Note that according to Eq. (36) or Eq. (37), $P(\lambda, 1) = \lambda$ i.e. $\lambda_c = P(\lambda_c, 1)$ determines the magnitude of the interaction observed in experiments with mesons at low energies, when $\xi = 0$, $x_c = 1$. If this quantity is considered known and one considers Eq. (38) as specifying λ in terms of λ_c, g_0^2 and L, then Eq. (38) and Eq. (39) constitute the usual means for renormalizing the amplitude of meson-meson scattering, which, as is apparent, also occurs outside the framework of the perturbation theory.

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Nonlinearity of the Field in Conformal Reciprocity Theory

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The first version of Born and Infeld's nonlinear electrodynamics and the variability of the gravitational constant are deduced from the conformally covariant gravitational equations derived from a certain generalized reciprocity law which is based on group theory and yields a nonlocal field theory. A correspondence principle is established between relativity theory and reciprocity theory.

I. FIELD EQUATIONS

1. LET k BE Einstein's gravitational constant, $ds^2 = g_{ik}dx^i dx^k$ be the element of interval with the metric $g_{ik} = g_{ki}$, $g = \| g_{ik} \|$ $(i, k = 1, \ldots, 4)$. Let Γ_{ik}, $\Gamma = \Gamma_r^r$ be the contracted curvature tensor and the curvature scalar constructed from Weyl's conformal connection[1] Γ_{kl}^i, Riemannian in the metric $u_{ik} = \Psi g_{ik}$, and of weight zero with respect to the g_{ik} ($\Psi = E^{-2\psi}$ is of weight -1 with respect to the g_{ik}). Let $p_{ik} = (\partial p_k/\partial x^i - \partial p_i/\partial x^k)$ be the electromagnetic field of absolute magnitude $P/\sqrt{2} = P'\sqrt{k/2}$. Let $P_{ki} = P_{ik} = p_{ir}p_k^r/P$, $P = P_r^r$,

$$L_{ik} = \Gamma\Gamma_{ik} + PP_{ik},\ L = L_r^r = \Gamma^2 + P^2, \quad (1)$$

$$Q_{ik} = \Gamma Q'_{ik} = \Gamma(\Gamma_{ik} - {}^1/_4\Gamma g_{ik}),$$
$$S_{ik} = PS'_{ik} = P(P_{ik} - {}^1/_4 P g_{ik}). \quad (2)$$

The Γ_{ik}, P_{ik} (as well as the p_{ik}) depend only on the ratios of the g_{ik} (and therefore on those of the of the u_{ik}). Variation of $L\sqrt{g}$ with respect to g_{ik} and p_i gives[2] the gravitational $Q_i^k\sqrt{g}$ and electromagnetic $S_i^k\sqrt{g}$ energy-momentum tensor densities of weight zero and current-charge vector densities $s^i\sqrt{g}$ and $s'^i\sqrt{g}$ of weight zero. This leads,[2,3] to the conformal covariant equations which satisfy the reciprocity principle[2-12]

$$Q_{ik} = -S_{ik} \text{ or } L_{ik} = {}^1\!/\!_4\, L g_{ik}, \quad (3)$$

$$\partial(p^{ik}\sqrt{g})/\partial x^k = s^i\sqrt{g}. \quad (4)$$

when $\partial(\Gamma^2\sqrt{g})/\partial p_i = 0$, Equations (4) become

$$\partial(p^{ik}\sqrt{g})/\partial x^k = 0; \quad (4')$$

Eq. (3) and (4) describe the gravitational, and electromagnetic fields respectively.

2. Equations (3) and (4), which are conformal in V_4 with metric g_{ik}, are Riemannian in the manifold U_4 with metric u_{ik}. The general form of the gravitational equations (3) and their variational derivation are independent of the factor Ψ and therefore of the arbitrary vector $\psi_i = \partial\psi/\partial x^i$ (where $\psi = -\frac{1}{2}\ln\Psi$) which defines the corresponding conformal affinity of Weyl. Elsewhere[13], gravitational equations of the form $Q_{ik} = 0$ are derived from $\Gamma^2\sqrt{g}$ in the special case $\Psi = \Gamma$. We have $L\sqrt{g} = \sqrt{\overset{0}{g}}$, where the metric $\overset{0}{g}_{ik} = g_{ik}L^{1/2}$, $\overset{0}{g} = \|\overset{0}{g}_{ik}\|$. It would seem that Ψ is a function of the absolute magnitude of the conformal curvature tensor of Weyl for the congruences noted by Vranceanu[1]. Ψ is directly related to the conformal generalization of the scalar field X of Rumer[14] or more generally with a scalar meson field, since the meson field is conformal[15-19]. In the conformal six-dimensional theory, the equation for Ψ is a generalization of the scalar Klein-Gordon equation[14,20-22]. Conformal reciprocity theory includes the theory of particles with spin 0, 1, and 2 (scalar mesons, photons, gravitons). Let R_{ik} and R be the contracted curvature tensor and the curvature scalar constructed from the Riemannian affinities $|\overset{i}{kl}|$ with respect to the metric $g_{ik} = U_{ik}/\Psi$. We shall have the symbol ( ); denote covarient differentiation with respect to the Riemannian affinity $|\overset{i}{kl}|$. We have[1]

$$\varphi_{ik} = (\psi)_{;i;k} + \psi_i\psi_k - \frac{1}{2}\psi^r\psi_r g_{ik},$$
$$\varphi = \varphi^r_r = g^{rs}(\psi)_{;r;s} - \psi^r\psi_r, \tag{5}$$
$$\Gamma_{ik} = R_{ik} + 2\varphi_{ik} + \varphi g_{ik}, \quad \Gamma = R + 6\varphi.$$

The tensor Q'_{ik} breaks up into a pure gravitational tensor $R_{ik} - (\frac{1}{4})Rg_{ik}$ and a meson tensor $2(\varphi_{ik} - \frac{1}{4}\varphi g_{ik})$. If we introduce the vector $g_i = p_i/\Psi^{1/2}$, we obtain a similar decomposition of S'_{ik}. Since Q_{ik} and S_{ik} are symmetric and $Q^r_r = S^r_r = 0$, their conformal divergences reduce to the Riemannian ones. The conservation laws for the energy-momentum and charge-current which follow from Equations (3) and (4), respectively, are

$$(Q^k_i\sqrt{g})_{;k} = -(S^k_i\sqrt{g})_{;k}, \quad \partial(s^i\sqrt{g})/\partial x^i = 0. \tag{6}$$

3. Let $U^k_i = L^k_i/2\Gamma$, $V^k_i = L^k_i/2P$,

$$U = U^r_r = L/2\Gamma, \quad V = V^r_r = L/2P. \tag{7}$$

The gravitational equations (3) become

$$U_{ik} = \frac{1}{4}Ug_{ik} \text{ or } V_{ik} = \frac{1}{4}Vg_{ik}, \tag{8}$$
$$L = \Gamma^2 + P^2 = 2\Gamma U = 2PV. \tag{9}$$

With the aid of conformal transformation of the metric g_{ik} of the manifold V_4 into the metrics $g^*_{ik} = g_{ik}U$ and $g^{**}_{ik} = -ig_{ik}V$, we obtain $(i^2 = -1)$

$$L^* = 2\Gamma^*, \quad L^{**} = 2iP^{**}, \quad \Gamma^{**} = -iP^*,$$
$$U^* = -iV^{**} = 1. \tag{9'}$$

Setting $(1 - P^2/U^2)^{1/2} = (1 - \Gamma^2/V^2)^{-1/2} = \epsilon$, Eq. (9) leads to

$$(a = \pm 1), \tag{10}$$

and thus Eq. (9) has two pairs of solutions Γ, P and V, U

$$\underset{1}{\Gamma} = \underset{1}{U}(1+\varepsilon), \quad \underset{1}{P} = \underset{1}{V}(1-\varepsilon)$$
$$\text{and } \underset{2}{\Gamma} = \underset{1}{U}(1-\varepsilon), \quad \underset{1}{P} = \underset{2}{V}(1+\varepsilon). \tag{11}$$

Now setting $U = -if$, $V = in$, and defining the scalars J, N, H, F, γ, j, h and the antisymmetric tensors h_{ik}, f_{ik} by the relations

$$\Gamma/J = J/N = P/H = H/F = \gamma/j = j/n$$
$$= p/h = h/f = p_{ik}/h_{ik} = h_{ik}/f_{ik} = \sqrt{\varepsilon}, \tag{12}$$

we arrive at the following expression for L [see Eq. (9)]:

$$\underset{00}{L} = 2\underset{1}{U^2}(1+\varepsilon) = 2\underset{1}{V^2}(1-\varepsilon) = 4h_1 = 4\varepsilon l_1,$$
$$\underset{01}{L} = 2U^2(1-\varepsilon) = 2\underset{2}{V^2}(1+\varepsilon) = 4h_2 = -4\varepsilon l_2, \tag{13}$$

where

$$\varepsilon = 1/\mu = (1 + P^2/f^2)^{1/2} = (1 - F^2/f^2)^{-1/2}, \tag{14}$$

$$\varepsilon = 1/\mu = (1 + \Gamma^2/n^2)^{1/2} = (1 - N^2/n^2)^{-1/2}. \tag{14'}$$

Setting $s = 1/w = (\epsilon + 1)/(\epsilon - 1)$, we obtain $\epsilon = 1/\mu = (s+1)/(s-1)$; ϵ and s are of weight zero. Conversely, Eq. (7) follows from L [see Eq. (1)], (13), and (14) or (14′); to each general solution Γ, P of Eq. (3) and (4′) there corresponds one value of U and V. Eq. (9) follows from (13) and (17). The same results are obtained by the conformal generalization of the five-dimensional theory[14, 19-22]

4. The quantities ϵ and μ in Eq. (14) are just the dielectric constant and permeability of the field as defined in the first version of Born and Infeld's nonlinear electrodynamics[7,8,23-25]. The quantities $f_{ik} = \mu p_{ik}$ and $p_{ik} = \epsilon f_{ik}$ are Born and Infeld's electromagnetic field of zero weight with absolute magnitudes $F/\sqrt{2} = F\sqrt{k/2}$, $P/\sqrt{2} = P\sqrt{k/2}$, and $U/\sqrt{2} = -if/\sqrt{2} = -ip\,\mu/\sqrt{2}$ is the "maximum" electromagnetic field ($F^2 \leq 0$, $P^2 \leq 0$, $f^2 \leq 0$, $p^2 \leq 0$). Similarly, $-iV/\sqrt{2} = n/\sqrt{2} = \gamma\mu/\sqrt{2}$ is the "maximum" gravitational field. The electromagnetic equations of Born and Infield are a special case of the nonlinear electromagnetic equations (4′), valid for $p_{ik} = \epsilon f_{ik}$, $P = \epsilon F$, $f = \text{const}$, and Eq. (14). The Born-Infeld equations are derived from the Lagrangian $l_2 = -\frac{1}{2} f^2\,(1-\mu) = -\mu h_2$, and therefore from the Hamiltonian $h_2 = -\frac{1}{2} f^2\,(1-\epsilon) = -\epsilon l_2$. The quantities l_2, h_2, l_1, h_1 are equivalent to L from the conformal point of view. Thus the arbitrariness in the choice of the lagrangian l_2 and the Hamiltonian h_2 in classical nonlinear electrodynamics is removed, and the irrational form of Eq. (13) reduces to the quadratic form of Eq. (9). The gravitational equations (3) are a generalization of the gravitational equations previously derived by the author[2] from the Langragian density $(\Gamma^2 + F^2)\sqrt{g}$. These two set of equations become identical for $p_{ik} = f_{ik}$. As is known, the mutual transformation of particles into each other, the interaction of different fields, and the polarization of fields lead necessarily to a nonlinear theory; the nuclear mass defect[26] also follows from the nonlinear gravitational equations. Nonlinearity also makes it possible to avoid infinite self-masses and dipole difficulties in nuclear theory, as well as to derive the equations of motion from the field equations.

5. Let us consider the case of a static spherical distribution of a field and a particle. Let a be a constant proportional to the point charge $e_0 = ec/\sqrt{k}$, $C^2 = 2\alpha^2/C'$ be a constant of integration, and $\rho = x$ be the radius vector with the origin at the center of the particle.[1] In polar coordinates we have

$$\underset{1}{P} = 2P^1_1 = 2P^4_4,\ P^2_2 = P^3_3 = 0,$$
$$\underset{1}{U} = -if_1 = 2\Gamma^2_2 = 2\Gamma^3_3, \tag{15}$$
$$\varepsilon \underset{1}{U} = -i\underset{1}{p} = 2\ \Gamma^1_1 = 2\Gamma^4_4.$$

We obtain two solutions $\rho_{1,2} = CE^{\pm\omega/2}$, which correspond, respectively to the internal and external regions of a sphere of radius $C \approx a(\rho_1\rho_2 = C^2)$. Let $\rho'^2 = \cosh\omega$, $\rho''^2 = \sinh\omega$. We have $\epsilon = 1/\mu = \pm\cot h\omega$ (depending on whether $\rho \gtreqless C$), $s = 1/\overline{w} = \rho^4/C^4 = E^{\pm 2\omega}$. The gravitational equations are identical with (9); we have $4(\Gamma^1_1)^2 - 4(\Gamma^2_2)^2 = -\underset{1}{P^2}$ [see Eq. (28′) below]. In the zero-weight metric $u_{ik} = g_{ik}\,\Psi = \tilde{g}_{ik}$, we have

$$\underset{1}{\tilde{\Gamma}} = 2(\tilde{\Gamma}^1_1 + \tilde{\Gamma}^2_2) = \mp C',\ \underset{2}{\tilde{\Gamma}} = -2(\tilde{\Gamma}^1_1 - \tilde{\Gamma}^2_2) = \pm 2\alpha^2 C^2/\rho^4,$$
$$\tilde{P}^2 = \underset{1}{\tilde{\Gamma}}\,\underset{2}{\tilde{\Gamma}} = C'^2/s = -4\alpha^4/\rho^4,\ -i\tilde{f}_1 = \mp 2\alpha^2\rho''^2/\rho^2, \tag{16}$$
$$-i\tilde{p}_1 = \mp 2\alpha^2\rho'^2/\rho^2,\ \underset{1}{\tilde{n}} = \mp 2\alpha^2\rho''^2/C^2,\ \underset{2}{\tilde{n}} = \pm 2\alpha^2 C^2\rho''^2/\rho^4.$$

The existence of a continuous solution $\underset{1}{\tilde{\Gamma}} = \mp C'$ and a divergent one $\underset{2}{\tilde{\Gamma}}$ is analogous to the theory of the double solution of a wave equation (de Broglie[27]). In the present (static and spherical case) we therefore have $\Psi = \Gamma/C'$. In terms of the metrics $g^*_{ik} = g_{ik}\,\underset{1}{U}$, $g^{**}_{ik} = -ig_{ik}\,\underset{2}{V}$, we obtain

$$-i\underset{1}{f^*} = \underset{2}{n^*} = 1,\ -i\underset{1}{p^*} = \underset{2}{\gamma^{**}} = \varepsilon,$$
$$-i\underset{1}{F^*} = \underset{2}{N^{**}} = 1/\rho'^2,\ -i\underset{1}{P^*} = \underset{2}{\Gamma^{**}} = 1/\rho''^2. \tag{16′}$$

II. THE GROUP OF CONFORMAL QUANTITIES

1. Let us define the general scalar curvatures $\underset{\beta\alpha}{A} = \underset{\beta\alpha}{A_r^r}$, $\underset{\beta\alpha}{a}$, Lagrangians $\underset{\beta\alpha}{L} = \underset{\beta\alpha}{L_r^r}$, and tensors $\underset{\beta\alpha}{A_i^k}$, $\underset{\beta\alpha}{G_i^k}$, $\underset{\beta\alpha}{Q_i'^k}$, $\underset{\beta\alpha}{A_i'^k}$, $\underset{\beta\alpha}{L_i^k}$, $\underset{\beta\alpha}{l_i'^k}$, $\underset{\beta\alpha}{L_i'^k}$ by the relations

$$\underset{\beta\alpha}{A} / \underset{\beta+1,\alpha}{A} = \underset{\beta\alpha}{a} / \underset{\beta+1,\alpha}{a} = \sqrt{\varepsilon}, \quad \underset{\beta\alpha}{A} / \underset{\beta,\alpha+1}{A} = \underset{\beta\alpha}{a} / \underset{\beta,\alpha+1}{a} = i\sqrt{s}, \tag{17}$$

$$\underset{\beta\alpha}{L} = \underset{\beta\alpha}{A^2} + \underset{\beta,\alpha+1}{A^2} = -2i\underset{\beta\alpha}{A}\,\underset{\beta+2,\alpha+1}{a} = 2i\,\underset{\beta,\alpha+1}{A}\,\underset{\beta+2,\alpha}{a}, \quad \underset{\beta\alpha}{L} = \varepsilon\,\underset{\beta+1,\alpha}{L} = -s\underset{\beta,\alpha+1}{L}, \tag{18}$$

$$\underset{\beta,\alpha-1}{A_i^k} + \underset{\beta,\alpha+1}{A_i^k} = -(1-s)\underset{\beta\alpha}{G_i^k}/i\sqrt{s}, \quad \underset{\beta,\alpha-1}{A_i^k} - \underset{\beta,\alpha+1}{A_i^k} = -(1+s)\underset{\beta\alpha}{A_i^k}/i\sqrt{s}, \tag{19}$$

$$\underset{\beta\alpha}{G_i^k} = \underset{\beta\alpha}{A_i^k} - \frac{1}{2}\underset{\beta\alpha}{A}\delta_i^k, \quad \underset{\beta\alpha}{Q_i'^k} = \underset{\beta\alpha}{A_i^k} - \frac{1}{4}\underset{\beta\alpha}{A}\,\delta_i^k, \quad \underset{\beta,\alpha-1}{A_i'^k} / \underset{\beta\alpha}{A_i^k} = \underset{\beta\alpha}{A_i^k} / \underset{\beta,\alpha+1}{A_i'^k} = i\sqrt{s}, \tag{20}$$

$$\underset{\beta\alpha}{L_i^k} = \underset{\beta\alpha}{A}\,\underset{\beta\alpha}{A_i^k} + \underset{\beta,\alpha+1}{A}\,\underset{\beta,\alpha+1}{A_i^k}, \quad \underset{\beta\alpha}{l_i'^k} = \underset{\beta\alpha}{G}\,\underset{\beta\alpha}{G_i^k} + \underset{\beta,\alpha+1}{G}\,\underset{\beta,\alpha+1}{G_i^k}, \quad \underset{\beta\alpha}{L_i'^k} = \underset{\beta\alpha}{A'}\,\underset{\beta\alpha}{A_i'^k} + \underset{\beta,\alpha+1}{A'}\,\underset{\beta,\alpha+1}{A_i'^k} \tag{21}$$

(β and α are integers, positive or negative, including 0). For $\beta = 0, 1, 2$, we identify the quantities $\underset{\beta,2\alpha}{A}$ with $\underset{\alpha+1}{\Gamma}$, $\underset{\alpha+1}{J}$, $\underset{\alpha+1}{N}$, respectively; similarly the $\underset{\beta,2\alpha}{a}$ are identified with $\underset{\alpha+1}{\gamma}$, $\underset{\alpha+1}{j}$, $\underset{\alpha+1}{n}$, $= -i\underset{\alpha+1}{V}$, the quantities $\underset{\beta,2\alpha+1}{A}$ with $\underset{\alpha+1}{P}$, $\underset{\alpha+1}{H}$, $\underset{\alpha+1}{F}$, the quantities $\underset{\beta,2\alpha+1}{a}$ with $\underset{\alpha+1}{p}$, $\underset{\alpha+1}{h}$, $\underset{\alpha+1}{f}$, $= i\underset{\alpha+1}{U}$; then from Eq. (17) we have

$$\underset{\alpha}{P} / \underset{\alpha}{H} = \underset{\alpha}{H} / \underset{\alpha}{F} = \sqrt{\varepsilon}, \quad \underset{\alpha}{\Gamma} / \underset{\alpha}{P} = \underset{\alpha}{P} / \underset{\alpha+1}{\Gamma} = i\sqrt{s}. \tag{17'}$$

From Eq. (18), to each pair of values of $\underset{\beta\alpha}{A}$, $\underset{\beta+2,\alpha}{a}$ there correspond four values of $\pm\underset{\beta,\alpha-1}{A}$, $\pm\underset{\beta,\alpha+1}{A}$ and $\pm\underset{\beta+2,\alpha-1}{a}$, $\pm\underset{\beta+2,\alpha+1}{a}$. For instance, to each pair of $\underset{\alpha}{P}$, $\underset{\alpha}{U}$ there correspond four values of $\pm\underset{\alpha}{\Gamma}$, $\pm\underset{\alpha+1}{\Gamma}$ and $\pm\underset{\alpha}{V}$, $\underset{\alpha+1}{V}$ such that $\underset{\alpha}{\Gamma}\,\underset{\alpha+1}{\Gamma} = \underset{\alpha}{P^2}$, $\underset{\alpha}{\Gamma} + \underset{\alpha+1}{\Gamma} = 2\underset{\alpha}{U}$; conversely, to each pair of values of $\underset{\alpha}{\Gamma}$, $\underset{\alpha}{V}$ there correspond four values of $\pm\underset{\alpha}{P}$, $\pm\underset{\alpha-1}{P}$ and $\pm\underset{\alpha}{U}$, $\pm\underset{\alpha-1}{U}$ such that $\underset{\alpha}{P}\,\underset{\alpha-1}{P} = \underset{\alpha}{\Gamma^2}$, $\underset{\alpha}{P} + \underset{\alpha-1}{P} = 2\underset{\alpha}{V}$.

2. We have

$$s = \Big(1 - 4\underset{\beta\alpha}{a^2} / \underset{\beta-1,\alpha+1}{A^2}\Big)^{1/2} = \Big(1 + 4\underset{\beta,\alpha+1}{a^2} / \underset{\beta-1,\alpha+1}{A}\Big)^{-1/2},$$
$$\varepsilon = \Big(1 + \underset{\beta\alpha}{A^2} / \underset{\beta+2,\alpha}{a}\Big)^{1/2} = \Big(1 - \underset{\beta+2,\alpha}{A^2} / \underset{\beta+2,\alpha}{a^2}\Big)^{-1/2}, \tag{22}$$

$$2i\underset{\beta\alpha}{a} / \underset{\beta-2,\alpha+1}{A} = 1 - s, \quad -2i\underset{\beta\alpha}{a} / \underset{\beta,\alpha+1}{A} = 1 + s, \tag{23}$$

$$\underset{\beta\alpha}{L} / \underset{\beta+2,\alpha}{a^2} = 2i\underset{\beta,\alpha+1}{A} / \underset{\beta+2,\alpha}{a} = -2(1-\varepsilon), \quad \underset{\beta\alpha}{L} / \underset{\beta+2,\alpha+1}{a^2} = -2i\underset{\beta\alpha}{A} / \underset{\beta+2,\alpha+1}{a} = -2(1+\varepsilon), \tag{24}$$

$$\underset{\beta,\alpha-1}{Q_i'^k} / \underset{\beta\alpha}{Q_i'^k} = \underset{\beta\alpha}{Q_i'^k} / \underset{\beta,\alpha+1}{Q_i'^k} = 1/i\sqrt{s}, \quad \underset{\beta\alpha}{L_i^k} = -\underset{\beta\alpha}{l_i'^k} + {}^1/_2\underset{\beta\alpha}{L}\delta_i^k = \underset{\beta\alpha}{L_i'^k}, \tag{25}$$

$$\underset{\beta\alpha}{A} = -\underset{\beta\alpha}{G} = \underset{\beta\alpha}{A'}, \quad \underset{\beta\alpha}{L} = \underset{\beta\alpha}{l'} = \underset{\beta\alpha}{L'}, \tag{26}$$

$$\underset{\beta+2,\alpha-1}{a} - \underset{\beta,\alpha-1}{a} = \underset{\beta+2,\alpha+1}{a} + \underset{\beta,\alpha+1}{a} = -i\underset{\beta\alpha}{A}, \quad \underset{\beta-2,\alpha+1}{A} - \underset{\beta-2,\alpha-1}{A} = \underset{\beta,\alpha+1}{A} + \underset{\beta,\alpha-1}{A} = 2\underset{\beta\alpha}{a}, \tag{27}$$

$$\underset{\beta\alpha}{A^2} - \underset{\beta\alpha}{a^2} + \underset{\beta+2,\alpha}{a^2} = 0, \quad 4\underset{\beta+1,\alpha}{a^2} - \underset{\beta,\alpha+1}{A^2} + \underset{\beta,\alpha-1}{A^2} = 0, \quad \underset{\beta\alpha}{L} = \underset{\beta\alpha}{a^2} - \underset{\beta+2,\alpha}{a^2} + \underset{\beta,\alpha+1}{a^2} - \underset{\beta+2,\alpha+1}{a^2}. \tag{28}$$

The quantities in Eq. (23) and (24) and the normalized scalars and Lagrangians of (26) are obtained by contraction of the tensor of (19), (20), and (21), and Eq. (28) follow from (27). From (28) we have ($\underset{\alpha}{f}\underset{\alpha}{\mu} = \underset{\alpha}{q}$, $\underset{\alpha}{n}\underset{\alpha}{\mu} = \underset{\alpha}{m}$, $\underset{\alpha}{f}\sqrt{\underset{\alpha}{\mu}} = \underset{\alpha}{q}'$, $\underset{\alpha}{n}\sqrt{\underset{\alpha}{\mu}} = \underset{\alpha}{m}'$):

$$\underset{\alpha}{P}^2 - \underset{\alpha}{p}^2 + \underset{\alpha}{f}^2 = 0\ (\underset{\alpha}{F}^2 - \underset{\alpha}{f}^2 + \underset{\alpha}{q}^2 = 0), \qquad (28')$$
$$\underset{\alpha}{\Gamma}^2 - \underset{\alpha}{\gamma}^2 + \underset{\alpha}{n}^2 = 0\ (\underset{\alpha}{N}^2 - \underset{\alpha}{n}^2 + \underset{\alpha}{m}^2 = 0);$$

$$\underset{\alpha}{P}^2 - \underset{\alpha-1}{P}^2 - 4\underset{\alpha}{j}^2 = 0\ (\underset{\alpha}{F}^2 - \underset{\alpha-1}{F}^2 - 4\underset{\alpha}{m}'^2 = 0), \qquad (28'')$$
$$\underset{\alpha}{\Gamma}^2 - \underset{\alpha-1}{\Gamma}^2 - 4\underset{\alpha}{h}^2 = 0\ (\underset{\alpha}{N}^2 - \underset{\alpha-1}{N}^2 - 4\underset{\alpha}{q}'^2 = 0).$$

In (18) we have $\underset{00}{L} = 4h_1$, $\underset{10}{L} = 4l_1 = \underset{00}{\mu}\underset{01}{L}$, $L = 4h_2 = -wL$, $\underset{00}{L} = 4l_2 = \mu w\underset{00}{L}$,

$$h_1 + l_1 = {}^1\!/_2\,\underset{1}{J}^2, \quad h_1 - l_1 = -{}^1\!/_2\,\underset{1}{H}^2, \qquad (18')$$
$$h_2 + l_2 = -{}^1\!/_2\,\underset{2}{J}^2, \ h_2 - l_2 = {}^1\!/_2\,\underset{1}{H}^2.$$

The conformal transformations

$$\overset{\beta\alpha}{\gamma}_{ik} = g_{ik}\,\varepsilon^{\beta/2}(-s)^{\alpha/2}\ (g_{ik} = \overset{00}{\gamma}_{ik}) \qquad (29)$$

permute separately the scalars $\underset{\beta\alpha}{A}$, $\underset{\beta\alpha}{a}$, the scalars of (26), the tensors $\underset{\beta\alpha}{Q}{}_i'^k$ and the tensors of (21). For arbitrary β and α, the gravitational equations

$$\underset{\beta\alpha}{L}{}_i^k = \frac{1}{4}\underset{\beta\alpha}{L}\delta_i^k, \quad \text{or} \quad \underset{\beta\alpha}{l}{}_i'^k = \frac{1}{4}\underset{\beta\alpha}{l}'\delta_i^k \qquad (30)$$
$$\text{or} \quad \underset{\beta\alpha}{L}{}_i'^k = \frac{1}{4}\underset{\beta\alpha}{L}'\delta_i^k$$

are therefore equivalent, from the conformal point of view, to Eq. (3). Owing to relations (25) and (26), they are all equivalent for fixed α and β. The components of any tensor density of arbitrary weight and order, for instance the scalars $\underset{\beta\alpha}{A}$ and $\underset{\beta\alpha}{a}$, those of (26), the tensors $\underset{\beta\alpha}{Q}{}_i'^k$, and those of (21) satisfy the relation

$$\underset{\beta+\beta',\,\alpha+\alpha'}{T}\ \underset{\beta-\beta',\,\alpha-\alpha'}{T} = \underset{\beta\alpha}{T}^2. \qquad (31)$$

3. The normalized fundamental scalars of the nonlinear conformal reciprocity theory form a group which is isomorphic to the cubic dihedral (projective) group D_6; in quantum mechanics they are therefore replaced by operator groups. Expressing the relations between the scalars $\underset{\beta\alpha}{A}$, $\underset{\beta\alpha}{a}$, for instance in terms of the quantities Γ, P, U, and V with indices 1 and 2, we obtain the groups B_2, B_1, B_{12}, A_2, A_1, and A_{12}.

$$(B_2): \quad \underset{2}{\Gamma}/\underset{1}{U} = \underset{1}{P}/\underset{1}{V} = 2/(1-s) = 1-\varepsilon,$$
$$-\underset{2}{N}/\underset{1}{U} = -\underset{1}{F}/\underset{1}{V} = 2/(1+s) = 1-\mu,$$

and their inverse quantities and scalars are $p_1/i\underset{1}{U} = \underset{1}{P}/\underset{1}{F} = \underset{1}{\Gamma}/\underset{1}{N} = \underset{2}{\Gamma}/\underset{2}{N} = (s+1)/(s-1) = \epsilon$, $(s-1)/(s+1) = \mu$.

$$(B_1): \quad \underset{1}{\Gamma}/\underset{1}{U} = \underset{1}{P}/\underset{2}{V} = 2/(1-w) = 1+\varepsilon,$$
$$\underset{1}{N}/\underset{1}{U} = \underset{1}{F}/\underset{2}{V} = 2/(1+w) = 1+\mu,$$

and their inverse scalars are $-\epsilon$, $-\mu$.

$$(B_{12}): \quad \underset{1}{P}^2/\underset{1}{U}^2 = \underset{1}{\Gamma}^2/\underset{1}{V}^2 = \underset{2}{\Gamma}^2/\underset{2}{V}^2 =$$
$$-4w/(1-w)^2 = 1-\varepsilon^2,$$
$$-\underset{1}{F}^2/\underset{1}{U}^2 = -\underset{1}{N}^2/\underset{1}{V}^2 = -\underset{2}{N}^2/\underset{2}{V}^2$$
$$= 4w/(1+w^2) = 1-\mu^2,$$

and their inverse quantities are ϵ^2, μ^2.

$$(A_2): \quad 2\underset{1}{U}/\underset{2}{\Gamma} = 2\underset{1}{V}/\underset{1}{P} = 2/(1-\varepsilon) = 1-s,$$
$$2\underset{1}{U}/\underset{1}{\Gamma} = 2\underset{2}{V}/\underset{1}{P} = 2/(1+\varepsilon) = 1-w,$$

and their inverse quantities and scalars are $-\underset{1}{\Gamma}/\underset{2}{\Gamma} = -\underset{1}{P}/\underset{2}{P} = -\underset{1}{V}/\underset{2}{V} = -\underset{1}{U}/\underset{2}{U} = (\epsilon+1)/(\epsilon-1) = s$, w.

$$(A_1): \quad -2\underset{1}{U}/\underset{2}{N} = -2\underset{1}{V}/\underset{1}{F} = 2/(1-\mu) = 1+s,$$
$$2\underset{1}{U}/\underset{1}{N} = 2\underset{2}{V}/\underset{1}{F} = 2/(1+\mu) = 1+w,$$

and their inverse quantities and scalars are $-s$, $-w$.

$$(A_{12}): \quad -4\underset{1}{U}^2/\underset{2}{J}^2 = -4\underset{1}{V}^2/\underset{1}{H}^2 = -4\mu/(1-\mu)^2$$
$$= 1-s^2,$$
$$4\underset{1}{U}^2/\underset{1}{J}^2 = 4\underset{2}{V}^2/\underset{1}{H}^2 = 4\mu/(1+\mu)^2 = 1-w^2,$$

and their inverse quantities and scalars are s^2, w^2.

More generally, the groups A and B describe the normalized fields (curvatures) of (23) and the normalized Lagrangians of (24) for *arbitrary* indices β and α. Interchange of the conformal factors $\epsilon \leftrightarrow s$, $\mu \leftrightarrow w$ causes interchange of the groups according to $B_1 \leftrightarrow A_1$, $B_2 \leftrightarrow A_2$, $B_{12} \leftrightarrow A_{12}$. In the same way $\epsilon \leftrightarrow -\epsilon$, $s \leftrightarrow w$ leads to $B_2 \leftrightarrow B_1$. Similarly, $s \leftrightarrow -s$, $\epsilon \leftrightarrow \mu$ leads to $A_2 \leftrightarrow A_1$. The elements of the groups A_{12} and B_{12} are products of the elements of A_1, A_2 and B_1, B_2. In this way we again obtain, for arbitrary distributions and motion of matter, the cubic dihedral group D_6 as described in the author's dissertation[9] (as well as elsewhere[11, 28]), independent of linear electrodynamics, in the capacity of the basis for a certain group algebra (the subalgebra of a generalization of the Dirac algebra) and a certain isomorphic operator or field-function group which corresponds to a spherical distribution of particle and field. The group $D_6 = C_3 C_2'$ is a direct product of the two cyclic subgroups C_3, C_2'. The group C_3 corresponds to a triplet: the kinematic variables (space-time), the 4-current variables (3-current–charge), and the dynamical variables (momentum–energy) (and therefore to a pair of C_2: 4-current–kinematic and dynamical variables). The subgroup C_2' corresponds to the pair: gravitation-electromagnetism (and therefore essentially to a triplet C_3': electricity–gravitation–magnetism).

III. CONSTANT AND VARIABLE QUANTITIES

1. Let us set $\delta = \sqrt{k}/c$. Let, in Kalantarov's[10] system of units, $\epsilon_0 \epsilon'$ be the dielectric constant (of dimensionality $Q\Phi^{-1}L^{-1}T$), $\mu_0 \mu'$ the magnetic permeability (of dimensionality $Q^{-1}\Phi L^{-1}T$), $q_0 = e'\sqrt{\epsilon_0}/\delta$, $\phi_0 = \phi_0'/C = e''/\sqrt{\epsilon_0}\delta c$ the electric and magnetic units of charge (of dimensionality Q and Φ)(canonical conjugates), and $e_0 = e/\delta$ the mechanical ("gravitational") unit of charge [of dimensionality $(Q\Phi L T^{-1})^{1/2}$]. In this system of units the dimensionalities of the fields P, H, F (as also the appropriate mixed components) are QL^{-2}, $(Q\Phi L^{-3}T^{-1})^{1/2}$, $\Phi L^{-1}T^{-1}$, respectively. Let a be the radius of a charged elementary particle, and $W = m/\delta^2$ be its energy in gravitational (mechanical) units. We have the following "conformal" relations with the constant coefficients $(\epsilon_0 \epsilon')^{1/2} = (\mu_0 \mu' c^2)^{-1/2}$, $\delta\lambda = 1/\delta'\lambda'$ (where ϵ', μ', λ, λ' are dimensionless)[10]

$$q_0/e_0 = e_0/\varphi_0' = (\varepsilon'\varepsilon_0)^{1/2}(e'/e = e/e'' = \sqrt{\varepsilon'}), \quad (32)$$

$$a/e_0 = e_0/W = \lambda\delta \; (a/e = e/m = \lambda)$$

and similar relations for the "quanta" of time $b_0 = b/c$ (Ambartsumian, Ivanenko[29]), momentum, and current, related by powers of the constant "conformal" factor $\delta c \lambda = \sqrt{k}\lambda$. Since h is the quantum of action, and $l^2 = hk/c \approx 10^{-64}$ cm^2, the canonically conjugate quantities are related by expressions of the type

$$b_0 W = h \quad \text{or} \quad bm = l^2. \quad (32')$$

2. We have the associations P_α, H_α, $F_\alpha \leftrightarrow \Gamma_\alpha$, J_α, N_α and P_α, $\Gamma_{\alpha+1}$, $P_{\alpha+1} \leftrightarrow \Gamma_\alpha$, P_α, $\Gamma_{\alpha+1}$, which are determined by by the conformal factors $\sqrt{\epsilon} = 1/\sqrt{\mu}$, $i\sqrt{s} = i/\sqrt{w}$ (and similarly for all $A_{\beta\alpha}$ and $a_{\beta\alpha}$). The close relation deduced above between the field f_{ik} and the curvature tensor Γ_{ik} is in some sense similar to the relation noted within the framework of Fock's[30] quantum theory. The conformal relations (17′)[or the more general relations (17), (31)] correspond exactly to the "conformal" relations (32). The constants $\epsilon'\epsilon_0$ and $\lambda^2 k$ [or $(\epsilon'\epsilon_0)^{1/2}$ and $\lambda\delta$] become the variable quantities $\epsilon_1 = 1/\sqrt{\epsilon_2} = \epsilon\epsilon_0 c = (\mu\mu_0 c)^{\;1}$ and $k_1 = 1/k_2 = sk = k/w$. The conform invariant quantity $\sqrt{\epsilon_1} = 1/\sqrt{\epsilon_2}$ relates the magnitudes of the triplet C_3' (and therefore those of the pair C_2'). The conform invariant quantity $\sqrt{k_1} = 1/\sqrt{k_2}$ relates the magnitudes of the triplet C_3 (and therefore of the pair C_2).

3. The quantities ϵ and s are associated with themselves as a result of reciprocity. The replacements $s \leftrightarrow w$ and $\epsilon \leftrightarrow -\epsilon$ are equivalent, as is similarly true for the replacements $\epsilon \leftrightarrow \mu$ and $s \leftrightarrow -s$. The variability of the dielectric constant (or the nonlinearity of electrodynamics) and the variability of the gravitational constant (postulated by Dirac and developed in the projective theory of relativity[31-34]) are equivalent, in a certain sense, and follow automatically from the gravitational equations of the conformal reciprocity theory developed above and therefore from the Lagrangian density $L\sqrt{g} = (\Gamma^2 + P^2)\sqrt{g}$. We shall differentiate between two types of nonlinearity. The first kind of nonlinearity of the gravitational field is due to the nonlinearity of the gravitational curvature tensor Γ_{ik} with respect to the electromagnetic potentials g_{ik} (or $u_{ik} = \Psi g_{ik}$) and their first derivatives and to the nonlinearity of the electromagnetic curvature tensor P_{ik} with respect to the electromagnetic potentials g_i (or $p_i = \Psi^{1/2} g_i$) and their first derivatives. Nonlinearities of the second kind arise from the fact that the Lagrangian L is quadratic (that is, the density $L\sqrt{g}$ is conform invariant) and therefore the gravitational equations (3) and equations (9) are quadratic in Γ, P (and ϕ). The second kind of nonlinearity (and therefore the existence of "maximum" fields) reflects the fact that it is inconsistent to treat the separate fields, or the quantities C_3 and C_3' (as well as meson, spinor, etc. fields) alone, and thus reflects the reciprocity principle inherent in the structure of the field equations and in the many-dimensional conformal metric (see Section IV).

4. The "maximum" electromagnetic field $U = -if = L/2\Gamma$ (and all $\underset{\alpha}{U} = -i\underset{\alpha}{f}$) and the "maximum gravitational field" $-iV = n = -iL/2P$ (and all $-i\underset{\alpha}{V} = \underset{\alpha}{n}$), as opposed to those of the Born-Infeld theory, are variable and given by the conformal field equations. As a nonlinear theory, the conformal reciprocity theory bears the same relation to the Born-Infeld theory of relativity, based on a variable maximum velocity $v_4 = cv_4' = i(g_{44})^{1/2}$, bears to the special theory in which the maximum velocity c is a constant. With the aid, however, of the conformal transformations $\underset{\alpha}{g^*_{ik}} = g_{ik}\underset{\alpha}{U}$, $\underset{\alpha}{g^{**}_{ik}} = -ig_{ik}\underset{\alpha}{V}$, the fields $\underset{\alpha}{U}$ and $-i\underset{\alpha}{V}$ in terms of the metrics $\underset{\alpha}{g^*_{ik}}$ (and $\underset{\alpha}{g^{**}_{ik}}$, respectively) become quantum constants $\underset{\alpha}{U^*} = -i\underset{\alpha}{V^{**}}$, which we have taken above equal to we have taken above equal to unity. In the metrics $\underset{1}{g^*_{ik}}$ and $\underset{2}{g^{**}_{ik}}$, L becomes linear: $L^* = 2\Gamma^*$ (Einstein's Lagrangian), $L^{**} = 2iP^{**}$ [see Eq. (9′)]. The gravitational equations (4) become

$$L_i^{*k} - \frac{1}{2}\Gamma^*\delta_i^k = 0 \quad \text{or} \quad L_i^{**k} - \frac{1}{2}P^{**}\delta_i^k = 0. \tag{33}$$

Similar results hold for arbitrary β or α. In analogy with the (nonconformal covariant) theory of the Broglie's double solution, the continuous fields (F^*, N^{**}) and singular fields ($-i\underset{1}{P^*} \to \underset{1}{\Gamma^{**}} \to \infty$ as $\epsilon \to \infty$, $s \to 1$, and analogously $-i\underset{0}{P^*} \to \infty$ as $s \to \infty$, $\epsilon \to 1$ and $\underset{1}{\Gamma^{**}}$ is finite) are solutions of the same conformal equations.

The metric $\underset{1}{g^*_{ik}}$ corresponds to the Born-Infeld theory. A "point" particle (singularity of the matter field), described by the quantities p^*_{ik}, Γ^{**}_{ik} is only the first approximation to an actual "smeared out" particle, identified with the matter field described by the quantities F^*_{ik}, N^{**}_{ik}. In the case $P \neq 0$ and $F \neq 0$, the limiting value $\epsilon = \mu = 1 (s = 1/w = \infty)$, corresponding to $\underset{1}{U} \to -i\underset{1}{V} \to \infty$ and $\underset{2}{V} \to \underset{1}{F}/2$, implies that $\underset{1}{\Gamma} \to \underset{1}{N} \to \infty$, $\underset{00}{L} \to \infty$, $\underset{2}{\Gamma} \to \underset{2}{N} \to 0$, and $\underset{01}{L} \to \underset{1}{F}^2$. The scalar $l_2 = \underset{01}{L}/4$ therefore approaches the Maxwell Lagrangian $F^2/4$. In the case $\epsilon = \mu = 1 (s = 1/w = \infty)$ and $P = F = 0$, we have $\underset{1}{U} \to \underset{1}{\Gamma}/2$, $\underset{00}{L} \to 4\underset{1}{U}^2 \to \underset{1}{N}^2 \to \underset{1}{\Gamma}^2$, and $\underset{01}{L} \to \underset{2}{N} \to \underset{2}{\Gamma} \to 0$.

IV. THE METRIC OF RECIPROCITY THEORY

1. Let $ds = ds_1/f$, $d\bar{s} = id\bar{s}_1$, $d\bar{\bar{s}} = (ds\,d\bar{s})^{1/2} = (id\bar{s}_1\,ds_1/f)^{1/2}$ be, respectively, the kinematic (space-time), dynamic (momentum-energy) and 4-current (3-current—charge) "world" intervals. Let $dS = idS_1/f$, $d\bar{S} = d\bar{S}_1$, $d\bar{\bar{S}} = (dS\,d\bar{S})^{1/2} = (id\bar{S}_1\,dS_1/f)^{1/2}$ be the corresponding "proper intervals ($\underset{1}{f} = f$, $\underset{1}{p} = p$, $\underset{1}{F} = F$, $\underset{1}{P} = P$). Let

$$d\tau^2 = d\bar{s}_1^2 + ds_1^2 + 2g_1\,d\bar{s}_1\,ds_1, \tag{34}$$
$$dT^2 = -(d\bar{S}_1^2 + dS_1^2 + 2G_1\,d\bar{S}_1 dS_1).$$

The total conformal interval (with the same dimensions as $d\bar{s}^2$, that is those of the energy-momentum tensor) will be

$$d\tau'^2 = d\tau^2 + dT^2 = 0; \tag{35}$$

$d\tau^2$ is a "kinematic" quantity (C), a "dynamic" quantity (D), or an "isobaric" quantity, depending on whether $d\tau^2$ is less than, greater than, or equal to zero. Let $g_1 = G_1 = 0$. If $d\tau^2 < 0 (d\bar{s}^2 < f^2\,ds^2)$ or, respectively, $d\tau^2 > 0 (d\bar{s}^2 > f^2\,ds^2)$, there exists a standard coordinate system in which

$$(C)\quad d\bar{S_1} = 0,\ ds_1 = p^* dS_1,\ ds_1 = iP^* dS_1, \tag{36}$$
$$(d\bar{S} = 0,\ ds = \varepsilon dS,\ d\bar{s} = F ds = P dS);$$
$$(D)\quad ds_1 = 0,\ d\bar{S}_1 = p^* d\bar{s}_1,\ dS_1 = iP^* d\bar{s}_1,$$
$$(ds = 0,\ d\bar{S} = \varepsilon d\bar{s},\ dS = F d\bar{S}/f^2 = P d\bar{s}/f^2) \tag{36′}$$

are the conformal relations between the "world" and "proper" kinematic and dynamic intervals. The interval (35) is therefore equivalent to (28′). When the "world" and "proper" kinematic and dynamic quantities are interchanged, Eq. (36) and (36′) are interchanged.

By reciprocity with the space interval $d\sigma^2$ or the momentum interval dl^2, the dynamic interval $d\bar{s}^2$ probably corresponds to r rows of four variables $\overset{a}{x}{}^i$ (where $a = 1, 2, \ldots, r$) analogous to the variables x^i (space-time) of the intervals ds^2. Then $d\bar{s}_1^2$ is a quadratic form in the variables $d\overset{a}{x}{}^i$, the metric $d\overset{a}{s}$, or the Pfaffian $d\overset{a}{\varphi}$ (real or matrix); the product $2g_1 d\bar{s}_1 ds_1$ is accordingly replaced by $2\lambda_{aik} d\overset{a}{x}{}^i dx^k$, $2\lambda_a d\overset{a}{s} ds$, or $2\lambda_a' d\overset{a}{\varphi} d\varphi$; similar statements hold for the proper quantities $d\bar{S}_1^2$ and $2G_1 d\bar{S}_1 dS_1$.

2. From the above we obtain a correspondence principle between relativity theory and reciprocity theory. The fact that the interval $d\tau^2$ breaks up into a dynamic $d\bar{s}^2$ and kinematic ds^2 interval and mixed terms $d\bar{s}ds$, corresponds exactly to the general relativistic case, that is, the splitting of the $d\bar{s}^2 = dl^2 - dW^2/v_4^2 + 2h'_\gamma dl^\gamma dW$ $(\gamma = 1, 2, 3)$ into the intervals dl^2 (momentum), dW^2 (energy), and mixed terms, as well as the splitting of the interval $ds^2 = d\sigma^2 - v_4^2 dt^2 + 2l'_\gamma dx^\gamma dt$ into the space interval $d\sigma^2$, the time interval dt^2, and mixed terms. The case $G_1 = g_1 = 0$ corresponds to the general relativistic static case.

From the above we obtain relations between the zero weight quantities $F/f = P/p = iF^*$ and $v/v_4 = u/u_4 = iv$,

$$\varepsilon = 1/\mu, \quad s = 1/w \text{ and } z = 1/z' = (x+1)/(x-1) = (1 + u^2/v_4^2)^{1/2} \tag{37}$$
$$= (1 - v^2/v_4^2)^{-1/2}, \quad x = 1/x' = (z+1)/(z-1).$$

We also obtain definite relations between the conformal many-dimensional components of the "proper" electromagnetic field ("dynamic" field $F = d\bar{s}/ds$ corresponding to the force $q_0 F = q_0 d\bar{s}/ds$; the "kinematic" field $f = f ds/ds$ equal to the maximum field which corresponds to the maximum force $q_0 f$; the field $q = f\mu$) and between the conformal components of the velocity: the spatial velocity $v = d\sigma/dt$ ($= dW/dl$ for $v_4 =$ const), the time-component of the velocity $v_4 = v_4 dt/dt = v_4 dl/dl$ (= maximum velocity = the general relativistic velocity of light $v_4 = cv_4' = -i(g_{44})^{1/2}$) and the velocity $w_0 = v_4 z'$. Similarly, the "world" electromagnetic field ("dynamical" field $P = \epsilon F = d\bar{s}/dS$, "kinematic" field $p = \epsilon f = f ds/dS$, and the field $f = p\mu$) corresponds to the world velocity [spatial velocity $u = zv = d\sigma/ds'$ ($= dW/d\bar{s}$ for $v_4 =$ const), the time-component of the velocity $u_4 = zv_4 = v_4 dt/ds'$ ($= v_4 dl/ds$ for $v_4 =$ const), and the velocity $v_4 = u_4 z'$ ($ds = iv_4 ds'$)]. The absolute values of Equation (28′) of the "world" and "proper" electromagnetic fields correspond, therefore, to the absolute values of the world and proper velocities: $u^2 - u_4^2 + v_4^2 = 0$ $(v^2 - v_4^2 + w_0^2 = 0)$. The relations

$$\underset{D}{d\bar{s}} = \mu \underset{D}{d\bar{S}}, \quad \underset{C}{ds} = \varepsilon \underset{C}{dS} \tag{38}$$

[see Eq. (36), (36′)] are analogous to the contraction of length and slowing down of time for a moving particle; the first relation indicates the mass defect of a particle interacting with the aggregate, and the second, the corresponding elongation of the interval; the invariant product $\underset{C}{ds}\underset{D}{d\bar{s}} = \underset{C}{dS}\underset{D}{d\bar{S}}$ is analogous to an invariant space-time hypervolume.

Under the conformal transformation (29), the electromagnetic quantities P, H, F and p, h, f, which are related by the many-dimensional relations (34)–(38), transform into a column of the matrices $\|A\|$ and $\|a\|$. Thus, Eq. (34)–(38) and the correspondence between the fields and velocities are valid for arbitrary indices β, α, or for gravitational and other fields, as well as the electromagnetic.

3. We can formulate our conclusions more generally. Let $\overset{\beta\alpha}{V}_4$ be the manifold with the interval $\overset{\beta\alpha}{ds}_1$ (where β, $\alpha = 0, 2, 4$); $\overset{\beta 4}{V}_4$, $\overset{\beta 2}{V}_4$, $\overset{\beta 0}{V}_4$ and $\overset{4\alpha}{V}_4$, $\overset{2\alpha}{V}_4$, $\overset{0\alpha}{V}_4$ are the kinematic, 4-current, and dynamic manifolds, and correspondingly the electric ("world"), gravitational ("mixed"), and magnetic ("proper") manifolds. For $G_1 = g_1 = 0$, let $\overset{\beta\alpha}{ds}_1 = \overset{\beta\alpha}{ds} f^{\alpha/4}$ in absolute value. For $\beta\alpha = 00$, 04, 40, and 44 the interval $\overset{\beta\alpha}{ds}_1$ coincides with $d\bar{S}_1$, dS_1, $d\bar{s}_1$ and ds_1 respectively, and $\overset{\beta\alpha}{ds}$ coincides with $d\bar{S}$, dS, $d\bar{s}$, and ds. In the special coordinate systems corresponding to Eq. (36) (case C) and (36′) (case D), respectively, we have

$$(C) \qquad \|\overset{\beta\alpha}{ds'}\| = ds \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & q^{1/2} \\ F & F^{1/2} & 1 \end{pmatrix}$$
$$\overset{\beta\alpha}{ds}_1 = \overset{\beta\alpha}{ds'} f^{(\alpha+\beta-4)/4} \tag{39}$$
$$(\overset{44}{ds}_1/f = \overset{44}{ds} = \overset{44}{ds'} = ds_1/f = ds)$$

$$(D)\qquad \|\overset{\beta\alpha}{ds'}\| = d\bar{S}\begin{pmatrix} 1 & F^{1/2} & F \\ q^{1/2} & 0 & 0 \\ q & 0 & 0 \end{pmatrix} \qquad (39')$$

$$\overset{\beta\alpha}{ds_1} = \overset{\beta\alpha}{ds'} f^{-(\alpha+\beta)/4}$$

$$(\overset{00}{ds_1} = \overset{00}{ds} = \overset{00}{ds'} = d\bar{S}_1 = d\bar{S})$$

In case C [see Eq. (39)] $\overset{44}{ds'}$, $\overset{42}{ds'}$, $\overset{40}{ds'}$ and $\overset{44}{ds'}$, $\overset{24}{ds'}$, $\overset{04}{ds'}$ are obtained from $ds = \overset{44}{ds'}$ by multiplication by the conformal factors 1, $F^{1/2} = (-F_0 w)^{1/2}$, $F = -F_0 w$ and 1, $q^{1/2} = (f\mu)^{1/2}$, $q = f\mu$. These conformal relations, based on the reciprocity of the conformal factors $w = 1/s$ and $\mu = 1/\epsilon$, F and q, can be combined with Eq. (17′) and (32). In case D [Eq. (39′)] the indices 0 and 4 commute, so that the world and proper, kinematic and dynamic manifolds commute (in particular $\overset{44}{ds} = ds$ and $\overset{00}{ds} = d\bar{S}$).

4. In case C we have the conformal relation

$$\overset{4\alpha}{ds'} / \overset{2\alpha}{ds'} = \overset{2\alpha}{ds'} / \overset{0\alpha}{ds'} = 1/q\ (= \varepsilon / f),$$
$$\overset{\beta 4}{ds'} / \overset{\beta 2}{ds'} = \overset{\beta 2}{ds'} / \overset{\beta 0}{ds'} = 1/F\ (= -s/F_0) \qquad (40)$$

and the more general conformal relations, which follow from (46) [or (48), (48′)] of the form

$$\underset{C}{ds}\underset{D}{ds} = \underset{C}{dS}\underset{D}{dS} \quad \text{or} \quad \underset{C}{\overset{44}{ds}}\,\underset{D}{\overset{40}{ds}} = \underset{C}{\overset{04}{ds}}\,\underset{D}{\overset{00}{ds}} \qquad (41)$$

In case D the indices 0 and 4 are interchanged in (40). More generally, the components T of any relative tensor of zero order and arbitrary weight (depending on $r' \gtrless 4$ variables) satisfy, in $\overset{\beta\alpha}{V_4}$, the conformal relations

$$\overset{4\alpha}{T}\overset{0\alpha}{T} = (\overset{2\alpha}{T})^2,\ \overset{\beta 4}{T}\overset{\beta 0}{T} = (\overset{\beta 2}{T})^2,\ (\underset{C}{\overset{44}{T}}\underset{D}{\overset{40}{T}} = \underset{C}{\overset{04}{T}}\underset{D}{\overset{00}{T}}). \qquad (42)$$

In particular, for the components of F, q we have $\underset{C}{\overset{42}{F}} = \underset{C}{\overset{24}{q}} = \underset{D}{\overset{02}{F}} = \underset{D}{\overset{20}{q}} = i$.

Eqs. (40) [as well as (41) and (42)] generalize Eqs. (32), (32′) with the constant conformal factors $\lambda\delta$, $(\epsilon'\epsilon_0)^{1/2}$, which are valid for an elementary particle as a whole; they also generalize the uncertainty relations, and in particular Eqs. (41) generalize Eqs. (32′) for canonically conjugate quantum quantities. Eqs. (40)–(42) reflect to some extent the uncertainty of field quantities, as noted by Bogoliubov and co-workers[35,36], Markov[37], Born, Raiskii[38], Yukawa and others.

5. The above conformal theory generalizes the reciprocity principle[2-12, 28], which leads to covariance of physical laws [see Eqs. (3), (4), (6), and (35)] with respect to the variables of the manifold $\overset{\beta\alpha}{V_4}$, and therefore with respect to pairs of opposite variable forms of the existence (I) and attributes (fundamental properties) (II) of matter. In (I) we include the space-time variables, in I′ the space and proper variables (in the relativistic sense), in II the kinematic and dynamic variables, and in II′ the "world" (electric) and "proper" (magnetic) variables. Each pair corresponds to canonically conjugate quantities. Mixed products, that is elements of the type $dx_\gamma dt$ [space-time ($\gamma = 1, 2, 3$], $dt\, ds$ (world-proper element), $d\bar{s}\, ds$ (4-current), $ds\, dS$ ("gravitational" element), correspond to "self-conjugate" quantities. With respect to I, the kinematic variables II break up into coordinates and time, the dynamical variables into momentum and energy, and the 4-current into the 3-current and charge. The elements of II break up in the same way with respect to II′, I′. The conformal factor $\sqrt{k_1} = 1/\sqrt{k_2}$ (or $\delta\sqrt{s} = \delta/\sqrt{w}$) thus relates the variables of II and the 4-current (that is the fields $\underset{\alpha}{\Gamma}$, $\underset{\alpha+1}{\Gamma}$, $\underset{\alpha}{P}$), and the conformal factor $\sqrt{\epsilon_1} = 1/\sqrt{\epsilon_2}$ [or $\sqrt{\epsilon\epsilon_0} = \mu\mu_0 c^2)^{-1/2}$] relates the variables of II′ and the mixed (gravitational) variables (in other words the fields $\underset{\alpha}{\Gamma}$, $\underset{\alpha}{N}$, $\underset{\alpha}{J}$, or $\underset{\alpha}{P}$, $\underset{\alpha}{F}$, $\underset{\alpha}{H}$). I, II, I′, and II′, correspond to the cyclic groups C_2' or C_3, or to the Pauli groups[9, 10, 11, 28]. The complete abstract group[9, 10, 11, 28] (their direct product) is represented by groups of operators or functions and the elements of a certain isomorphic group algebra which is a generalization of the Dirac algebra.

The above theory can be extended to the case of complex Langrangians[39,40], nonlinear Infeld action functions[25], nonsymmetric g_{ik}[7,8,23-25], as well as to the cases of meson and spinor fields[7,8,15,23,37,41-44].

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