tude of the output power is determined by the relationship $W = E^2\omega_{21}\theta/2Q$ where θ is the volume of the resonator which is in order of magnitude 10 cm^3. Substituting here the given quantities we have $W = 7 \times 10^{-3}$ erg/sec. To produce self-oscillation it is essential that $B > 2/Q$. Let us take therefore a value $B = 10^{-3}$. Then from Eq. (19) we have $D_0 = Bh/4\pi p^2\tau \approx 10^9$ cm^{-3}. The number of molecules falling on an area of 1 cm^2 is $D_0 v = 4 \times 10^{13}$ cm^{-2} sec^{-1}. It should be noted that in real operating conditions the molecules are concentrated in a narrow beam whose area is significantly smaller than the transverse area of the resonator. The value that has been used above is characteristic of an average molecular density in the resonator. The total number of molecules which enter the resonator in a unit time is $N = D_0 vS \approx 10^{14}$ sec^{-1} where S is the transverse area of the resonator. The order of magnitude of the power emitted by the molecule is the same as the input power of the generator $Nh\omega/2 = 7 \times 10^{-3}$ erg/sec. This value agrees with the value obtained above.

Let us evaluate the effect of a varying resonator temperature on the generator stability. The order of magnitude of resonator frequency variation due to unconstant temperature is determined by the following relationship:

$$\Delta\omega_p / \omega_p = \Delta R / R = \alpha\Delta T, \qquad (27)$$

where α is the linear expansion coefficient, R the radial dimension of the resonator, and ΔT is the accuracy with which the resonator temperature is maintained. For example for invar we have $\alpha = 1.5 \times 10^{-6}$. For small values of transit phase

$$\Delta\omega = \Delta\omega_p (6Q / \omega_{21}\tau) \approx 4 \cdot 10^{-3} \Delta\omega_p. \qquad (28)$$

From (27) and (28) it follows that in order to maintain a stability of 5×10^{-11} near the self-oscillation threshold it is essential to maintain the temperature constant with an accuracy of 10^{-2}. With an increase in beam intensity, the accuracy with which it is necessary to maintain the temperature can be decreased by one order of magnitude.

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On the Nonlinear Generalization of the Meson and Spinor Field Equations

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Exact periodic solutions of the nonlinear generalized Klein-Gordon and Dirac equations are considered. The energy of the nonlinear classical meson field is compared with that derived from quantum theory.

THE NECESSITY for nonlinear generalizations of scalar, spinor, and other field equations, as well as the possible importance of the nonlinearities in specific effects, makes desirable detailed examinations of this problem. Here we consider a purely classical unquantized scalar or pseudo-scalar neutral meson field. We make use of the exact solution of nonlinear wave equations in terms of elliptic functions, which are used to calculate the total energy and momentum of the field. Further, the expression obtained is expanded in a series in terms of a small parameter of nonlinearity. We can consider also the case in which this parameter is large. It is of interest to compare the expressions obtained in the case of weak nonlinearity with the results of a perturbation-theory treatment of the nonlinear quantum field.

We also consider the nonlinear spinor equation.

1. NONLINEAR MESODYNAMICS

Let us take

$$L = -\tfrac{1}{2}\{(\nabla\varphi)^2 - \varphi_t^2 + \Phi(\varphi)\}, \qquad (1.1)$$

to be the fundamental Lagrangian of a nonlinear scalar field, where the nonlinearity of the problem is given by the arbitrary function $\Phi(\varphi)$. We attempt to find a solution of the corresponding nonlinear equation

$$\varphi_{tt} - \varphi_{nn} + F(\varphi) = 0, \quad F(\varphi) = \frac{1}{2}\frac{d}{d\varphi}\Phi(\varphi) \qquad (1.2)$$

in the form

$$\varphi = \varphi(\sigma), \quad \sigma = k_\mu x_\mu, \quad (k_4 = i\omega, \ x_4 = it, \ c \equiv 1). \qquad (1.3)$$

We shall restrict our considerations to periodic $\varphi(\sigma)$, which are known to exist at least if $\Phi(\varphi)$ is a polynomial of no higher than fourth order.

We shall find the time-average of the energy and momentum densities

$$\overline{H} = \frac{1}{2T}\int_0^T \{(\nabla\varphi)^2 + \varphi_t^2 + \Phi(\varphi)\}\,dt,$$

$$\mathbf{G} = -\frac{1}{T}\int_0^T (\nabla\varphi)\,\varphi_t\,dt. \qquad (1.4)$$

After inserting (1.3) into (1.2) and (1.4), we obtain

$$\overline{H} = a(k^2 + K_0^2)/\omega, \qquad \mathbf{G} = a\mathbf{k}, \qquad (1.5)$$

$$(\omega^2 - k^2)\,\varphi_\sigma^2 + \Phi(\varphi) = h \equiv \text{const},$$

$$K_0^2 = h\omega/2a, \quad a = (\omega/T)\int_0^T \varphi_\sigma^2\,dt. \qquad (1.6)$$

Let us apply the general expressions obtained to certain specific cases

$$F(\varphi) = k_0^2\varphi, \qquad (1.7)$$

$$F(\varphi) = k_0^2\varphi + \alpha\varphi^2, \qquad (1.8)$$

$$F(\varphi) = k_0^2\varphi + \beta\varphi^3. \qquad (1.9)$$

For the linear case (1.7) we have

$$\varphi = \varphi_0\begin{pmatrix}\cos(\sigma - \sigma_0)\\ \sin(\sigma - \sigma_0)\end{pmatrix}, \quad \varphi_0 = \text{const}, \qquad (1.10)$$

where σ is given by (1.3), and $\omega^2 - k^2 = k_0^2$. This leads to

$$a = \varphi_0^2\omega/2, \quad h = k_0^2\varphi_0^2, \quad K_0 = k_0,$$

$$\overline{H} = a\omega, \quad \mathbf{G} = a\mathbf{k}. \qquad (1.11)$$

Similarly, in the nonlinear case (1.8) we obtain

$$\varphi = \varphi_0\begin{pmatrix}\mathrm{cn}^2(\sigma - \sigma_0)\\ \mathrm{sn}^2(\sigma - \sigma_0)\end{pmatrix} + \varphi_1; \quad \varphi_0, \varphi_1 \equiv \text{const}, \qquad (1.12)$$

where

$$\omega^2 - k^2 = \left(\frac{k_0}{2}\right)^2\left[\frac{\varepsilon}{3}\left(\frac{\alpha\varphi_0}{k_0^2}\right) + \sqrt{1 - \frac{1}{3}\left(\frac{\alpha\varphi_0}{k_0^2}\right)^2}\right],$$

$$\varphi_1 = -\frac{k_0^2}{2\alpha}\left[\left(1 + \frac{\alpha\varphi_0}{k_0^2}\right) - \sqrt{1 - \frac{1}{3}\left(\frac{\alpha\varphi_0}{k_0^2}\right)^2}\right],$$

$$k_1^2 = \varepsilon\frac{\alpha\varphi_0}{6(\omega^2 - k^2)}, \quad \varepsilon = \begin{pmatrix}+1\\ -1\end{pmatrix},$$

$$h = k_0^2\left[1 + \frac{2}{3}\left(\frac{\alpha\varphi_1}{k_0^2}\right)\right]\varphi_1^2,$$

$$a = l\frac{\omega\varphi_0^2}{2}, \quad K_0^2 = k_0^2\left[1 + \frac{2}{3}\left(\frac{\alpha\varphi_1}{k_0^2}\right)\right]\left(\frac{\varphi_1}{\varphi_0}\right)^2\frac{1}{l},$$

$$l = \frac{8}{15}\left\{(2k_1^2 - 1)\left(\frac{k_1'}{k_1}\right)^2 + \left[3k_1'^2 + \frac{(1 - 2k_1^2)^2}{k_1^2}\right]\left[\frac{E(k_1)}{K(k_1)} - k_1'\right]\frac{2}{k_1^2}\right\}.$$

In the nonlinear case (1.9) we have, analogously,

$$\varphi = \varphi_0\begin{pmatrix}\mathrm{cn}(\sigma - \sigma_0)\\ \mathrm{sn}(\sigma - \sigma_0)\end{pmatrix}, \ \varphi_0 = \text{const}, \qquad (1.13)$$

where

$$\omega^2 - k^2 = \begin{pmatrix}k_0^2 + \beta\varphi_0^2\\ k_0^2 + \tfrac{1}{2}\beta\varphi_0^2\end{pmatrix}, \quad k_1^2 = \varepsilon\frac{\beta\varphi_0^2}{2(\omega^2 - k^2)}.$$

Here the functions (1.5), (1.6) become

$$h = k_0^2(1+\beta\varphi_0^2/2k_0^2)\varphi_0^2,\quad a = l\omega\varphi_0^2/2(\omega^2-k^2),$$
$$l = \frac{2}{3}\left\{2-(1+k_1^2)\frac{1}{k_1^2}\left[1-\frac{E(k_1)}{K(k_1)}\right]\right\}, \tag{1.14}$$
$$K_0^2 = k_0^2[1+\beta\varphi_0^2/2k_0^2]\, l^{-1},$$

where $K(k_1)$, $E(k_1)$ are the complete elliptic integrals of the first and second kind, k_1 is the elliptic-function modulus*, and

$$k_1^2 + k_1'^2 = 1.$$

Let us now expand the expressions obtained in power series in the small parameters a and β, respectively, considering the nonlinearity to be weak. Then in case (1.8) the average energy of the two solutions is

$$\overline{H}_\alpha = {}^1/_2(\overline{H}_{\varepsilon=1}+\overline{H}_{\varepsilon=-1})_\alpha \tag{1.15}$$
$$= a\omega_0\left\{1-\frac{2}{9}\left[1+\frac{17}{15}\left(\frac{\omega_0}{k_0}\right)^2\right]\frac{\alpha}{\omega_0^2}\frac{a}{\omega_0}+\ldots\right\}$$

and when $k = 0$ (writing $\omega_0^2 = k^2 + k_0^2$), we have

$$\overline{H}_\alpha^0 = \frac{1}{2}(\overline{H}_{\varepsilon=1}^0+\overline{H}_{\varepsilon=-1}^0)_\alpha$$
$$= ak_0\left\{1-\frac{5}{12}\frac{67}{60}\frac{102}{100}\left(\frac{\alpha}{k_0^2}\right)^2\left(\frac{a}{k_0^2}\right)+\ldots\right\}.$$

Similarly, for case (1.9) we have

$$\overline{H}_\beta = {}^1/_2(\overline{H}_{\varepsilon=1}+\overline{H}_{\varepsilon=-1})_\beta = a\omega_0\left\{1+\frac{3}{8}\left(\frac{\beta a}{\omega_0^3}\right)\right.$$
$$\left.-\frac{3}{4}\left[1-\frac{23}{48}\left(\frac{\omega_0}{k_0}\right)^2\right]\left(\frac{\beta a}{\omega_0^3}\right)^2+\ldots\right\}, \tag{1.16}$$

and when $k = 0$,

$$\overline{H}_\beta^0 = {}^1/_2(\overline{H}_{\varepsilon=1}^0+\overline{H}_{\varepsilon=-1}^0)_\beta$$
$$= ak_0\left\{1+\frac{4}{5}\left[\frac{3}{8}\frac{5}{4}\left(\frac{\beta a}{k_0^3}\right)-0.49\left(\frac{\beta a}{k_0^3}\right)^2\right]+\ldots\right\}.$$

We may consider more general cases in which the nonlinearity is given by a sum of terms such as (1.8) and (1.9). Then in the first approximation, the correction to the energy will be given by the sum of the corrections from each nonlinear term, namely,

$$\overline{H}^0 = \overline{H}_\alpha^0+\overline{H}_\beta^0 = ak_0\left\{1+\frac{4}{5}\left[\frac{3}{8}\frac{5}{4}\left(\frac{\beta a}{k_0^3}\right)-0.49\left(\frac{\beta a}{k_0^3}\right)-\frac{5}{12}\frac{67}{60}\left(1.02\cdot\frac{5}{4}\right)\left(\frac{\alpha}{k_0^2}\right)\left(\frac{a}{k_0}\right)\right]+\ldots\right\}. \tag{1.17}$$

Let us now consider the quantum theory of the scalar nonlinear field, which is usually treated by perturbation theory in a way similar to an anharmonic oscillator. This makes it possible to find an approximate expression for the energy in the form of a power series in the parameter of nonlinearity. In particular, by supplementing known results[1,2], we obtain a correction term proportional to β^2 for the energy, namely

$$\begin{aligned} H_q = k_0\Big\{&\left(n+\frac{1}{2}\right)+\frac{3\beta}{8k_0^3}\left[n^2+n+\frac{1}{2}\right]-\frac{5}{12}\left(\frac{\alpha}{k_0^2}\right)^2\frac{1}{k_0}\left[n^2+n+\frac{11}{30}\right]\\ &-\left(\frac{\beta}{k_0^3}\right)^2 2^{-10}\,[2\,(\sqrt{n^2(n+1)(n+2)}+\sqrt{(n+1)^3(n+2)}\\ &+\sqrt{(n+2)^3(n+1)}+\sqrt{(n+3)^3(n+1)}\,)^2\\ &+(n+1)(n+2)(n+3)(n+4)]+\ldots\Big\}\end{aligned} \tag{1.18}$$

and the amplitude a is related to the quantum number n by $a = n + \frac{1}{2}$.

Thanks to this relation, we can compare H_q with $\overline{H}^0$. It is not difficult to see that they do not coincide for arbitrary a, although comparing (1.17) for $a = 1$ with the expression

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*It is interesting to note that the solution of the nonlinear equation in case (1.9) can also be written in a form similar to the well-known Euler expression

$$\varphi = \varphi_0\begin{pmatrix}e_n(\sigma-\sigma_0)\\ e_n^*(\sigma-\sigma_0)\end{pmatrix},\quad e_n(z) = \mathrm{cn}\, z + i\,\mathrm{sn}\, z$$

$$\omega^2-k^2 = k_0^2+\beta\varphi_0^2,\quad k_1^2 = 2\beta\varphi_0^2/(\omega^2-k^2),\quad \varphi_0 \equiv \mathrm{const}$$

As $\beta \to 0$, we obtain the ordinary Euler expression $e_n(z) \to e^{iz}$.

$$H_q = k_0\left\{1 + \frac{3}{8}\frac{5}{4}\frac{\beta}{k_0^3} - 0{,}51\left(\frac{\beta}{k_0^3}\right)^2 - \frac{5}{12}\frac{67}{60}\left(\frac{\alpha}{k_0^2}\right)\frac{1}{k_0^2} + \ldots\right\},$$

obtained from (1.18) by setting $a = 1$ (that is, $n = 1/2$), we get

$$(\overline{H}^0 - k_0) \approx {}^4/_5\,(H_q - k_0). \qquad (1.19)$$

It should be emphasized that the classical and quantum results coincide only for a single amplitude, and that other values of the amplitude cause sharp differences. This leads to the evident conclusion that the nonlinear equations correspond to treating a single particle. We note again[3] that if we consider N similar particles whose total energy is N times the energy of each particle, then to make the energy expression and the equation itself have the same form as that for a single particle, the parameter of nonlinearity must be replaced by

$$\lambda'_n = \lambda_n / N^{(n/2-1)}, \qquad (1.20)$$

where λ_n is the coefficient of φ^n in the polynomial $\Phi(\varphi)$.

2. SOLUTION OF THE NONLINEAR SPINOR EQUATION

The nonlinear generalization of the spinor equation previously proposed by Ivanenko, Heisenberg, and Mirianashvili[5,1,7] is of particular interest in view of the possibility of using it as the basis of a general particle theory. In this section we establish the relation between the nonlinear Dirac and Klein-Gordon equations and find several of their exact solutions in terms of elementary functions.

Let us consider the nonlinear Dirac equation

$$[(\gamma_\mu\,\partial/\partial x_\mu) + a(\overline{\psi},\psi)]\,\psi = 0, \qquad (2.1)$$

where $a(\overline{\psi}, \psi)$ is an arbitrary scalar function of $\overline{\psi}$, ψ, and γ_μ. Let us restrict ourselves to the case in which $a(\overline{\psi}, \psi)$ is independent of the γ_μ matrices. We then have

$$a(\overline{\psi}, \psi) = a(\rho), \quad \rho = \overline{\psi}\psi = \overline{\psi}^\alpha\psi^\alpha. \qquad (2.2)$$

Further, let us assume that the equation is separable in terms of the spin and space coordinates in the form

$$\psi(s, x_\mu) = \chi(s)\,\Phi(x_\mu), \quad \overline{\chi}(s)\,\chi(s) = 1, \qquad (2.3)$$

where $\chi(s)$ is a constant unit spinor, and $\Phi(x_\mu)$ is a real function of the coordinates.* Then (2.1) becomes

$$(\gamma_\mu\,\partial\Phi/\partial x_\mu + A(\Phi))\,\chi(s) = 0,$$
$$A(\Phi) = a(\rho)\,\Phi. \qquad (2.4)$$

Applying the operator[4] $\gamma_\nu\,\partial/\partial x_\nu$ to equation (2.4), we obtain the nonlinear Klein-Gordon equation for $\Phi(x_\mu)$ [3,5,6]

$$\partial^2\Phi/\partial x_\mu^2 - B(\Phi) = 0,$$
$$B(\Phi) = A(\Phi)\,dA(\Phi)/d\Phi. \qquad (2.5)$$

Since $\Phi(x_\mu)$ is a real function, it follows from (2.5) that $B(\Phi)$ must also be a real function of Φ. After determining $\Phi(x_\mu)$ from (2.5), the spin part $\chi(s)$ is determined from (2.4). This should be done by considering only such solutions of (2.5) which lead to $\chi(s)$ independent of x_μ.

Thus the set (2.1) of first-order equations can be considered a linearization of the second-order equations (2.5) in the sense that (2.1) has a solution of the type given by (2.3) [with condition (2.2)] whose spatial part satisfies (2.5).

If Eq. (2.5) is given, its linearization reduces to finding $A(\Phi)$ from the equation

$$A(\Phi) = \pm\sqrt{h + 2\int_0^\Phi B(\Phi)\,d\Phi}, \qquad (2.6)$$
$$h = \text{const}.$$

As we see, the transition from (2.4) to (2.5) is unique, although the inverse transition is not unique (in view of the ± in front of the radical and the arbitrary constant h in (2.6)). To be specific we can, for instance, choose the positive sign and set

$$h = 0 \ [i.e.,\ A(0) = 0]. \qquad (2.7)$$

We shall attempt to find a solution of (2.4) of the form†

*It is not particularly difficult to find complex solutions, and they will be published in the immediate future.

†Equation (2.1) has the following particular solutions: the function $\psi = c\exp(ik_\mu x_\mu)$, $c = \text{const}$, with $k_\mu^2 = a^2$, and the constant spinor $-\psi_0$, which is a solution of the algebraic equation $a(\overline{\psi}_0\psi) = 0$.

$$\Phi(x_\mu) = \Phi(\sigma), \quad \sigma = k_\mu x_\mu, \tag{2.8}$$

so that Eqs. (2.4) and (2.5) become

$$(\gamma_\mu k_\mu \, d\Phi / d\sigma + A(\Phi)) \chi(s) = 0, \tag{2.9}$$

$$(k_\mu^2 \, d^2\Phi / d\sigma^2 - B(\Phi)) = 0. \tag{2.10}$$

It follows from this that in order for $\chi(s)$ to be independent of x_μ, it is necessary that

$$A(\Phi) = \lambda d\Phi / d\sigma, \tag{2.10'}$$

where λ is a constant that causes the determinant of the coefficients of Eq. (2.4) to vanish.

On the other hand, the first integral of (2.10) is of the form

$$\sqrt{k_\mu^2} \frac{d\Phi}{d\sigma} = \left[h' + 2\int_0^\Phi B(\Phi)\, d\Phi\right]^{1/2}, \quad h \equiv \text{const.} \tag{2.11}$$

According to (2.6), putting $h' = h$ makes (2.11) coincide with (2.10′) if

$$\lambda = \pm M \quad M = \sqrt{k_\mu^2}. \tag{2.11'}$$

For the spin part we obtain

$$(\gamma_\mu k_\mu + M) \chi(s) = 0, \tag{2.12}$$

whose solution is known[5]. We have thus shown that when condition (2.2) is satisfied, Eq. (2.1) has a solution of the type given by (2.3), (2.8).

Let us consider some examples; since the spin part $\chi(s)$ is always given by Eq. (2.12), we shall consider only the spatial part $\Phi(x_\mu)$.

Let us consider the nonlinear equation[1]

$$(\gamma_\mu \partial / \partial x_\mu + a\bar{\psi}\psi)\psi = 0, \tag{2.13}$$

whose solution of type (2.3), (2.7) is given, in view of (2.5), by the solution of

$$(\partial^2 / \partial x_\mu^2 - b\Phi^4)\Phi = 0, \quad b = 3a^2; \tag{2.14}$$

when (2.7) is satisfied, we have

$$\Phi = (k_\mu x_\mu + c)^{-1/2}, \quad c = \text{const}, \qquad k_\mu^2 = (2a)^2 = M^2. \tag{2.15}$$

Similarly, the spatial part of the solution of[7]

$$(\gamma_\mu \partial / \partial x_\mu) + a_0 + a_1 \bar{\psi}\psi)\psi = 0, \tag{2.16}$$

is given by the solution of

$$(\partial^2 / \partial x_\mu^2 - b_0 - b_1\Phi^2 - b_2\Phi^4)\Phi = 0, \qquad b_0 = a_0^2, \; b_1 = 4a_0a_1, \; b_2 = 3a_1^2; \tag{2.17}$$

when condition (2.7) is satisfied, we obtain

$$\Phi = \sqrt{-a_0/2a_1}\exp[{}^1/_4(k_\mu x_\mu + c)]\{\sinh[{}^1/_2(k_\mu x_\mu + c)]\}^{1/2} \qquad k_\mu^2 = (2a_0)^2 = M^2, \; a_0 \neq 0, \; a_1 \neq 0. \tag{2.18}$$

Let us now consider the inverse problem. The equation

$$(\partial^2 / \partial x_\mu^2 - b_0 - b_1\Phi^2)\Phi = 0 \tag{2.19}$$

can be linearized. In particular, according to (2.6), we have

$$\left(\gamma_\mu \partial\Phi / \partial x_\mu + \sqrt{h + \left(b_0 + \tfrac{1}{2} b_1\Phi^2\right)\Phi^2}\right)\chi(s) = 0, \qquad \psi = \chi(s)\Phi(x_\mu), \quad \bar{\chi}(s)\chi(s) = 1. \tag{2.20}$$

The spatial part of the solution of (2.20) is given by the solution of (2.19)[8]

$$\Phi = \Phi_0 Z(\sigma), \quad \Phi_0 = \text{const}, \tag{2.21}$$

where $Z(\sigma)$ is an elliptic function.

In particular, when condition (2.7) is satisfied, Eq. (2.21) leads to

$$\Phi = \sqrt{-b_0/2b_1}[\cos(k_\mu x_\mu + c)]^{-1}, \qquad k_\mu^2 = b_0^2 = M^2. \tag{2.22}$$

When $b_1 = 0$, we obtain the linear equation

$$(\partial^2 / \partial x_\mu^2 - b_0)\Phi = 0 \tag{2.23}$$

and the analogue, in the sense of the above linearization, of the Dirac equation

$$(\gamma_\mu \partial\Phi / \partial x_\mu + \sqrt{h + b_0\Phi^2})\chi(s) = 0. \tag{2.24}$$

The spatial part of (2.24) is

$$\Phi = c_1 \cos\sigma + c_2 \sin\sigma. \tag{2.25}$$

Here $h = -b_0(c_1^2 + c_2^2)$. In particular, when condition (2.7) is satisfied, we have

$$c_2 = \pm ic_1,\ \Phi = ce^{i\sigma} + c^*e^{i\sigma} \qquad (2.26)$$

which gives the solution to the ordinary Dirac equation.

It is also of interest to investigate the nonlinear generalization of the Duffin-Kemmer equation

$$(\beta_\mu \partial / \partial x_\mu - c(\overline{\varphi}, \varphi))\varphi = 0, \qquad (2.27)$$

where the β_μ are Kemmer-Duffin matrices, and $c(\overline{\varphi}, \varphi)$ is an arbitrary scalar function.

I consider it my duty to express my deep gratitude to Professor D. D. Ivanenko for constant attention to the work and to Professor Kh. Ia. Khristov for valuable comments.

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Statistical Theory of Systems of Charged Particles With Account of Short Range Forces of Repulsion

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The free energy of an electrically neutral system of charged particles (ions) has been found by taking into account the repulsive forces between them. The general expression obtained for the free energy of such systems is applied to its calculation for a concrete form of "long-range" and "short-range" forces.

THE EQUILIBRIUM STATE of each statistical system of N particles is entirely determined by a knowledge of the Gibbs distribution function D or the equivalent aggregate distribution function $F_s(x_1, x_2, \ldots, x_s)$ $(s = 1, 2, \ldots)$[1]. For example, the pair distribution function $F_s(x_1, x_2)$ permits us to find the thermodynamic potential of the system, knowing which we can solve any problem pertaining to the state of thermodynamic equilibrium.

Ways of finding the correlation functions of systems of particles both with Coulomb (slowly decreasing with distance) potential of interaction $\Phi^0(r)$, and with molecular (rapidly decreasing with distance) interaction $\Phi^1(r)$, were first developed by Bogoliubov[1]. However, the problem of the construction of an expansion by which we could find the correlation functions in the case of a system with an interaction containing both Coulomb and short-range forces, remained unsolved.

Making use of an equation with variational derivatives, we[2] succeeded in outlining a method of finding the correlation functions both for systems with interaction $\Phi^0(r)$ or $\Phi^1(r)$, and for the "additive" interaction $\Phi(r) = \Phi^0(r) + \Phi^1(r)$ which contains, in the terminology of Vlasov[3], both "short-range" and "long-range" forces. In this paper, following Ref. 2, we define as the thermodynamic potential the free energy $F(\Phi)$ of a system of charged particles with explicit account of the short-range repulsive forces between them, which enables us to de-