]

$$(\Box^2 - k_0^2 - \lambda\varphi^2)\varphi = 0.$$

have been studied in more or less detail.

Here consideration is given to all possible variants of pseudoscalar mesodynamics with third-order self-action. Asymptotic expressions are obtained for the nonrelativistic potentials of point nucleons for the various types of self-action. They all show the presence of movable singularities for a definite sign of λ. The conditions for saturation are examined in connection with the sign of λ. Studies are also made of the wave solutions of the nonlinear equations that correspond to plane waves, and of the question of the definiteness of the energy density.

1. GENERAL RELATIONS

The initial assumptions adopted are as follows:

(1) General covariance of all equations in the four-dimensional space; (2) conservation of energy and momentum in the free field; (3) the possibility of the passage to the limit $k_0 \to 0$ for arbitrary nonlinearities without the appearance of additional divergences.

The first-order[4] Lagrangian most convenient for our purposes is taken in the form

$$L^{(1)} = \alpha\,(\psi^+ (\beta_\lambda \partial/\partial x_\lambda + k_0)\,\psi + (\Lambda/2)\,\psi^+\beta'\psi \cdot \psi^+\beta''\psi), \tag{1.1}$$

where

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu - \delta_{\mu\nu}\beta_\rho - \delta_{\rho\nu}\beta_\mu = 0, \tag{1.2}$$

$$\psi^+ = \psi^* R_4, \quad R_\mu = 2\beta_\mu^2 - I. \tag{1.3}$$

The matrix ψ is either a complete (in the case of a separable mixture of fields) or a contracted (in the case of a pure field) undor of the second rank.[5]

From postulate (1) it follows obviously that $\psi^+\beta'\psi \times \psi^+\beta''\psi$ is an invariant contraction of the operators β' and β'' belonging to the complete Kemmer group G_{126}, and also that these two matrices can differ only by diagonal factors belonging to the commutative subgroup of G_{126}. These latter, again from the requirement of covariance, must be expressible in terms of the restricted basis I, R_5 ($R_5 = \prod_{\mu=1}^{4} R_\mu$)[6]

$$\beta' = R'\beta, \quad \beta'' = R''\beta, \tag{1.4}$$

$$R' = a'I + b'R_5, \quad R'' = a''I + b''R_5. \tag{1.5}$$

The variation in $L^{(1)}$ to obtain the field equations is carried out independently with respect to the potentials and the intensities,

$$\psi^{(I)} = (1/2k_0)\,(I - R_5)\,\psi, \quad \psi^{(II)} = (1/2)\,(I + R_5)\,\psi, \tag{1.6}$$

and therefore requires precise specification of the commutative properties of β with respect to the separation operators. The possible cases are

$$\text{(A)}\ \ (I \pm R_5)\,\beta\,(I \mp R_5) = 0, \qquad \text{(B)}\ \ (I \pm R_5)\,\beta\,(I \pm R_5) = 0.$$

In case (A) the operator β is diagonal in the space $\psi^{(I)}\psi^{(II)}$, but not so in case (B).

It is now easy to obtain the field equations for the cases (A) and (B) from the Lagrangian (1.1).

2. ON THE METHOD OF MERGING THE FIELDS

The problem of merging two spinor fields with nonlinear self-action is very elementary, so that we confine ourselves to some brief remarks.

The order of the self-action is lowered by the merging. If the order of the self-action in the spinor equations is n, then the order of the self-action in the boson equations obtained by merging them in pairs is simply $(n + 1)/2$.

The merging of Dirac fields with third-order self-action gives the general result

$$\beta_\lambda \partial\psi/\partial x_\lambda + k_0\psi + \Lambda_1 S\psi^+R'\psi + \Lambda_2 S\beta_\mu\psi^+R''\beta_\mu\psi + \Lambda_3\, S[\beta_\mu\beta_\nu]\,\psi^+R'''\,[\beta_\mu\beta_\nu]\,\psi + (\widetilde{NL}) = 0, \tag{2.1}$$

where $(\widetilde{NL})$ denotes nonlinear terms formed from those stated by a Larmor transformation, and

$$S = (1/2)^4 \prod_{\mu=1}^{4} (I + R_\mu)$$

is a scalar projection operator. The total number of constants Λ is fixed by the choice of the original nonlinearity.

An essential point is that in a pure pseudoscalar field a quadratic self-action is impossible, and the ps-field obtained by the merging of Dirac fields with self-action of the type $\lambda\varphi^3$ is linear. A paper by Heisenberg[7] has proposed for the universal description of quantum fields the nonlinear spinor equation

$$\gamma_\lambda \partial\psi/\partial x_\lambda + \lambda\psi\psi^+\psi = 0. \tag{2.2}$$

In addition to the serious criticism of this theory, given by Kita,[8] it must be remarked that the merging of such fields does not give the needed result for meson fields for the following reasons:

1) There is no term in k_0, and the connection between the intensities and the potentials in the meson equations depends on λ; also the field intensity diverges for $\lambda \to 0$. Thus the entire scheme does not conform to the correspondence principle.

2) The second-order self-action obtained by the merging gives nonlinear corrections in static fields with central symmetry only for the practically uninteresting case of scalar fields.

3. PSEUDOSCALAR FIELDS

The pseudoscalar theory is obviously of the greatest interest for practical purposes, both because of the pseudoscalar nature of π-mesons and because for this type the linear approximation gives the least bad results.

Let us consider the different operators β in the nonlinear terms of the wave equations of a ps field with self-action of the third order. For such fields the densities with $\beta = [\beta_\mu \beta_\nu]$ and $\beta = \beta_5 \beta_\mu$ are in general equal to zero.

The self-action with $\beta = \{\beta_\mu \beta_\nu\}$ refers to a type of weak gravitational self-actions, and we shall not consider it. There remain the cases

$$\text{(A)}\ \beta = I,\ R_5, \tag{3.1}$$

$$\text{(B)}\ \beta = \beta_\mu,\quad R_5\beta_\mu. \tag{3.2}$$

For the R-operators we confine ourselves to the simplest types (the other possibilities are equivalent to pairwise combinations of the nonlinearities obtained below, with the introduction of two constants λ_1 and λ_2):

$$1.\quad R' = R'' = I, \tag{3.3}$$

$$2.\quad R' = R'' = (1/2)(I + R_5), \tag{3.4}$$

$$3.\quad R' = R'' = (1/2)(I - R_5), \tag{3.5}$$

$$4.\quad R' = (1/2)(I \pm R_5),\quad R'' = (1/2)(I \mp R_5). \tag{3.6}$$

Cases A_1 and A_2 lead to extremely complicated equations; in the static approximation they are equivalent and give oscillating potentials with movable singularities. These equations will not be considered in what follows. The remaining cases give the following equations for real fields (besides the wave equations we give the expressions for the corresponding energy-momentum tensors, which are needed in what follows; the question of the derivation of these equations will not be considered owing to lack of space):

$$\text{I.}\quad (\Box^2 - k_0^2)\varphi - \lambda\varphi^3 = 0 \qquad (A_3) \tag{3.7}$$

$$T_{\mu\nu}^{(NL)} = T_{\mu\nu(0)}^{(L)} + (k_0^2/2)\,\delta_{\mu\nu}\varphi^2 (1 + \lambda\varphi^2/2). \tag{3.8}$$

$$\text{II.}\quad (\Box^2 - k_0^2)\varphi - \lambda\varphi\,(\partial\varphi/\partial x_\lambda)^2 = 0 \qquad (B_3) \tag{3.9}$$

$$T_{\mu\nu}^{(NL)} = (T_{\mu\nu(0)}^{(L)} + (k_0^2/2)\,\delta_{\mu\nu}(\exp(\lambda\varphi^2) - 1))\exp(-\lambda\varphi^2). \tag{3.10}$$

$$\text{III.}\quad (\Box^2 - k_0^2(1 + \lambda\varphi^2))\varphi - \lambda\varphi\,(\partial\varphi/\partial x_\lambda)^2/(1 + \lambda\varphi^2) = 0 \quad (A_4,\ B_1,\ B_2) \tag{3.11}$$

$$T_{\mu\nu}^{(NL)} = (T_{\mu\nu(0)}^{(L)} - (k_0^2/2\lambda)\,\delta_{\mu\nu}(1 - \lambda^2\varphi^4))/(1 + \lambda\varphi^2). \tag{3.12}$$

$$\text{IV.}\quad (\Box^2 - k_0^2(1 + \lambda\varphi^2))\varphi - 2\lambda\varphi\,(\partial\varphi/\partial x_\lambda)^2/(1 + \lambda\varphi^2) = 0 \quad (B), \tag{3.13}$$

$$T_{\mu\nu}^{(NL)} = (T_{\mu\nu(0)}^{(L)} + (k_0^2/2\lambda)\,\delta_{\mu\nu}(1 + \lambda\varphi^2)^2 \ln(1 + \lambda\varphi^2))/(1 + \lambda\varphi^2)^2. \tag{3.14}$$

Here

$$T_{\mu\nu(0)}^{(L)} = -(\partial\varphi/\partial x_\mu)(\partial\varphi/\partial x_\nu) + {}^1\!/_2\delta_{\mu\nu}(\partial\varphi/\partial x_\lambda)^2, \tag{3.15}$$

$$\varphi = (1/2)(I - R_5)\psi,\quad \psi^+ = \psi R_4. \tag{3.16}$$

4. STATIC SOLUTIONS[9]

For $\lambda > 0$ Schiff's equation has the asymptotic solution[10]

$$\varphi(\mathrm{x}) \approx g/x \ln(x/x_0), \tag{4.1}$$

where

$$x_0 = g\lambda^{1/2}. \quad (4.2)$$

In the limit $k_0 \to 0$, corresponding to $|\mathbf{x}| \to 0$ (more precisely, $|\mathbf{x}| \ll k_0^{-1}$), Eqs. II to IV take the form

$$\ddot{\varphi} - \lambda\dot{\varphi}^2 F(\varphi) = 0, \quad \dot{\varphi} = d\varphi/d\varphi_L, \quad (4.3)$$

where

$$\nabla^2\varphi_L(\mathbf{x}) = 0.$$

Generally speaking, one must take a dipole solution

$$\varphi_L(\mathbf{x} - \mathbf{x}_0) = f(\sigma, \mathbf{x} - \mathbf{x}_0)/|\mathbf{x} - \mathbf{x}_0|^3,$$

providing directional and noncentral forces between the singularities of the field.

Integration of Eq. (4.3) (under the condition $\varphi(\mathbf{x}) \to 0$ for $|\mathbf{x}| \to \infty$) leads to a general solution of the form:

$$\int_0^{\varphi(\mathbf{x})} d\xi \exp\left(-\lambda \int_0^{\xi} F(\xi')\, d\xi'\right) = \varphi_L(\mathbf{x}). \quad (4.4)$$

This solution can be applied with regard to $\varphi(\mathbf{x})$ only in cases III and IV:

$$\int_0^{\varphi(\mathbf{x})} d\xi \exp(-\lambda\xi^2/2) = \varphi_L(\mathbf{x}) \quad \text{(II)}, \quad (4.5)$$

$$\varphi(\mathbf{x}) = \lambda^{-1/2} \sin(\lambda^{1/2}\varphi_L) \quad \text{(III)}, \quad (4.6)$$

$$\varphi(\mathbf{x}) = \lambda^{-1/2} \tan(\lambda^{1/2}\varphi_L) \quad \text{(IV)}, \quad (4.7)$$

When the sign of λ is changed, hyperbolic functions appear in Eqs. (4.6) and (4.7). All three of these solutions are extremely interesting because of the presence of the movable singular point $|\mathbf{x}| = x_0 \sim \lambda^{1/2}$ for $\lambda > 0$. In case II the potential diverges for $|\mathbf{x}| \to x_0$ and is simply nonexistent for $|\mathbf{x}| < x_0$. In case III the potential is everywhere finite and for $|\mathbf{x}| < x_0$ oscillates between the values $\pm\lambda^{-1/2}$; the frequency of the oscillations becomes infinite for $|\mathbf{x}| \to 0$. Analagous oscillations, but of a discontinuous nature, occur in case IV. For $\lambda < 0$ the potentials (4.5) and (4.6) diverge monotonically and relatively weakly at the origin, and the potential (4.7) is finite at the origin and equal to $\lambda^{-1/2}$.

Saturation in heavy nuclei is provided by those potentials for which

$$\varphi(n\varphi_L) < n\varphi(\varphi_L), \quad (4.8)$$

this condition being satisfied by the various cases as follows:

$$\text{I — for } \lambda > 0, \quad \text{II — for } \lambda < 0, \quad \text{III — for } \lambda > 0, \quad \text{IV — for } \lambda < 0.$$

It must be noted that the existence of the potential of a charge in a nonlinear theory does not make it possible to obtain from it the potential of a dipole by differentiating; this approach is based on the principle of superposition and is not admissible in a nonlinear theory. Cap[11] obtained erroneous results just for this reason.

5. WAVE SOLUTIONS

Let us study the solutions of the wave equations I — IV that depend on a phase $\varphi = \varphi(\theta)$ with a constant wave vector k_μ

$$\theta(\mathbf{x}, t) = k_\lambda x_\lambda = -\omega t + \mathbf{kx}, \quad (5.1)$$

$$k_\lambda^2 = -k_0^2. \quad (5.2)$$

For $\lambda \to 0$ such solutions go over into ordinary linear plane waves. For convenience in the discussion we introduce the amplitude φ_0 explicitly and go over everywhere to dimensionless quantities $\xi = \varphi/\varphi_0$ and, according to circumstances, $\bar{\lambda} = \lambda(\varphi_0/k_0)^2$ or $\lambda_0 = \lambda\varphi_0^2$. We shall carry out the study in the phase plane (η, ξ), where $\eta = d\xi/d\theta$.

Then the following results are obtained.

I. For this case the equation of the phase contours is

$$\eta^2 + \xi^2 (1 + \lambda\xi^2/2) = C. \tag{5.3}$$

a) $\lambda > 0$. There is one singular point of the type of a center, at $\eta = \xi = 0$; to it there corresponds the value $C_0 = 0$. The process is always periodic. The solution reduces to the elliptic integral

$$(1 + \bar{\lambda})^{-1/2} F(\Phi, k) = -\theta(x) + C', \tag{5.4}$$

where

$$\Phi = \arccos(\varphi/\varphi_0),\ k^2 = \bar{\lambda}/2(1+\bar{\lambda}). \tag{5.5}$$

b) $\lambda < 0$. There are three singular points, $\eta = \xi = 0$ and $\eta = 0$, $\xi_0 = \pm \bar{\lambda}^{-1/2}$ — a center and two saddle-points. The equation of the separatrix passing through the saddle-points (cf. Fig. 1, A) is

$$\eta^2 - (1/2)(1 - \bar{\lambda}\xi^2) = 0. \tag{5.6}$$

Wave processes are possible only for

$$1/2\bar{\lambda} > C > 0. \tag{5.7}$$

The solution is again obtained in the form of an elliptic integral

$$(1 - \bar{\lambda}/2)^{-1/2} F(\Phi, k) = \theta(x) + C', \tag{5.8}$$

where

$$\Phi = \arcsin(\varphi/\varphi_0), \quad k^2 = \bar{\lambda}/2(1 - \bar{\lambda}/2). \tag{5.9}$$

The process is periodic only under the condition $\bar{\lambda} \leq 1$ or $\varphi_0 \leq k_0/\lambda^{1/2}$. The energy density $\mathcal{E}$ is not positive definite; $\mathcal{E} < 0$ in the region of aperiodic motions for large ξ.

II. The equation of the phase contours is

$$\eta^2 - C\exp(\lambda_0\xi^2) = \lambda_0^{-1}. \tag{5.10}$$

a) $\lambda > 0$. There is a singular point of the center type, $\eta = \xi = 0$ ($C_0 = -\lambda_0^{-1}$), and a separatrix at $C_{lim} = 0$, with the equation $\eta^2 = \lambda_0^{-1}$. Beyond the separatrix $C > 0$ and the motions are aperiodic (cf. Fig. 1, B).

The general form of the solution (for arbitrary λ) is

$$\int_0^{\varphi/\varphi_0} d\xi\,(C\exp(\lambda_0\xi^2) + \lambda_0^{-1})^{-1/2} = \theta(x) + C'. \tag{5.11}$$

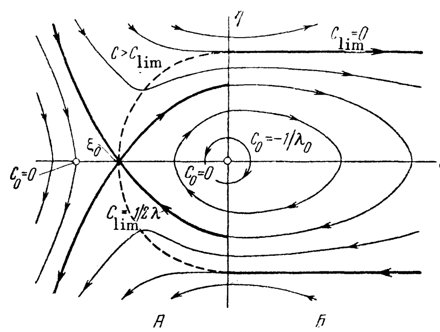

FIG. 1

To the separatrix there corresponds a solution of the form

$$\varphi(x) = \pm\lambda^{-1/2}\theta(x) + C'. \tag{5.12}$$

The asymptotic solutions for $\lambda > 0$ and $C \gg 0$ reduce to Kramp functions.

b) $\lambda < 0$. The center remains and the separatrix disappears. All motions are periodic. For any sign of λ the energy density remains positive definite in the entire phase plane (η, ξ) [cf. Eq. (3.10)].

III. The equation of the phase contours is

$$\eta^2 - (C - \xi^2)(1 + \lambda_0\xi^2) = 0. \tag{5.13}$$

a) $\lambda > 0$. There is a center at $\eta = \xi = 0$. All the phase contours are closed. The general solution is

$$(1 + \lambda_0)^{-1/2} F(\Phi, k) = -\theta(x) + C', \tag{5.14}$$

where

$$\Phi = \arccos(\varphi/\varphi_0), \qquad k^2 = \lambda_0/(1 + \lambda_0).$$

The energy density is positive definite inside a finite region which contracts to the center for $c|\mathbf{k}|/\omega \to \infty$.

b) $\lambda < 0$. Two new singular points appear at $\eta = 0$, $\xi_0 = \pm\lambda_0^{-1/2}$, through which there passes a separatrix ($C_{lim} = -1/\lambda_0$) (cf. Fig 2, A)

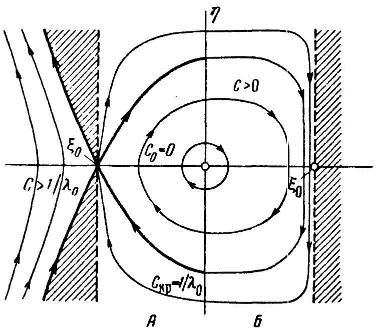

FIG. 2

$$\eta^2 - (1/\lambda_0)(1 - \lambda_0\xi^2)^2 = 0. \tag{5.15}$$

The energy density is always positive definite in the region of the oscillatory processes.

IV. The equation of the phase contours is

$$\eta^2 - (C - (1/\lambda_0)\ln(1 + \lambda_0\xi^2))(1 + \lambda_0\xi^2)^2 = 0. \tag{5.16}$$

a) $\lambda > 0$. There is a center at $\eta = \xi = 0$ ($C_0 = 0$). All the phase curves are closed. $\mathcal{E} > 0$ everywhere.

b) $\lambda < 0$. In this case no points with $|\xi| \geq 1/\lambda_0$ exist in the plane (η, ξ). Inside the allowed region (cf. Fig 2, B) all the curves are closed and $\mathcal{E} > 0$.

APPENDIX

1. The conversion of the original matrix wave equation into an ordinary second-order differential equation requires the use of projection operators constructed from the reflection matrices of the Kemmer algebra.[12,13] The well known contractions are given by the following method. Let us consider the wave solutions of the nonlinear equation

$$\beta_\lambda \partial\psi/\partial x_\lambda + k_0\psi + \lambda R'\beta_{\mu,\ldots,\nu}\psi \cdot \psi^+ R''\beta_{\mu,\ldots,\nu}\psi = 0, \tag{A.1}$$

$$\psi = \psi(\theta), \quad \theta(x) = k_\lambda x_\lambda. \tag{A.2}$$

The wave function can be put in this case in the form

$$\psi(x) = k_0(\psi_1(\theta) + (\beta_\lambda k_\lambda/k_0)\psi_2(\theta))a, \tag{A.3}$$

$$\psi^+(x) = k_0 a^* R_4(\psi_1(\theta) - (\beta_\lambda k_\lambda/k_0)\psi_2(\theta)). \tag{A.4}$$

In the case of the s or ps field the constant undor a is simply given by

$$a = (1/2)(I - R_5)E, \tag{A.5}$$

and in the case of a vector or pseudovector field it is

$$a = (1/16)(I - R_5)S_\mu(I + R_5/R_\mu)(2 + M_1)E, \tag{A.6}$$

where E is the unit undor, $M_1 = \sum_{\mu=1}^{4} R_\mu$, and S_μ is an operator specifying the spin state of the wave. Using this, we find that

$$\psi^+ R''\beta_{\mu,\ldots,\nu}\psi = k_0^2(A_{\mu,\ldots,\nu}\psi_1^2 - (k_\lambda/k_0)A_{\mu,\ldots,\nu\lambda}\psi_1\psi_2 - (k_\lambda k_\sigma/k_0^2)A_{\mu,\ldots,\nu\lambda\sigma}\psi_2^2), \tag{A.7}$$

where

$$A_{\mu,\ldots,\nu} = a^* R_4 R''\beta_{\mu,\ldots,\nu}a, \tag{A.8}$$

$$A_{\mu,\ldots,\nu\lambda} = a^* R_4 R''\beta_{\mu,\ldots,\nu}\beta_\lambda a, \tag{A.9}$$

$$A_{\mu,\ldots,\nu\lambda\sigma} = a^* R_4 R''\beta_{\mu,\ldots,\nu}\beta_\lambda\beta_\sigma a. \tag{A.10}$$

are constant tensors.

The simplest example is that of a pseudoscalar wave in a field with $\beta_{\mu,\ldots,\nu} = I$ and $R' = R'' = (\tfrac{1}{2}k_0)(I - R_5)$; here, setting the diagonal and off-diagonal parts of Eq. (A.1) separately equal to zero, we get the system (with $\dot\psi = d\psi/d\theta$)

$$\dot\psi_2 + \psi_1 + \lambda A\psi_1^3 = 0, \tag{A.11}$$

$$\dot\psi_1 - \psi_2 = 0, \tag{A.12}$$

equivalent to Schiff's equation.[1]

To obtain static solutions possessing spherical symmetry, we apply the substitution

$$\psi(\mathrm{x}) = (k_0\psi_1(x) - (\beta, \mathrm{x}^0)\psi_2(x))a, \tag{A.13}$$

$$\psi^+(\mathbf{x}) = a^* R_4 (k_0 \psi_1(x) + (\beta, \mathbf{x}^0)\psi_2(x)). \tag{A.14}$$

The quantity a is taken in the form (A.5), or, in the case of a v or pv field, in the form (A.6) with $S = 0$, $S_4 = 1$, and so on.

The complete picutre of the interaction of nonlinear meson waves with a singularity in the meson field — a nucleon — is extremely complicated and requires special investigation. We therefore confine ourselves to a remark regarding the specifically nonlinear scattering of a meson by a nucleon. Let the solution of the nonlinear equation

$$\beta_\lambda \partial\psi/\partial x_\lambda + k_0\psi + \lambda R'\beta\psi \cdot \psi^+ R''\beta\psi = 0 \tag{A.15}$$

be a sum of the following form [we refrain as a matter of principle from the introduction of any sort of interaction terms into the nonlinear theory; both the nucleon itself and the wave incident on it are already contained in Eq. (A.15)]

$$\psi(x) = \psi_0(\mathbf{x}) + \psi'(\mathbf{x})\exp(-i\omega t), \tag{A.16}$$

where $\psi_0(\mathbf{x})$ is a solution of the static equation

$$\beta, \nabla\psi_0 + k_0\psi_0 + \lambda R'\beta\psi_0\psi_0^+ R''\beta\psi_0 = 0. \tag{A.17}$$

Assuming $\psi' \ll \psi_0$, we get for the determination of ψ' in the first approximation the linear equation

$$\beta, \nabla\psi' + (k_0 - \beta_4\omega/c)\psi' + \lambda R'\beta\psi' \cdot \psi_0^+ R''\beta\psi_0 + \lambda R'\beta\psi_0(\psi_0^+ R''\beta\psi' + \psi^{+\prime} R''\beta\psi_0) = 0. \tag{A.18}$$

Proceeding further to choose ψ' in the form

$$\psi'(\mathbf{x}) = a\exp(i\mathbf{k}\mathbf{x}) + \psi''(\mathbf{x}) \tag{A.19}$$

and solving the resulting equation for ψ', one can convince oneself that ψ'' is spherically symmetric at large distances, and the nonlinear scattering is almost always isotropic.

The second-order equation for a weak meson wave colliding with a nucleon at rest can in many cases be reduced to the form

$$(\nabla^2 - k_0^2)\psi - (n^2(\mathbf{x})/c^2)\partial^2\psi/\partial t^2 = 0, \tag{A.20}$$

where $n \to 1$ for $\lambda \to 0$.

This equation contains a curious effect of the nonlinear capture of rays falling on the source with impact parameter smaller than a certain value l_{cr}. In this process the rays wind up spirally around the singularity of the field.

[1] L. Schiff, Phys. Rev. 84, 1 (1951).
[2] B. Malenka, Phys. Rev. 85, 686 (1952).
[3] R. Fornaguera, Nuovo cimento 1, 132 (1955).
[4] F. Belinfante, Phys. Rev. 74, 779 (1948).
[5] F. Belinfante, Physica 6, 849 (1939).
[6] A. A. Borgardt, J. Exptl. Theoret. Phys. (U.S.S.R.) 24, 24 (1953).
[7] W. Heisenberg, Zeitschr. f. Naturforsch. 9A, 292 (1954).
[8] H. Kita, Prog. Theor. Phys. 15, 83 (1956).
[9] A. A. Borgardt, Dokl. Akad. Nauk SSSR 110, 42 (1956), Soviet Phys. "Doklady" 1, 524 (1957).
[10] V. Iaichnitsyn, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 1082 (1956), Soviet Phys. JETP 4, 925 (1957).
[11] F. Cap, Phys. Rev. 95, 287 (1954).
[12] A. A. Borgardt, J. Exptl. Theoret. Phys. (U.S.S.R.) 24, 284 (1953).
[13] I. Lujiwara, Prog. Theor. Phys. 10, 589 (1953).

Translated by W. H. Furry