$\rho^t(\mathbf{p}_c;\, q_c\nu_c q_d\nu_d)$ in just the same way that $\Delta\sigma(\mathbf{n}_c)$ is related to $\rho'(\mathbf{n}_c,\, p_c;\, 0,0,0,0)$. They would give the mean values of certain spin operators in an ensemble of particles with momenta in the range $(\mathbf{p}_c,\, \mathbf{p}_c + \Delta\mathbf{p})$. But the mean values so obtained would depend not only on the nature of the spin state, but also on the number of particles in the ensemble. Therefore the quantity used to characterize just the spin state is the mean value of the operator $\hat{A}^{q\nu}$ (for particle c, for example), calculated for one particle:

$$\overline{A^{q\nu}}(\mathbf{p}_c) = (i_c \| \hat{A}^q \| i_c)\, \rho_{\Pi T}(\mathbf{n}_c,\, p_c';\, q,\, \nu,\, 0,\, 0) / \Delta\sigma(\mathbf{n}_c) = (i_c \| \hat{A}^q \| i_c)\, \rho(\mathbf{n}_c,\, p_c;\, q\nu 00) / \rho(\mathbf{n}_c,\, p_c;\, 0,0,0,0). \quad \text{(II.5)}$$

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220

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Electric Monopole Transitions of Atomic Nuclei

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Processes arising in $E0$ transitions in nuclei are considered. Equations are derived for the probabilities of shell electron conversion, pair production, two-photon transitions, and electron scattering cross section involving excitation of the $E0$ nuclear transition. The calculations are carried out with Coulomb functions of the electron with allowance for the finite dimensions of the nucleus. The conversion probability in an $E0$ nuclear transition is compared with the competing $E2$ and $M1$ nuclear transitions. Some estimates are given for the $E0$ nuclear transition matrix element for various single-particle and collective nuclear models.

SINGLE-PHOTON NUCLEAR TRANSITIONS between states of zero spin are forbidden by the law of conservation of angular momentum. In this case the radiation transition occurs by emission of two (or more) quanta. A distinction is made between two types of transitions: the $M0$-transition, with a change in parity $(0^{\pm} \to 0^{\mp})$, and the $E0$ transition, in which parity remains unchanged $(0^{\pm} \to 0^{\pm})$. In the second case, shell electron conversion or pair production is possible in addition to the two-photon transition. The $E0$ conversion differs substantially from other multipole conversion processes in that the monopole potential is localized in the region of the nucleus, while in other conversion processes the region of the nucleus makes a relatively small contribution. Since the Coulomb potential on the surface of the nucleus is approximately mc^2 even at low values of Z, the motion of the electron in the vicinity of the nucleus is essentially relativistic. The calculations must thus be made with relativistic electron functions. In addition, the field in the vicinity of the nucleus differs substantially from the field of a point charge, leading to a considerable local variation in the electron wave function compared with the functions in the field of a point charge. The calculations of the monopole conversion probability must therefore contain allowances for the finite dimensions of the nucleus (in the case of a point nucleus, the $E0$ conversion transition is strictly forbidden). $E0$ nuclear transitions with electron conversion and pair production were studied in many investigations[1-7]. However, Fowler's calculation[1] was made in a non-relativistic approximation, while the remaining works[2-7] are based on the Born approximation or fail to allow for the finite dimensions of the nucleus. It is interesing to find out how substantially the effect of the nuclear dimensions influence the results obtained. We shall give below the calculated values of the probabilities of many processes connected with the $E0$ nuclear transition. The nucleus is considered as a sphere of radius $R_0 = r_0 A^{1/3} = 1.2 \times 10^{-13} A^{1/3}$ cm, with a uniform volume charge density. The shielding effect of the atomic electrons is neglected. In the processes considered here the electrons do not have too high an energy, so that $kR_0 \ll 1$ (k is the electron wave vector), thus restricting the applicability of the results to electron energies $\varepsilon \lesssim 15$ Mev for heavy nuclei. Retardation is neglected. The calculations are in relativistic units. The wave functions of the electron in the continuous energy spectrum are normalized to a unity energy interval. The formalism of the spherical spinors, developed by Berestetskii *et al*[8], will henceforth be used everywhere. The initial state of the system is designated by the index 1, the final state by 2.

1. WAVE FUNCTION OF AN ELECTRON IN THE FIELD OF A FINITE NUCLEUS

Our calculations require wave functions of an electron in the field of a finite nucleus. These functions can be readily obtained by joining on the surface of the nucleus the regular solution of the Dirac equation for the region $r < R_0$ for each partial wave (j, λ, $l = j + \lambda$), namely

$$\psi_{j\lambda\mu} = A_{j\lambda}\begin{cases} i\hat{Y}^{-\lambda}_{j\mu}F_{j\lambda}(r) \\ \hat{Y}^{\lambda}_{j\mu}G_{j\lambda}(r)\end{cases}$$

with the general solution to the equation in the region $r \geqslant R_0$, which is a superposition of the regular $\psi_{j\mu\lambda}(\gamma)$ and irregular $\psi_{j\mu\lambda}(\gamma)$ solutions of the Dirac equation for the field of a point charge (see Refs. 9–11). The constant $A_{j\lambda}$ is determined from the boundary conditions. The first to use this method to calculate the effect of the finite dimensions was Sliv[12]. Let us evaluate the constant $A_{j\lambda}$ with an accuracy to approximately $(kR_0)^2$:

$$A_{j\lambda} = \left\{\frac{g_{j\lambda}(\gamma)f_{j\lambda}(-\gamma) - f_{j\lambda}(\gamma)g_{j\lambda}(-\gamma)}{G_{j\lambda}f_{j\lambda}(-\gamma) - F_{j\lambda}g_{j\lambda}(-\gamma)}\right\}_{r=R_0},$$

$$F_{j\lambda} = r^{j-\lambda}\Phi_1^{(2\lambda)}(x);\quad G_{j\lambda} = r^{j+\lambda}\Phi_2^{(2\lambda)}(x);$$

$$x = r/R_0;\quad \Phi_1^{(2\lambda)} = \sum_{\nu=0}^{\infty} a_\nu^{(2\lambda)}x^{2\nu};$$

$$\Phi_2^{(2\lambda)} = \sum_{\nu=0}^{\infty} b_\nu^{(2\lambda)}x^{2\nu}.$$

The series Φ_1 and Φ_2 converge rapidly and if an approximate accuracy of 5% is specified, one need merely take the following terms (at $x = 1$):

$\lambda = +1/2$:

$$\Phi_1^{(+)} = 1 - \omega^2/6 + Ze^2\omega/15 + \omega^4/120,$$

$$R_0\Phi_2^{(+)} = \omega/3 - Ze^2/10 - \omega^3/30;$$

$\lambda = -1/2$:

$$R_0\Phi_1^{(-)} = -\omega/3 + Ze^2/10 + \omega^3/30,$$

$$\Phi_2^{(-)} = 1 - \omega^2/6 + Ze^2\omega/15 + \omega^4/120.$$

Here $\omega = 1.5\, Ze^2 + \varepsilon R_0$, where ε is the total electron energy.

For further calculations it is convenient to normalize the functions $\psi_{j\lambda}(\gamma)$ so that the normalization coincides with that required for the analogous process in the point-charge representation. The functions $f_{j\lambda}(\pm\gamma, r)$ and $g_{j\lambda}(\pm\gamma, r)$ for the bound and free state of the electrons have been derived by many workers (see, for example, Refs. 9–11).

The constants $A_{j\lambda}$ for the continuous and discrete spectrum are readily obtained by substituting the functions.

1) *Electron in bound state;* shell n, j, λ;

$$A_{j\lambda n} = N_{jn} R_0^{\gamma - j - 1/2} S_{j\lambda}(R_0);$$

here at $\lambda = -1/2$

$$S_{j\lambda} \approx \frac{(1-\varepsilon^2)^{1/2}\,[n'(n'+2\gamma)-(N-\varkappa)^2]}{(1-\varepsilon)^{1/2}\,[n'+2\gamma+N-\varkappa]\,\Phi_2^{(-)} + (1+\varepsilon)^{1/2} R_0 \Phi_1^{(-)}\,[N-2\gamma-n'-\varkappa]},$$

at $\lambda = +1/2$

$$S_{j\lambda} \approx \frac{(1-\varepsilon^2)^{1/2}\,[n'(n'+2\gamma)-(N-\varkappa)^2]}{(1-\varepsilon)^{1/2}\,[n'+2\gamma+N-\varkappa]\,R_0\Phi_2^{(+)} + (1+\varepsilon)^{1/2} \Phi_1^{(+)}\,[N-2\gamma-n'-\varkappa]},$$

$$N_{jn} = \left(\frac{\Gamma(2\gamma+1+n')}{n'!\,N(N-\varkappa)}\right)^{1/2} \frac{1}{\Gamma(2\gamma+1)} \left(\frac{2Ze^2}{N}\right)^{\gamma+1/2},$$

$$N = \sqrt{n^2 - 2n'(|\varkappa| - \gamma)}, \quad \varkappa = 2\lambda(j+1/2), \quad \gamma = \sqrt{\varkappa^2 - Z^2e^4}.$$

2) *Electron in continuous spectrum of energy* $\varepsilon > 0$:

$$A_{j\lambda} = C_j R_0^{\gamma - j - 1/2} S_{j\lambda}(\varepsilon R_0);$$

at $\lambda = -1/2$

$$S_{j\lambda}(\varepsilon R_0) = \left\{\frac{(\varepsilon^2-1)^{1/2}\,[\operatorname{Re} Q_{j\lambda}(\gamma)\operatorname{Im} Q_{j\lambda}(-\gamma) - \operatorname{Re} Q_{j\lambda}(-\gamma)\operatorname{Im} Q_{j\lambda}(\gamma)]}{(\varepsilon-1)^{1/2}\Phi_2^{(-)}\operatorname{Im} Q_{j\lambda}(-\gamma) + (\varepsilon+1)^{1/2} R_0 \Phi_1^{(-)} \operatorname{Re} Q_{j\lambda}(-\gamma)}\right\}_{r=R_0},$$

at $\lambda = +1/2$

$$S_{j\lambda}(\varepsilon R_0) = \left\{\frac{(\varepsilon^2-1)^{1/2}\,[\operatorname{Re} Q_{j\lambda}(\gamma)\operatorname{Im} Q_{j\lambda}(-\gamma) - \operatorname{Re} Q_{j\lambda}(-\gamma)\operatorname{Im} Q_{j\lambda}(\gamma)]}{(\varepsilon+1)^{1/2}\Phi_1^{(+)}\operatorname{Re} Q_{j\lambda}(-\gamma) + (\varepsilon-1)^{1/2} R_0 \Phi_2^{(+)} \operatorname{Im} Q_{j\lambda}(-\gamma)}\right\}_{r=R_0},$$

$$C_j = \frac{(2p)^{\gamma}\,|\Gamma(\gamma + iZe^2\varepsilon/p)|\exp(\pi Z e^2 \varepsilon / 2p)}{\sqrt{\pi p}\,\Gamma(2\gamma+1)}, \quad e^{2i\eta} = -\frac{\varkappa - iZe^2/p}{\hat{\gamma} + iZe^2\varepsilon/p},$$

$$Q_{j\lambda} = e^{i\eta}(\hat{\gamma} + iZe^2\varepsilon/p)\,{}_1F_1(\hat{\gamma}+1+iZe^2\varepsilon/p;\ 2\hat{\gamma}+1;\ 2ipR_0);\ \hat{\gamma} = \pm\gamma.$$

3) *Electron at negative level of continous spectrum* $(-\varepsilon)$:

$$A_{j\lambda} = B_j R_0^{\gamma - j - 1/2} S_{j\lambda}(\varepsilon R_0);$$

at $\lambda = -1/2$

$$S_{j\lambda}(\varepsilon R_0) = \left\{\frac{(\varepsilon^2-1)^{1/2}[\operatorname{Re} P_{j\lambda}(\gamma)\operatorname{Im} P_{j\lambda}(-\gamma) - \operatorname{Im} P_{j\lambda}(\gamma)\operatorname{Re} P_{j\lambda}(-\gamma)]}{(\varepsilon+1)^{1/2}\Phi_2^{(-)}\operatorname{Re} P_{j\lambda}(-\gamma) + (\varepsilon-1)^{1/2} R_0 \Phi_1^{(-)}\operatorname{Im} P_{j\lambda}(-\gamma)}\right\}_{r=R_0},$$

at $\lambda = +\frac{1}{2}$

$$S_{j\lambda}(\varepsilon R_0) = \left\{\frac{(\varepsilon^2-1)^{1/2}[\operatorname{Re} P_{j\lambda}(\gamma)\operatorname{Im} P_{j\lambda}(-\gamma) - \operatorname{Im} P_{j\lambda}(\gamma)\operatorname{Re} P_{j\lambda}(-\gamma)]}{(\varepsilon-1)^{1/2}\Phi_1^{(+)}\operatorname{Im} P_{j\lambda}(-\gamma) + (\varepsilon+1)^{1/2}R_0\Phi_2^{(+)}\operatorname{Re} P_{j\lambda}(-\gamma)}\right\}_{r=R_0},$$

$$B_j = \frac{(2p)^{\gamma}\,|\,\Gamma(\gamma - iZe^2\varepsilon/p)\,|\exp(-\pi Ze^2\varepsilon/2p)}{\sqrt{\pi p}\,\Gamma(2\gamma+1)}, \quad e^{2i\eta} = \frac{\varkappa - iZe^2/p}{\hat{\gamma} - iZe^2\varepsilon/p},$$

$$P_{j\lambda}(\hat{\gamma}) = e^{i\eta}(\hat{\gamma} - iZe^2\varepsilon/p)\,{}_1F_1(\hat{\gamma}+1 - iZe^2\varepsilon/p;\ 2\hat{\gamma}+1;\ 2ipR_0).$$

2. MATRIX ELEMENT OF ELECTRIC MONOPOLE

If retardation is neglected, the matrix element of the monopole assumes the form:

$$\langle 2\,|\,\hat{H}_{E0}\,|\,1\rangle = e^2\sum_{i=1}^{Z}\int \psi_2^+ u_2^+(\xi_i)\frac{1}{|\mathbf{r}-\xi_i|}\psi_1 u_1(\xi_i)\,(d\xi_i)\,(d\mathbf{r}),$$

where ψ_2 and ψ_1 are the electron wave functions, while u_2 and u_1 are the functions of the nucleus in the final and initial states respectively. Simple calculations yield for the matrix element of a transition of the type $(1, j\lambda) \to (2, j\lambda)$:

$$\langle 2\,|\,\hat{H}_{E0}\,|\,1\rangle = e^2 A_{j\lambda}(1)\,A^*_{j\lambda}(2)\,R_{j\lambda}R_0^{2j+1},$$

where $R_{j\lambda}$ is the nuclear matrix element. With an accuracy to quantities on the order of 0.03 and less, $R_{j\lambda}$ equals:

$$R_{j\lambda} \approx R_j = [(2j+1)(2j+2)]^{-1}\left\langle u_2^+\left|\sum_{i=1}^{Z}\left(\frac{\xi_i}{R_0}\right)^{2j+1}\right|u_1\right\rangle + \frac{(j+1)^{-1}\omega_1\omega_2 - \omega_1^2 - \omega_2^2}{4(j+1)(2j+3)(2j+4)} \times \left\langle u_2^+\left|\sum_{i=1}^{Z}\left(\frac{\xi_i}{R_0}\right)^{2j+3}\right|u_1\right\rangle + \ldots$$

The second term in R_j amounts to 0.1 of the first term even at $Z = 90$ and $\varepsilon \approx 30$. Hereinafter we shall denote

$$\left\langle u_2^+\left|\sum_{i=1}^{Z}\left(\frac{\xi_i}{R_0}\right)^{2j+1}\right|u_1\right\rangle = \rho_j.$$

3. PROBABILITIES OF $E0$ TRANSITIONS

Conversion of shell electron from state (n, j, λ).

$$W = \frac{2\pi e^4}{[(2j+1)(2j+2)]^2}C_j^2 N_{jn}^2\,|\,S_{j\lambda}(1)\,S_{j\lambda}(2)\,|^2 R_0^{4\gamma}\rho_j^2.$$

In the case of the K, LI, and LII shells, the nuclear matrix elements are the same and the ratios of the corresponding values of W are independent of the properties of the nucleus:

$$\frac{W_K}{W_{LI}} = \frac{[2(\gamma+1)]^{\gamma+1}[(2+2\gamma)^{1/2}+1]}{2(2\gamma+1)}\left(\frac{P_K}{P_{LI}}\right)^{2\gamma-1} \times \exp\left(\pi Ze^2\left[\frac{\varepsilon_K}{P_K} - \frac{\varepsilon_{LI}}{P_{LI}}\right]\right)\left|\frac{S_K(1)\,S_K(2)}{S_{LI}(1)\,S_{LI}(2)}\right|^2,$$

$$\frac{W_{LI}}{W_{LII}} = \frac{\sqrt{2+2\gamma}-1}{\sqrt{2+2\gamma}+1}\left|\frac{S_{LI}(1)\,S_{LI}(2)}{S_{LII}(1)\,S_{LII}(2)}\right|^2.$$

The conversion of the $LIII$ shell electron is determined by the quantity $(Ze^2/N)^{2\gamma+1}p^{2\gamma-1}R_0^{4\gamma}$ and for $\gamma \sim 4$ it is approximately $10^8 - 10^{10}$ times smaller than the probability of the K-electron conversion. Numerical estimates of the probability of the L and K-electron conversion are given in the end of the article.

Paired Conversion of Monopole

The probability of the $E0$ conversion with pair production can be obtained by calculating the matrix element of an $E0$ transition with electron functions

$$\psi_{-\mathbf{p}_1\nu_1}(\mathbf{r}) = \sum_{j\mu\lambda} e^{i\pi(j-\lambda-1)/2} \times [\hat{Y}^{\lambda}_{j\mu}(-\mathbf{p}_1)]^*_{-\nu_1}\psi_{j\mu\lambda}(\mathbf{r}, -\varepsilon_1)\,e^{-i\delta_{j\lambda}(1)},$$

$$\psi_{\mathbf{p}_2\nu_2}(\mathbf{r}) = \sum_{j\mu\lambda} e^{i\pi(j-\lambda-1)/2} \times [\hat{Y}^{\lambda}_{j\mu}(\mathbf{p}_2)]^*_{-\nu_2}\psi_{j\mu\lambda}(\mathbf{r}, \varepsilon_2)\,e^{-i\delta_{j\lambda}(2)},$$

in the asymptotic approximation $\psi_{p_2\nu_2}$ and $\psi_{p_1\nu_1}$ contain plane and converging spherical waves (normalization of $\psi_{j\mu\lambda}$ by $\delta(\varepsilon - \varepsilon')\delta_{\mu\mu'}\delta_{jj'}\delta_{\lambda\lambda'}$). Considering that the contribution of waves with $j > 1/2$ is negligibly small, and taking only $j = 1/2$ into account, we obtain for the pair-production probability and for the angular correlation of the positron with the electron:

$$W_{\text{pair}} = \frac{2^{4\gamma-1}e^4R_0^{4\gamma}}{36\pi\,|\Gamma(2\gamma+1)|^4}\left(\int\sum_\lambda |S_\lambda(1)\,S_\lambda(2)|^2\,F(Z\Delta\varepsilon_1\varepsilon_2)\,\delta(\Delta-\varepsilon_1-\varepsilon_2)\,d\varepsilon_1\,d\varepsilon_2\right)\rho_j^2,$$

$$F(Z\Delta\varepsilon_1\varepsilon_2) = (p_1p_2)^{2\gamma-1}\exp\left(\pi Ze^2\left[\frac{\varepsilon_2}{p_2}-\frac{\varepsilon_1}{p_1}\right]\right)\left|\Gamma\left(\gamma+iZe^2\frac{\varepsilon_2}{p_2}\right)\Gamma\left(\gamma+iZe^2\frac{\varepsilon_1}{p_1}\right)\right|^2.$$

For Ca^{40}_{20}, Zr^{90}_{40}, and Po^{214}_{84} the integral was evaluated numerically to be 49, 1.2, and 0.017 respectively. The function of the angular correlation $N(\theta)$ of the positron with the electron at fixed energy $\varepsilon_1(\varepsilon_2)$ is determined by the square of the modulus of the elements:

$$N(\theta) \sim \sum_{\nu_1\nu_2}\left|\sum_{\mu\lambda}\exp\{i(\delta_{j\lambda}(2)-\delta_{j\lambda}(1))\}\,[\hat{Y}^\lambda_{j\mu}(\mathbf{p}_2)]_{-\nu_2}[\hat{Y}^\lambda_{j'\mu}(-\mathbf{p}_1)]^*_{-\nu_1}A_{j\lambda}(1)A^*_{j\lambda}(2)\right|^2.$$

From which we find: $(j = 1/2)$ $(\lambda = \pm 1/2)$

$$N(\theta) = \frac{1}{2}\sum_\lambda |S_\lambda(1)\,S_\lambda(2)|^2 - [S_+(1)\,S_-(1)\,S_+(2)\,S_-(2)]\cos\varphi_1\cos\theta_{\mathbf{p}_1\mathbf{p}_2},$$

where the phase φ_1 is determined with accuracy to terms in $(pR_0)^2$ by the following equation

$$e^{i\varphi_1} = \left[\frac{(1-iZe^2/p_1)}{(1+iZe^2/p_1)}\frac{(1+iZe^2/p_2)}{(1-iZe^2/p_2)}\right]^{1/2}.$$

As $Z \to 0$, the equations for the conversion probability for the angular correlation become the known equations obtained in the Born approximations[9].

Inelastic Scattering of Electrons

Unlike the preceding case, the matrix element is calculated with the following electron functions

$$\psi_{\mathbf{p}_1\nu_1}(\mathbf{r}) = \sum_{j\mu\lambda} e^{i\pi(j-\lambda-1)/2}[\hat{Y}^\lambda_{j\mu}(\mathbf{p}_1)]^*_{-\nu_1}\psi_{j\mu\lambda}(\mathbf{r}\varepsilon_1)\,e^{i\delta_{j\lambda}(1)},$$

$$\psi_{\mathbf{p}_2\nu_2}(\mathbf{r}) = \sum_{j\mu\lambda} e^{i\pi(j-\lambda-1)/2}[\hat{Y}^\lambda_{j\mu}(\mathbf{p}_2)]^*_{-\nu_2}\psi_{j\mu\lambda}(\mathbf{r}\varepsilon_2)\,e^{-i\delta_{j\lambda}(2)},$$

and $\psi_{p_1\nu_1}$ contains a plane wave and divergent waves in its asymptotic form. Restricting ourselves again to terms with $j = 1/2$, we obtain for the scattering cross section and for the angular distribution $(j = 1/2;\ \lambda = \pm 1/2)$:

$$\sigma = \frac{\pi 2^{4\gamma}e^4R_0^{4\gamma}}{9|\Gamma(2\gamma+1)|^4}\,p_1^{2\gamma-3}p_2^{2\gamma-1}\,|\Gamma(\gamma+iZe^2\varepsilon_1/p_1)\,\Gamma(\gamma+iZe^2\varepsilon_2/p_2)|^2$$
$$\times\exp\left(\pi Ze^2\left[\frac{\varepsilon_1}{p_1}+\frac{\varepsilon_2}{p_2}\right]\right)\sum_\lambda|S_\lambda(1)\,S_\lambda(2)|^2\rho_j^2,$$

$$N(\theta) = \frac{1}{2}\sum_\lambda|S_\lambda(1)\,S_\lambda(2)|^2 + [S_+(1)\,S_-(1)\,S_+(2)\,S_-(2)]\cos\varphi_2\cos\theta_{\mathbf{p}_1\mathbf{p}_2},$$

where φ_2 is found with an accuracy to terms in $(pR_0)^2$ from

$$e^{i\varphi_2} \approx \left[\frac{(1-iZe^2/p_1)}{(1+iZe^2/p_1)}\frac{(1-iZe^2/p_2)}{(1+iZe^2/p_2)}\right]^{1/2}.$$

By way of illustration let us show the calculations of the cross sections for several elements at electron energies 10 and 20 mc^2. We shall represent the cross section as

$$\sigma = (\sigma_0/4\pi)\,(1 + b\cos\theta)\,\rho_j^2\,10^{-30}\ (\text{cm}^2/\text{sterad}).$$

The values of b and σ_0 are given in Table 1.

TABLE I.

Nucleus	Energy 10 mc^2		Energy 20 mc^2	
	σ_0	b	σ_0	b*
Ca_{20}^{40}	0.054	0.93	0.34	1
Ge_{32}^{72}	1.08	0.97	3.2	1
Zr_{40}^{90}	0.98	0.97	3.7	1
Pd_{46}^{106}	2.52	0.99	7.4	1
Po_{84}^{214}	67	0.98	190	1

*At an energy $E = 20$ it is important that the phase shift due to the finite dimensions of the nucleus be taken into account. This was not done in this estimate.

4. EFFECT OF FINITE NUCLEAR DIMENSIONS

It is interesting to determine to what extent results of calculations with the Coulomb functions of the point nucleus are charged by accounting for the finite dimensions of the nucleus. We obtain for the ratio of the probabilities, calculated for the point nucleus and finite nucleus:

$$\varkappa\left(\frac{\text{point}}{\text{finite}}\right) = \left[\frac{3}{\gamma(2\gamma+1)}\right]^2 F \left| \langle 2 \left| \sum_{i=1}^{Z} \left(\frac{\xi_i}{R_0}\right)^{2\gamma} \right| 1 \rangle \middle/ \langle 2 \left| \sum_{i=1}^{Z} \left(\frac{\xi}{R_0}\right)^2 \right| 1 \rangle \right|^2;$$

For the conversion, scattering, and pair production processes, F assumes the following form: for the $(nj\lambda)$ shell electron conversion

$$F_\lambda \approx \frac{[(\varepsilon_1+1)^{1/2}(\varepsilon_2+1)^{1/2}(n'+\varkappa-N)\operatorname{Re} Q_{j\lambda}(\gamma 2)+(\varepsilon_2-1)^{1/2}(1-\varepsilon_1)^{1/2}(n'+N-\varkappa)\operatorname{Im} Q_{j\lambda}(\gamma 2)]^2}{[S_\lambda(1)\,S_\lambda(2)]^2},$$

for electron scattering ($\lambda = \pm \frac{1}{2}$)

$$F = \frac{\sum_\lambda [(\varepsilon_1+1)^{1/2}(\varepsilon_2+1)^{1/2}\operatorname{Re} Q_{j\lambda}(\gamma 1)\operatorname{Re} Q_{j\lambda}(\gamma 2)+(\varepsilon_1-1)^{1/2}(\varepsilon_2-1)^{1/2}\operatorname{Im} Q_{j\lambda}(\gamma 1)\operatorname{Im} Q_{j\lambda}(\gamma 2)]^2}{\sum_\lambda [S_\lambda(1)\,S_\lambda(2)]^2},$$

for pair production

$$F = \frac{\sum_\lambda |(\varepsilon_1+1)^{1/2}(\varepsilon_2-1)^{1/2}\operatorname{Re} P_{j\lambda}(\gamma 1)\operatorname{Im} Q_{j\lambda}(\gamma 2)+(\varepsilon_1-1)^{1/2}(\varepsilon_2+1)^{1/2}\operatorname{Im} P_{j\lambda}(\gamma 1)\operatorname{Re} Q_{j\lambda}(\gamma 2)|^2}{\sum_\lambda [S_\lambda(1)\,S_\lambda(2)]^2}$$

The corrections for the finite size of the nucleus, calculated for the case of conversion, are given at the end of the article. The values of $\varkappa$ for scattering and for $E = 10\ mc^2$ are:

Nucleus:	Ca_{20}^{40}	Ge_{32}^{72}	Zr_{40}^{90}	Pd_{46}^{106}	Po_{84}^{214}
$\varkappa$:	0,98	0,97	1,0	1,16	1,87

5. TWO-QUANTUM NUCLEAR TRANSITION

In the case of a 0–0 nuclear transition, radiation transition is possible only by emission of two or more photons simultaneously. The photon spectrum now becomes continuous. Similar two-quantum nuclear transitions were investigated earlier by Sachs[13], Schwinger[5], and Goldberger[14]. It was assumed in all these investigations that the nucleus emits quanta upon transition through one virtual state, the energy of which was chosen arbitrarily in the estimate. We shall give below a somewhat different estimate of the probability of a two-photon nucleus, taking into account a group of virtual states of the nucleus.

In recent studies of photo-nuclear processes, a gigantic maximum of the "resonance" type was observed in the cross sections of the (γn) and (γp) reactions at energies $\hbar\omega_{res} \approx 60\ A^{-1/3}$ Mev. The presence of such a nuclear "resonance" can be considered as a result of a sharp increase in the density of the "dipole" levels of the nucleus in the region of the "resonance" energy (by "dipole" levels we understand here levels reached by a nucleus, originally in the ground state, by absorption of a dipole quantum). It is natural to propose that in a nuclear transition with emission of two dipole quanta the greatest contribution is due to the virtual transitions at the levels near the "dipole resonance" of the nucleus.

Within the framework of these assumptions, we obtain for $W_{\gamma\gamma}$ and $dW_{\gamma\gamma}$:

$$dW_{\gamma\gamma} = 2\pi \sum_{\lambda_1\lambda_2} |\langle 2 | \hat{H}_{\gamma\gamma} | 1\rangle|^2 \rho_{\gamma_1}\rho_{\gamma_2}\delta(\Delta - k_1 - k_2)\, dk_1 dk_2;$$

$$|\langle 2|\hat{H}_{\gamma\gamma}|1\rangle|^2 = \left|\sum_{\nu} \frac{\langle 2|\hat{H}_{\gamma}|\nu\rangle\langle\nu|\hat{H}_{\gamma}|1\rangle}{\Delta - k_1 - \varepsilon_{\nu}}\right|^2 \approx$$

$$\approx \frac{1}{(\varepsilon_{res} + k_1 - \Delta)^2}\left|\sum_{\nu}\langle 2|\hat{H}_{\gamma}|\nu\rangle\langle\nu|\hat{H}_{\gamma}|1\rangle\right|^2;$$

$\hat{H}_{\gamma}$ is the operator of interaction with the dipole quantum

$$\hat{H}_{\gamma} = \sum_i e\xi_i \mathbf{E}_{\lambda}; \quad \mathbf{E}_{\lambda} = ik\sqrt{2\pi/k}\,\boldsymbol{\varepsilon}_{\lambda}\exp\{i\mathbf{k}\boldsymbol{\xi} - i\omega t\},$$

$\boldsymbol{\varepsilon}_{\lambda}$ is the quantum polarization vector, and k is the wave number of the quantum;

$$dW_{\gamma\gamma} = \frac{8e^4 k_1^3 k_2^3 R_0^4}{27\pi(\varepsilon_{res} + k_1 - \Delta)^2}\left|\langle 2\right|\sum_{i,j}^{Z}(\xi_i\xi_j/R_0^2)\left|1\rangle\right|^2\delta(\Delta - k_1 - k_2)\,dk_1 dk_2,$$

$$W_{\gamma\gamma} = \frac{8e^4 R_0^4}{27\pi}\left(\int\frac{k_1^3k_2^3\delta(\Delta - k_1 - k_2)}{(\varepsilon_{res} + k_1 - \Delta)^2}dk_1dk_2\right)\left|\langle 2\right|\sum_{i,j}(\xi_i\xi_j/R_0^2)\left|1\rangle\right|^2.$$

The integral can be calculated, but the end result is quite cumbersome. Taking it into account that $\varepsilon_{res} > \Delta$, it is easy to obtain the upper and lower estimates of the integral:

$$\int\frac{k_1^3k_2^3\delta(\Delta - k_1 - k_2)\,dk_1dk_2}{(\varepsilon_{res} + k_1 - \Delta)^2} = \frac{\Delta^5}{140}\left(\frac{\Delta}{\varepsilon_{res}}\right)^2 S,$$

where

$$\left\{1 + \frac{a}{2-a} - \frac{a^2}{2(2-a)^2}\right\} \leqslant S \leqslant \left\{1 + \frac{a}{1-a} - \frac{a^2}{(1-a)^2}\right\}, \quad a = \frac{\Delta}{\varepsilon_{res}}.$$

Thus

$$W_{\gamma\gamma} = \frac{2e^4R_0^4}{945\pi}\Delta^5\left(\frac{\Delta}{\varepsilon_{res}}\right)^2 S\,|\langle 2|\sum_{ij}(\xi_i\xi_j/R_0^2)|1\rangle|^2;$$

The matrix element of the transition can be broken up into two parts:

$$\langle 2\Big|\sum_{i,j}(\xi_i\xi_j/R_0^2)\Big|1\rangle = \langle 2\Big|\sum_i(\xi_i/R_0)^2\Big|1\rangle + \langle 2\Big|\sum_{i>j}(\xi_i\xi_j/R_0^2)\Big|1\rangle.$$

According to Levinger's calculation[15] we obtain for the diagonal matrix elements of such an operator

$$\langle 2\Big|\sum_{i>j}\xi_i\xi_j\Big|1\rangle \approx -\frac{1}{2}\langle 2\Big|\sum_i\xi_i^2\Big|1\rangle.$$

Assuming that this equation holds approximately also for the nondiagonal element, we obtain

$$W_{\gamma\gamma} \approx e^4\frac{\Delta^5}{1890\pi}\left(\frac{\Delta}{\varepsilon_{res}}\right)^2 R_0^4 S\,|\langle 2|\sum_i(\xi_i/R_0)^2|1\rangle|^2.$$

The function of the quantum angular correlation $W(\theta)$ is identical with that obtained for two electric dipole quanta in the $0\to 1\to 0$ cascade and is of the form $W(\theta) = 1 + \cos^2\theta$, where θ is the angle between the wave vectors of the quanta.

6. RELATIVE CONTRIBUTION OF THE $M1$, $E2$, AND $E0$ TRANSITIONS OF THE NUCLEUS

In the case of a nuclear transition between states with equal spins and parities, it is possible to have a $E0$-transition along with $E2$ and $M1$ transitions. It would be interesting to compare the probabilities of the conversions of the shell electrons for various multipoles. We shall give below the calculated values of the ratios of the probabilities, W_{E0} to W_{M1} and to W_{E2}, for K-electron conversion.

To determine the conversion probability in the $E2$ transition we shall consider only the contribution of the charge of the transition. Actually, at energies on the order of $\lesssim Ze^2$ the contribution of the transition current is approximately $(Ze^2)^2$ of the corresponding contribution of the transition charge. At greater transition energies $k \gg Ze^2$ both terms are of the same order, but they have different signs (see Ref. 9). Thus, the estimate of the $E2$ transition made with the aid of the scalar potential differs in this case from the accurate value by a fact known not to exceed 4, a circumstance that can be allowed for in the estimate. Consequently

$$W_{E2} = 2\pi e^4\frac{k^6}{225}\sum_{j_2\lambda_2}[\mathrm{I}T_1 + \mathrm{II}T_2]\,|\langle 2\|E2\|1\rangle|^2;$$

$$T_1 = (2j_2+1)^{1/2}(2j_2-2\lambda_2+1)^{1/2}w\left(j_2\tfrac{1}{2}\,2j_1-\lambda_1;\ j_2-\lambda_2 j_1\right)C^{j_1-\lambda_1 0}_{j_2-\lambda_2 020},$$

$$T_2 = (2j_2+1)^{1/2}(2j_2+2\lambda_2+1)^{1/2}w\left(j_2\tfrac{1}{2}\,2j_1+\lambda_1;\ j_2+\lambda_2 j_1\right)C^{j_1+\lambda_1 0}_{j_2+\lambda_2 020},$$

$$\mathrm{I} \approx \int_{R_0}^{\infty} g_{j_2\lambda_2}(r)\,g_{j_1\lambda_1}(r)\,F_2(kr)\,r^2dr,\quad \mathrm{II} \approx \int_{R_0}^{\infty} f_{j_2\lambda_2}(r)\,f_{j_1\lambda_1}(r)\,F_2(kr)\,r^2dr,$$

where $F_L(kr)$ is the Hankel spherical function.

The integral of the II$E2$ transition is always approximately proportional to $(Ze^2)^2$ also for all Z less than 1, and we can therefore disregard henceforth integral II in the estimate, without causing an error greater than 2. The reduced matrix element of the $E2$ transition is determined by the equation

$$\langle\Omega^{\Lambda+}_{IM_2}(2)\Big|\sum_{i=1}^{Z}\xi_i^2\sqrt{4\pi}Y_{2M}(\theta_i\varphi_i)\Big|\Omega^{\Lambda-}_{IM_1}(1)\rangle = C^{IM_2}_{IM_1 2M}\langle 2\|E2\|1\rangle.$$

We can denote analogously

$$\langle \Omega_{1M}^{\Lambda +}(2) \Big| \sum_{i=1}^{Z} \xi_i^2 \Big| \Omega_{1M}^{\Lambda}(1) \rangle = \langle 2 \| E0 \| 1 \rangle,$$

where $\Omega_{1M_2}^{\Lambda}(2)$ and $\Omega_{1M_1}^{\Lambda}(1)$ are the wave functions of the nucleus in the final and initial states respectively. In the case of an $E2$ multipole conversion on the K shell, two final states of the electron are possible:

$$(j = 3/2,\ \lambda = +1/2,\ l = 2)$$
$$\text{and}\ (j = 5/2,\ \lambda = -1/2,\ l = 2).$$

To compare the probability of the $E2$ transition with the probability W_{E0} of the monopole transition, we shall use the value of the probability for the first transition multiplied by two, thus overestimating the total probability of the $E2$ transition. For the $M1$ transition $(j_1\lambda_1 1) \to (j_1\lambda_1 2)$ we have

$$W_{M1} = 4\pi \frac{e^4 k^4}{9} [w(j_1 1/2\, 1 1;\ 1 j_1)]^2$$
$$\times (\mathrm{I} + \mathrm{II})^2 |\langle 2 \| M1 \| 1 \rangle|^2,$$

where the reduced matrix element is determined analogously:

$$\langle \Omega_{1M_2}^{\Lambda +}(2) \Big| \sum_i \sqrt{4\pi} \left\{ \left(\frac{\mu_i}{2M} + \xi_i^2 \frac{\delta_p \Phi(\xi_i)}{3} \right) \sqrt{2}\, (\sigma_i \mathbf{Y}_{1M}^{-1}) + \frac{1}{M} (\mathbf{l}_i \mathbf{Y}_{1M}^{-1}) \right\} \Big| \Omega_{1M_1}^{\Lambda}(1) \rangle = C_{1 M_1 1 M}^{1 M_2} \langle 2 \| M1 \| 1 \rangle.$$

The operator $\sqrt{2}(\xi^2/3)[\sigma_i \mathbf{Y}_{1M}^{-1}(\theta_i\ \varphi_i)]$ takes into account the effect of the spin-orbital proton interaction in the $M1$ radiation, and this term plays an important role in the transitions of the type $1/2 + \to 1/2 +$; $\delta_p = 1$ for the proton and $\delta_p = 0$ for the neutron, while μ_i is the algebraic magnetic moment of the nucleon in the Bohr magneton, M is the nucleon mass (2×10^3), and σ are the Pauli matrices.

$$\mathrm{I} = \int_{R_0}^{\infty} f_{j_1\lambda_1}(2)\, g_{j_1\lambda_1}(1)\, F_1(kr)\, r^2\, dr,$$
$$\mathrm{II} = \int_{R_0}^{\infty} g_{j_1\lambda_1}(2)\, f_{j_1\lambda_1}(1)\, F_1(kr)\, r^2\, dr.$$

The integrals I and II of the conversion transitions can be readily estimated for two limiting values of the transition energy ($\Delta \equiv k$): $k \ll Ze^2$ and $k \gg Ze^2$.

If $k \lesssim Ze^2$, we employ the Bessel-function expansion

$$F_L(kr) = -i \frac{(2L)!}{2^L L!} (kr)^{-L-1},$$

and if $k \gg Ze^2$ we carry out the calculation in the Born approximation. As a result we obtain for $k \lesssim Ze^2$:

for the $E2$ transition:

$$\mathrm{I} \approx -3ik^{-3} (\varepsilon_2 + 1)^{1/2} (\varepsilon_1 + 1)^{1/2} N_{j_1}(1)$$
$$\times C_{j_2}(2)\, R_0^{\gamma_1 + \gamma_2 - 3} \operatorname{Re} \Phi(\gamma_2),$$

for the $M1$ transition:

$$\mathrm{I} \approx -ik^{-2} (\varepsilon_2 - 1)^{1/2} (\varepsilon_1 + 1)^{1/2} N_{j_1}(1)\, C_{j_1}(2)\, R_0^{2(\gamma_1 - 1)} \operatorname{Im} \Phi(\gamma_2),$$
$$\mathrm{II} \approx ik^{-2} (\varepsilon_2 + 1)^{1/2} (1 - \varepsilon_1)^{1/2} N_{j_1}(1)\, C_{j_1}(2)\, R_0^{2(\gamma_1 - 1)} \operatorname{Re} \Phi(\gamma_2),$$

where

$$\Phi(\gamma_2) = \frac{e^{i\eta_2} (\gamma_2 + iZe^2 \varepsilon_2 / p_2)}{Ze^2 + ip_2}\, {}_2F_1\left(\gamma_2 + 1 + \frac{iZe^2\varepsilon_2}{p_2};\ 1;\ 2\gamma_2 + 1;\ \frac{2ip_2}{Ze^2 + ip_2} \right).$$

We shall assume furthermore that $\operatorname{Re} \Phi(\gamma_2) \approx \operatorname{Im} \Phi(\gamma_2) \approx 1/Ze^2$. If $k \gg Ze^2$ we can neglect the integrals II, since they are approximately proportional to $(Ze^2)^2$ I. We obtain in this case for $E2$ and $M1$:

$$E2:\ \mathrm{I}=i\frac{2}{\sqrt{\pi}}\left[\frac{(\varepsilon_2+1)(\varepsilon_1+1)}{k}\right]^{1/2}N_{j_1}R_0^{\gamma_1-1}\left(\frac{p_2}{k}\right)^{5/2}\frac{1}{p_2^2-k^2},$$

$$M1:\ \mathrm{I}\approx i\frac{2}{\sqrt{\pi}}\left[\frac{(\varepsilon_2-1)(\varepsilon_1+1)}{k}\right]^{1/2}N_{j_1}R_0^{\gamma_1-1}\left(\frac{p_2}{k}\right)^{3/2}\frac{1}{p_2^2-k^2}.$$

Using the values obtained for the integrals I and II, we obtain for the ratios of the $E2$ and $M1$ probabilities to the $E0$ transition probability:

For $k \lesssim Ze^2$

$$\frac{W_{E0}}{W_{E2}}\geqslant 3{,}5\cdot 2^{2(\gamma_1-\gamma_2)}R_0^{2(\gamma_1+1-\gamma_2)}\frac{[S_\lambda(1)S_\lambda(2)]^2}{(\varepsilon_1+1)(\varepsilon_2+1)}\left|\frac{\Gamma(2\gamma_2+1)}{\Gamma(2\gamma_1+1)}\right|^2\frac{p_2^{2(\gamma_1-\gamma_2)}}{[\mathrm{Re}\,\Phi(\gamma_2)]^2}\times\left|\frac{\Gamma(\gamma_1+iZe^2\varepsilon_2/p_2)}{\Gamma(\gamma_2+iZe^2\varepsilon_2/p_2)}\right|^2\left|\frac{\langle 2\|E0\|1\rangle}{\langle 2\|E2\|1\rangle}\right|^2,$$

For $k \gg Ze^2$

$$\frac{W_{E0}}{W_{E2}}\geqslant 2^{2\gamma_1+1}R_0^{2(\gamma_1-1)}(p_2^2-k^2)^2p_2^{2\gamma_1-6}\exp\left(\pi Ze^2\frac{\varepsilon_2}{p_2}\right)\times\left|\frac{\Gamma(\gamma_1+iZe^2\varepsilon_2/p_2)}{\Gamma(2\gamma_1+1)}\right|^2\frac{|S_\lambda(1)S_\lambda(2)|^2}{(\varepsilon_1+1)(\varepsilon_2+1)}\left|\frac{\langle 2\|E0\|1\rangle}{\langle 2\|E2\|1\rangle}\right|^2.$$

Taking it into account that the two last factors are approximately equal to unity, we obtain in accordance with the above estimate for $p \leqslant Ze^2$

$$W_{E0}/W_{E2}\geqslant 144\,|\langle 2\|E0\|1\rangle/\langle 2\|E2\|1\rangle|^2.$$

At low energies this ratio depends little on Z and is close to unity, even if we assume a value of approximately 0.1 for the ratio of the matrix elements. If the transition energy is higher, $\Delta \sim 1$, the ratio W_{E0}/W_{E2} depends greatly on the value of the nuclear charge. At small values of Z we have $W_{E2} \approx W_{E0}$, and at large values of Z $(Z > 60)$ we have $W_{E0} \gg W_{E2}$, if the reduced matrix elements are equal.

Analogously, we obtain the ratio of W_{E0}/W_{M1} for the same limiting cases. (The transition $(j = 1/2;\ \lambda = -1/2) \to (j = 3/2;\ \lambda = 1/2)$ is taken into account by the factor 2.)

For $k \lesssim Ze^2$

$$\frac{W_{E0}}{W_{M1}}\geqslant\frac{4[S_\lambda(1)S_\lambda(2)]^2}{[(\varepsilon_2-1)^{1/2}(\varepsilon_1+1)^{1/2}\,\mathrm{Im}\,\Phi(\gamma_2)-(1-\varepsilon_1)^{1/2}(1+\varepsilon_2)^{1/2}\,\mathrm{Re}\,\Phi(\gamma_2)]^2}\left|\frac{\langle 2\|E0\|1\rangle}{\langle 2\|M1\|1\rangle}\right|^2$$

and for $k \gg Ze^2$

$$\frac{W_{E0}}{W_{M1}}\geqslant 2^{2\gamma_1-1}\frac{[S_\lambda(1)S_\lambda(2)]^2}{(\varepsilon_1+1)(\varepsilon_2-1)}\frac{[p_2^2-k^2]^2}{p_2^{4-2\gamma_1}}\frac{R_0^{2(\gamma_1-1)}\exp(\pi Ze^2\varepsilon_2/p_2)\,|\Gamma(\gamma_1+iZe^2\varepsilon_2/p_2)|^2}{[\Gamma(2\gamma_1+1)]^2}\times\left|\frac{\langle 2\|E0\|1\rangle}{\langle 2\|M1\|1\rangle}\right|^2.$$

In this case the ratio W_{E0}/W_{M1} is also a rapidly-growing function of Z. If $Z \lesssim 50$, its approximate value is 10^{-2}. The estimates cited show that in the transition-energy region $k \lesssim Ze^2$ one can except a considerable inclusion of electric monopole in the K electron conversion in the case of nuclear transitions of the type $\mathrm{I}^{\pm} \to \mathrm{I}^{\pm}$. In the transition-energy region $k \gg Ze^2$, the contribution of the monopole may be substantial only for sufficiently large values of Z $(Z > 50)$.

7. ESTIMATE OF THE NUCLEAR TRANSITION MATRIX ELEMENT

The monopole nuclear matrix element, like that of other multipoles, can be estimated at the present time only within the framework of definite model representations. We shall give below the results of the estimate of the $E0$ matrix element both for the single-particle model, as well as for several models with collective motion of the nucleons. The estimates of the monopole matrix element are also given by Schiff[16].

1) *Single-Particle Model*

In the limiting case, when the change in the state of the nucleus occurs by a transition of one particle from state $\Omega_{IM}^{\Lambda}(1)$ into the state $\Omega_{IM}^{\Lambda}(2)$, the matrix element $\langle 2|\Sigma\xi_i^2|1\rangle$ reduces to the single-particle matrix element:

$$\langle 2\Big|\sum_{i=1}^{Z}\xi_i^2\Big|1\rangle=\left(\frac{Z}{A^2}+\delta_p\right)\langle\Omega_{IM}^{\Lambda}(2)\,|\,\xi^2\,|\,\Omega_{IM}^{\Lambda}(1)\rangle.$$

The core, the field of which contains the "external" nucleon, is assumed invariant. The term proportional to Z/A^2 results from the allowance for the effect of the recoil of the core, while $\delta_p = 1$ for the proton transition and $\delta_p = 0$ for the neutron transition. As a rough estimate

$$\langle\Omega_{IM}^{\Lambda+}(2)|\xi^2\,|\,\Omega_{IM}^{\Lambda}(1)\rangle\leqslant 0{,}6\,R_0^2.$$

2) *Hydrodynamic Polarization Oscillations of the Nucleus*

The monopole nuclear matrix element can also be estimated by representing the nucleus as a drop of a charged two-component nuclear liquid. Such a drop may be subject not only to radially-symmetrical pulsating surface oscillations but also to polarization oscillations of the proton and neutron components of the liquid. The frequency of these oscillations is considerably smaller than the frequency of the oscillations connected with the compressibility of the nucleus.

Comparing the classical and quantum-theoretical polarizabilities of a drop acted upon by a small perturbation $V = \lambda r^2 e^{i\omega t}$, we obtain an integral singular equation for the square of the modulus of the matrix element. The solution of this equation leads to the estimate:

$$\Big|\langle 2\Big|\sum_{i}^{Z}\xi_i^2\Big|1\rangle\Big|^2 g(\Delta)=\frac{3\pi\Gamma E_0^2\,Z\Delta R_0^5}{(\Delta^2-E_0^2)^2+\Gamma^2\Delta^2}$$

$$\times\left\{1-\frac{5}{3}\,\mathrm{Im}\,F(x)+\frac{5}{3}\,\frac{\Delta^2-E_0^2}{\Gamma\Delta}\,\mathrm{Re}\,F(x)\right\},$$

$$F(x)=\frac{[(6-x^2)(x\cos x-\sin x)+2x^2\sin x]}{x^2(x\cos x-\sin x)},$$

$$x=(k_0R_0)\sqrt{\frac{\Delta^2}{E_0^2}-1+i\,\frac{\Gamma\Delta}{E_0^2}},\qquad E_0=\frac{MV_0A}{4\pi NZe^2},$$

$$V_0=\frac{3}{4\pi}\,R_0^3;$$

$$(k_0R_0)\approx 2{,}08\left|\left[\frac{\varepsilon_{\mathrm{res}}^2}{E_0^2}-1-i\,\frac{\Gamma\varepsilon_{\mathrm{res}}}{E_0^2}\right]^{-1/2}\right|.$$

Here N is the number of neutrons in the nucleus, $A = N + Z$, M the nucleon mass, Γ and $\varepsilon_{\mathrm{res}}$ the width and "resonance" energy of the known dipole "resonance" appearing in (γn) and (γp) reactions, and $g(\Delta)$ the density of the nuclear levels that can be excited by the $E0$ transition of the nucleus at an energy Δ. The above estimate is correct for transition energies at which the spectrum of the nucleus becomes continuous.

3) *Surface Quadrupole Oscillations of the Nucleus*

Even-even nuclei with A ranging from 76 to 152 have apparently an energy spectrum corresponding to a hydrodynamic phonon nuclear excitation of the quadrupole type. This was first pointed out by Scharf-Goldhaber and Weneser[17]. However, another interpretation of the spectrum of these nuclei is possible. The problem can be solved by investigating the ratios of the photon-emission probabilities to the electron conversion probability for various nuclear transitions. Electric monopole transitions make possible still another clarification of the character of excitation of this group of nuclei. It would therefore be interesting, within the framework of this model, to estimate the matrix element of the $E0$ transition and to compare it with the matrix element of the $E2$ transition. We shall next describe the state of the nucleus by the following quantum numbers: ν – number of phonons, I, μ – spin and spin projection on the OZ axis. The wave function of the nucleus is $\chi^{\nu}_{I\mu}$. Depending on the number of the phonons, the spin of the even-even nucleus can assume the following values $\nu = 0, I = 0$; $\nu = 1, I = 2$; $\nu = 2, I = 0$, 2, and 4. Experimentally one observes the following sequence in the values of the nuclear spin in the fundamental, first, and second excited states

$$[I=0,\ I=2,\ I=0];$$
$$[I=0,\ I=2,\ I=2];\text{and }[I=0,\ I=2,\ I=4].$$

In the case of the spectrum of the first and second types, $E0$ transitions with electron conversion can occur between the second and ground state and between the first and second states of the nucleus respectively. A simple calculation, under the assumption that these states are phonon excitations of the nucleus, yields the following values of the matrix elements of the $E2$ and $E0$ transitions ($\hat{E}2$ and $\hat{E}0$ operators of the $E2$ and $E0$ multipoles):

$$\langle\chi^{0+}_{00}\,|\,\hat{E0}\,|\,\chi^{2}_{00}\rangle = \frac{3}{4\pi} ZR_0^2 \sqrt{\frac{2}{5}}\left(\frac{\hbar\omega}{2c_2}\right),$$

$$\langle\chi^{1+}_{2\mu}\,|\,\hat{E0}\,|\,\chi^{2}_{2\mu}\rangle = \frac{3}{4\pi} ZR_0^2\, 6 \sqrt{\frac{5}{4\pi}}\, C^{20}_{2020}\left(\frac{\hbar\omega}{2c_2}\right)^{3/2},$$

$$\langle\chi^{0+}_{00}\,|(\hat{E2})_M\,|\,\chi'_{2\mu}\rangle = \frac{3}{4\pi} ZR_0^2\left(\frac{\hbar\omega}{2c_2}\right)^{1/2}(-1)^M\,\delta_{\mu-M}\sqrt{\frac{4\pi}{5}},$$

$$\langle\chi^{0+}_{00}\,|\,(\hat{E2})_M\,|\,\chi^{2}_{2\mu}\rangle = \frac{3}{4\pi} ZR_0^2 \sqrt{8}\, C^{20}_{2020}\left(\frac{\hbar\omega}{2c_2}\right)\delta_{\mu-M}\,(-1)^M$$

$$\langle\chi^{1+}_{2\mu}\,|\,(\hat{E2})_M\,|\,\chi^{2}_{00}\rangle = \frac{3}{4\pi} ZR_0^2\frac{2}{5}\left(\frac{\hbar\omega}{2c_2}\right)^{1/2}\delta_{\mu M}\sqrt{2\pi},$$

$$\langle\chi^{1+}_{2\mu_1}\,|\,(\hat{E2})_M\,|\,\chi^{2}_{2\mu_2}\rangle = \frac{3}{4\pi} ZR_0^2\sqrt{\frac{8\pi}{5}}\, C^{2\mu_2}_{2\mu_1,\,2-M}\left(\frac{\hbar\omega}{2c_2}\right)^{1/2}(-1)^M.$$

We obtain for the ratio of the matrix elements of the competing transitions:

$$\langle\chi^{0+}_{00}\,\|\,\hat{E0}\,\|\,\chi^{2}_{00}\rangle\,/\,\langle\chi^{1+}_{2\mu_1}\,\|\,\hat{E2}\,\|\,\chi^{2}_{00}\rangle \approx 0.1,$$

$$\langle\chi^{1+}_{2\mu}\,\|\,\hat{E0}\,\|\,\chi_{2\mu}\rangle\,/\,\langle\chi^{1+}_{2\mu_1}\,\|\,\hat{E2}\,\|\,\chi^{2}_{2\mu_2}\rangle \lesssim 0.05.$$

It is assumed in the estimate that $(\hbar\omega/2c_2)^{1/2} \approx 0.18$ (see, for example, Refs. 18 and 19). For the matrix elements of the $E0$ transitions of two nuclei with different spectra we have

$$\langle\chi^{1+}_{2\mu}\,\|\,\hat{E0}\,\|\,\chi^{2}_{2\mu}\rangle\,/\,\langle\chi^{0+}_{00}\,\|\,\hat{E0}\,\|\,\chi^{2}_{00}\rangle \approx 0.5.$$

We obtain a numerical estimate of the order of magnitude of the matrix elements by putting $(\hbar\omega/2c_2)^{1/2} \approx 0.18$; then

$$\langle\chi^{0+}_{00}\,|\,\hat{E0}\,|\,\chi^{2}_{00}\rangle \approx 5\cdot 10^{-3} ZR_0^2,$$

$$\langle\chi^{1+}_{2\mu}\,|\,\hat{E0}\,|\,\chi^{2}_{2\mu}\rangle \approx 2\cdot 10^{-3} ZR_0^2.$$

8. CERTAIN RESULTS OF THE CALCULATIONS

By way of illustration let us give the calculated probabilities of the shell-electron conversion, pair conversion and $\gamma\gamma$ transition for the nuclei Ca^{40}_{20}, Ge^{72}_{32}, Zr^{90}_{40}, Pd^{106}_{46}, and Po^{214}_{84}.

Table 2 gives the values of the probabilities, divided by

$$|\langle 2\,|\sum_i(\xi_i/R_0)^2\,|\,1\rangle|^2 \text{ and } |\langle 2\,|\sum_{i,j}(\xi_i\xi_j/R_0^2)|\,1\rangle|^2$$

for the $\gamma\gamma$ transition.

TABLE II.

Nucleus	Δ, mc^2	$w_{\gamma\gamma}$	W_{pair}	w_K	$\frac{w_K}{w_{LI}}$	$\frac{w_{LI}}{w_{LII}}$	$\varkappa_K$	$\varkappa_{L\,II}$	$\varkappa_L$
Ca_{20}^{40}	6.7	$1.5\cdot10^9$	$\sim4\cdot10^7$	$5,05\cdot10^8$	7.8	342	0.99	0.965	0.97
Ge_{32}^{72}	1.4	$5.5\cdot10^4$	0	$2,56\cdot10^8$	7.4	198	0.96	0.83	0.945
Zr_{40}^{90}	3.5	$5.7\cdot10^6$	$\sim1.4\cdot10^6$	$1,16\cdot10^{10}$	7.6	102	0.87	0.88	0.88
Pd_{46}^{106}	2.27	$3.3\cdot10^6$	—	$3,54\cdot10^{10}$	6.9	71	0.88	0.87	0.87
Po_{84}^{214}	2.85	$5.6\cdot10^7$	$\sim8.7\cdot10^6$	$8,45\cdot10^{12}$	5.3	19.6	0.68	0.77	0.75

All the values of the probabilities are given in the usual units, sec^{-1}.

Data concerning $E0$ transitions of even-nuclei with A ranging between 60 and 160 are of particular interest. A study of the $E0$ transitions for this range of A between levels of nuclei with spin $I: 0^+ \to 0^+$ and $2^+ \to 2^+$ would make it possible to establish the character of the excited states of the nuclei.

Note added in proof (April 27, 1957): After this article went to press, we became acquainted with the work by Church and Weneser [E. Church, J. Weneser, Phys. Rev. 103, 1035 (1956)] which deals with the conversion of the shell electron in the $E0$ nuclear transition. They used numerical methods and took the screening effect into account. Since the calculated data are given in graphic form, direct comparison of the results is difficult. However, qualitative deductions and the values obtained in our work published for many nuclei are in good agreement with the values cited by Church and Weneser.

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221