Hydrodynamics of Solutions of Two Superfluid Liquids

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(Submitted to JETP editor February 15, 1956)

(J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 653-657 (April, 1957)

The hydrodynamical equations are obtained for solutions of two superfluid liquids on the basis of the conservation laws. In this case, three independent motions are possible in the liquid: a normal one with velocity v_n and two superfluid (potential) ones with velocities v_s' and v_s''. Three different types of sound vibrations can be propagated in such solutions.

WE CONSIDER here two superfluid liquids, for example, liquid He^4 and liquid He^6. In such a case, three possible motions are possible (in principle) in the liquid: the normal motion with velocity $\mathbf{v}_n$, and two superfluid motions with velocities $\mathbf{v}_s'$ and $\mathbf{v}_s''$. We shall show how the equations of this three-velocity hydrodynamics can be obtained from the conservation laws and from the principle of the potential character of the superfluid motion. In the derivation of these equations, we shall consider the liquid in a frame of reference in which the normal (non-superfluid) portion of the liquid is at rest. In this system, the total energy of the liquid can be written in the form

$$E = \rho \mathbf{v}_n^2 / 2 + (\mathbf{p}' + \mathbf{p}'') \mathbf{v}_n + \varepsilon. \tag{1}$$

Here we have adopted the following notation: ρ is the density of the liquid, $\mathbf{p}'$ and $\mathbf{p}''$ the respective momenta of the two superfluid motions, and ε the internal energy of the liquid. The latter is a function both of the thermodynamic variables density ρ_1, ρ_2 and entropy S, and of the relative velocities $\mathbf{v}_s' - \mathbf{v}_n$ and $\mathbf{v}_s'' - \mathbf{v}_n$ and is defined by a thermodynamic identity.

The form of the thermodynamic identity is established in the following way. We recall the expression for the total energy E which we used in the derivation of the hydrodynamical equations for He II[1]:

$$E = \rho \mathbf{v}_s^2 / 2 + \mathbf{p}' \mathbf{v}_s + \varepsilon'. \tag{2}$$

Here $\mathbf{p}'$ is the momentum of the relative motion in the reference system, which moves with velocity $\mathbf{v}_s$; it is expressed in terms of the total momentum of the liquid $\mathbf{j}$:

$$\mathbf{p}' = \mathbf{j} - \rho \mathbf{v}_s. \tag{3}$$

The energy is defined by the thermodynamic identity

$$d\varepsilon' = \mu' d\rho + T dS + (\mathbf{v}_n - \mathbf{v}_s, d\mathbf{p}'). \tag{4}$$

If we replace the momentum $\mathbf{p}'$ by the momentum $\mathbf{p}$ in the reference system which moves with velocity $\mathbf{v}_n$:

$$\mathbf{p} = \mathbf{j} - \rho \mathbf{v}_n = \mathbf{p}' + \rho (\mathbf{v}_s - \mathbf{v}_n), \tag{5}$$

the expression for the energy E reduces to the form

$$E = \rho \mathbf{v}_n^2 / 2 + \mathbf{p} \mathbf{v}_n + \varepsilon, \tag{6}$$

where the internal energy ε is already defined by a different identity than Eq. (4), namely,

$$d\varepsilon = \mu d\rho + T dS + \mathbf{p} d (\mathbf{v}_s - \mathbf{v}_n) \tag{7}$$

with the chemical potential μ connected with μ' by the relation

$$\mu = \mu' + (\mathbf{v}_n - \mathbf{v}_s)^2 / 2.$$

Thus, when we transform to the reference system associated with the normal motion, we must regard the internal energy ε as a function of the density, entropy and relative velocity. In this case, Eq. (7) is also a definition of the momentum of the relative motion $\mathbf{p}$.

In the case of three-velocity hydrodynamics, the thermodynamic identity is written, in analogy with Eq. (7), as

$$d\varepsilon = \mu_1 d\rho_1 + \mu_2 d\rho_2 + T dS + \mathbf{p}' d (\mathbf{v}_s' - \mathbf{v}_n) + \mathbf{p}'' d (\mathbf{v}_s'' - \mathbf{v}_n), \tag{8}$$

μ_1 and μ_2 are the chemical potentials of the components of the solution. The densities ρ_1 and ρ_2 are expressed in terms of the concentration c of the solution

$$\rho_1 = \rho c, \quad \rho_2 = \rho (1 - c). \tag{9}$$

We now write down the conservation laws for the energy E, the momentum of the liquid $\mathbf{j} = \mathbf{p}' + \mathbf{p}'' + \mathbf{v}_n$, the mass, and the entropy. For the energy we have:

$$\dot{E} + \operatorname{div} \mathbf{Q} = 0, \tag{10}$$

$\mathbf{Q}$ is the vector flow of energy; its form is unknown to us at present. The time derivative of the momentum j must be equal to the divergence of some tensor:

$$\dot{j}_i + \partial \Pi_{ik} / \partial x_k = 0. \quad (11)$$

The form of the symmetric tensor of the momentum flux Π_{ik} can be established as was done in the derivation of the hydrodynamical equations of a superfluid[1]. We represent the tensor Π_{ik} in terms of its value π_{ik} in a reference system moving with velocity v_n:

$$\Pi_{ik} = \rho v_{ni} v_{nk} + (p_i' + p_i'') v_{nk} + (p_k' + p_k'') v_{ni} + \pi_{ik}. \quad (12)$$

We shall give the explicit form of the tensor π_{ik} below.

The mass conservation laws are written as equations of continuity:

$$\dot{\rho}_1 + \operatorname{div}(\mathbf{p}' + \rho_1 \mathbf{v}_n + \mathbf{g}') = 0, \qquad \dot{\rho}_2 + \operatorname{div}(\mathbf{p}'' + \rho_2 \mathbf{v}_n + \mathbf{g}'') = 0, \quad (13)$$

these contain the unknown vectors $\mathbf{g}'$ and $\mathbf{g}''$. Inasmuch as the equation of continuity holds for all liquids,

$$\dot{\rho} + \operatorname{div} \mathbf{j} = 0, \quad (14)$$

the following relation exists between $\mathbf{g}'$ and $\mathbf{g}''$:

$$\mathbf{g}' + \mathbf{g}'' = 0. \quad (15)$$

We write the entropy equation of continuity in the form

$$\dot{S} + \operatorname{div}(S\mathbf{v}_n + \mathbf{f}) = 0 \quad (16)$$

with the unknown vector $\mathbf{f}$ in the energy flux.

We represent the equations for superfluid motion in such a fashion that the conditions curl $\mathbf{v}_s' = 0$ and $\mathbf{v}_s'' = 0$ are satisfied:

$$\dot{\mathbf{v}}_s' + \nabla\left(\varphi_1 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n \mathbf{v}_s'\right) = 0, \qquad \dot{\mathbf{v}}_s'' + \nabla\left(\varphi_2 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n \mathbf{v}_s''\right) = 0. \quad (17)$$

Here φ_1 and φ_2 are unknown functions at present.

Our problem now is this: using the conservation laws, to find the form of the unknown functions. For this purpose, we compute the time derivative of E and, with the aid of the hydrodynamical equations (13), (16) and (17), eliminate all terms that represent total divergences. In accord with Eq. (1), we have:

$$\begin{aligned}\dot{E} &= {}^1/_2 \dot{\rho}' \mathbf{v}_n^2 + \rho \mathbf{v}_n \dot{\mathbf{v}}_n + (\dot{\mathbf{p}}' + \dot{\mathbf{p}}'') \mathbf{v}_n + (\mathbf{p}' + \mathbf{p}'') \dot{\mathbf{v}}_n + \mu_1 \dot{\rho}_1 + \mu_2 \dot{\rho}_2 \\ &\quad + T\dot{S} + (\mathbf{p}', \dot{\mathbf{v}}_s' - \dot{\mathbf{v}}_n) + (\mathbf{p}'', \dot{\mathbf{v}}_s'' - \dot{\mathbf{v}}_n) = \\ &= -{}^1/_2 \dot{\rho}\, \mathbf{v}_n^2 + \dot{\mathbf{j}} \mathbf{v}_n + \mathbf{p}' \dot{\mathbf{v}}_s' + \mathbf{p}'' \dot{\mathbf{v}}_s'' + \mu_1 \dot{\rho}_1 + \mu_2 \dot{\rho}_2 + T\dot{S}.\end{aligned}$$

We further represent all the time derivatives here with the help of Eqs. (13)–(17):

$$\begin{aligned}\dot{E} &= -(\mu_1 - \mathbf{v}_n^2/2) \operatorname{div}(\mathbf{p}' + \rho_1 \mathbf{v}_n + \mathbf{g}') - (\mu_2 - \mathbf{v}_n^2/2) \operatorname{div}(\mathbf{p}'' + \rho_2 \mathbf{v}_n + \mathbf{g}'') \\ &\quad - \mathbf{p}' \nabla(\varphi_1 - \mathbf{v}_n^2/2 + \mathbf{v}_n \mathbf{v}_s') - \mathbf{p}'' \nabla(\varphi_2 - \mathbf{v}_n^2/2 + \mathbf{v}_n \mathbf{v}_s'') - v_{ni} \partial \Pi_{ik}/\partial x_k \\ &\quad - T \operatorname{div}(S\mathbf{v}_n + \mathbf{f}).\end{aligned}$$

Eliminating all total divergences, we ultimately get

$$\begin{aligned}\dot{E} &= -\operatorname{div}\Big\{(\mathbf{p}' + \rho_1 \mathbf{v}_n + \mathbf{g}')\left(\mu_1 - \frac{\mathbf{v}_n^2}{2}\right) + (\mathbf{p}'' + \rho_2 \mathbf{v}_n + \mathbf{g}'')\left(\mu_2 - \frac{\mathbf{v}_n^2}{2}\right) \\ &\quad + T(S\mathbf{v}_n + \mathbf{f})\Big\} + \Big\{-\rho \mathbf{v}_n \nabla \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n(\rho_1 \nabla \mu_1 + \rho_2 \nabla \mu_2 + S\nabla T) \\ &\quad - v_{ni} \frac{\partial \Pi_{ik}}{\partial x_k} - \mathbf{p}' \nabla(\mathbf{v}_n \mathbf{v}_s') - \mathbf{p}'' \nabla(\mathbf{v}_n \mathbf{v}_s'')\Big\} + \{\mathbf{p}' \nabla(\mu_1 - \varphi_1) \\ &\quad + \mathbf{p}'' \nabla(\mu_2 - \varphi_2) + \mathbf{g}' \nabla \mu_1 + \mathbf{g}'' \nabla \mu_2 + \mathbf{f} \nabla T\}.\end{aligned} \quad (18)$$

We now take up the transformation of the second brace in Eq. (18). In this case it is appropiate to represent the tensor π_{ik} in the following fashion ($\mathfrak{M}_{ik}$ is an unknown tensor):

$$\pi_{ik} = (-\varepsilon + \mu_1\rho_1 + \mu_2\rho_2 + TS)\,\delta_{ik} + \mathfrak{M}_{ik}. \quad (19)$$

After substitution of Π_{ik} in the form (12) with this π_{ik} the second brace reduces to

$$-\operatorname{div}\{(\mathbf{j}\,\mathbf{v}_n)\,\mathbf{v}_n + \mathbf{p}'\,(\mathbf{v}_n\mathbf{v}_s') + \mathbf{p}''\,(\mathbf{v}_n\mathbf{v}_s'')\}$$
$$- v_{ni}\frac{\partial}{\partial x_k}\{\mathfrak{M}_{ik} - p_k'\,(v_{si}' - v_{ni}) - p_k''\,(v_{si}'' - \dot{v}_{ni})\}.$$

Making use of this result, we get

$$\dot{E} = -\operatorname{div}\{(\mathbf{p}' + \rho_1\mathbf{v}_n + \mathbf{g}')\,(\mu_1 - \mathbf{v}_n^2/2) + (\mathbf{p}'' + \rho_2\mathbf{v}_n + \mathbf{g}'')\,(\mu_2 - \mathbf{v}_n^2/2)$$
$$+ \mathbf{v}_n\,(\mathbf{j}\mathbf{v}_n) + \mathbf{p}'\,(\mathbf{v}_n\mathbf{v}_s') + \mathbf{p}''\,(\mathbf{v}_n\mathbf{v}_s'')\} + \{\mathbf{p}'\,\nabla\,(\mu_1 - \varphi_1) + \mathbf{p}''\nabla\,(\mu_2 - \varphi_2) \quad (20)$$
$$+ \mathbf{g}'\,\nabla\,(\mu_1 - \mu_2) + \mathbf{f}\nabla T + v_{ni}\frac{\partial}{\partial x_k}\,[p_k'\,(v_{si}' - v_{ni}) + p_k''\,(v_{si}'' - v_{ni}) - \mathfrak{M}_{ik}]\Big\}.$$

In the absence of dissipation, the quantities φ_1, φ_2, f, g′ and $\mathfrak{M}_{ik}$ can depend only on the thermodynamic variables and the velocities, and cannot depend on their time and space derivatives. Examining isolated cases, such as: temperature equal to zero, small concentration of one of the components, *etc.*, we can establish the form of the unknown functions uniquely. To simplify the problem, we draw upon some physical considerations. Thus, in a superfluid, all the entropy is contained in the thermal excitations which partake only of the normal motion. Therefore, the entropy flow is equal to the product Sv_n, and the vector f (which we introduced into the entropy flow) must be set equal to zero. So far as the tensor $\mathfrak{M}_{ik}$ is concerned, we also know its form to a certain degree. Actually, if we introduce the liquid densities connected with the normal (ρ_n) and the two superfluid motions (ρ_s' and ρ_s''), then we can write the tensor Π_{ik} in the form

$$\Pi_{ik} = \rho_n v_{ni}\, v_{nk} + \rho_s' v_{si}' v_{sk}' + \rho_s'' v_{si}'' v_{sk}'' + p\delta_{ik} \quad (21)$$

(p = pressure).

The sum $\rho_n + \rho_s' + \rho_s''$ is equal to the total density of the liquid. The momenta of the relative motion are then equal to

$$\mathbf{p}' = \rho_s'\,(\mathbf{v}_s' - \mathbf{v}_n),\ \ \mathbf{p}'' = \rho_s''\,(\mathbf{v}_s'' - \mathbf{v}_n). \quad (22)$$

Taking (22) and (23) into account, we find that the tensor $\mathfrak{M}_{ik}$ is equal to*

$$\mathfrak{M}_{ik} = p_k'\,(v_{si}' - v_{ni}) + p_k''\,(v_{si}'' - v_{ni}). \quad (23)$$

It now remains for us only to consider the combination of the remaining terms in (20), which do not have the form of a divergence:

$$\mathbf{p}'\,\nabla\,(\varphi_1 - \mu_1) + \mathbf{p}''\nabla\,(\varphi_2 - \mu_2) + \mathbf{g}'\,\nabla\,(\mu_1 - \mu_2).$$

For this purpose, we make use of the limiting case in which the concentration of one of the components is close to zero. Then, as is well known (see Ref. 2) this component will partake entirely of the normal motion, *i.e.*, one of the momenta of the superfluid motion (p′) will be strictly equal to zero, except in the case in which and g′ = – g″ = 0 (see Ref. 2). Inasmuch as the vector g′ is equal to zero in this case for all values of the densities of the components, then, by means of simple calculations, we can conclude that this condition is always satisfied. In similar fashion, we find

$$\mu_1 = \varphi_1,\ \ \mu_2 = \varphi_2,\ \ \mathbf{g}' = 0. \quad (24)$$

Taking (21), (23) and (24) into account, we write down the final form of the hydrodynamical equations for the solution of two superfluid liquids. The equations of continuity:

$$\dot{\rho}_1 + \operatorname{div}\,(\mathbf{p}' + \rho_1\mathbf{v}_n) = 0,$$
$$\dot{\rho}_2 + \operatorname{div}\,(\mathbf{p}'' + \rho_2\mathbf{v}_n) = 0. \quad (25)$$

One of these can be substituted in the equation of continuity for the entire liquid:

$$\dot{\rho} + \operatorname{div}\mathbf{j} = 0,\ \ \mathbf{j} = \rho\mathbf{v}_n + \mathbf{p}' + \mathbf{p}''. \quad (26)$$

The equation of continuity for the entropy:

$$\dot{S} + \operatorname{div} S\mathbf{v}_n = 0. \quad (27)$$

The equations of superfluid motion:

$$\dot{\mathbf{v}}_s' + \nabla\Big(\mu_1 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n\mathbf{v}_s'\Big) = 0,$$
$$\dot{\mathbf{v}}_s'' + \nabla\Big(\mu_2 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n\mathbf{v}_s''\Big) = 0. \quad (28)$$

Equation of motion of the liquid as a whole:

$$\dot{j}_i + \partial\Pi_{ik}/\partial x_k = 0, \quad (29)$$

where the momentum flux tensor is equal to

$$\Pi_{ik} = \rho v_{ni}\, v_{nk} + (p_k' v_{si}' + v_{nk}\, p_i')$$
$$+ (p_k'' v_{si}'' + v_{nk}\, p_i'') + p\delta_{ik}, \quad (30)$$

*It is easy to see that the tensor $\mathfrak{M}_{ik}$ is symmetric, because of (22).

and the pressure is equal to

$$p = -\varepsilon + TS + \mu_1\rho_1 + \mu_2\rho_2. \quad (31)$$

Taking the thermodynamical identity (7) into consideration, we then obtain

$$dp = \rho_1 d\mu_1 + \rho_2 d\mu_2 + SdT - \mathbf{p}' d(\mathbf{v}_s' - \mathbf{v}_n) - \mathbf{p}'' d(\mathbf{v}_s'' - \mathbf{v}_n). \quad (32)$$

Finally, the law of conservation of energy takes the form

$$\dot{E} + \operatorname{div}\mathbf{Q} = 0,$$
$$\mathbf{Q} = (\mathbf{p}' + \rho_1\mathbf{v}_n)\left(\mu_1 - \frac{\mathbf{v}_n^2}{2}\right) + (\mathbf{p}'' + \rho_2\mathbf{v}_n)\left(\mu_2 - \frac{\mathbf{v}_n^2}{2}\right) + \mathbf{v}_n(\mathbf{j}\mathbf{v}_n) + \mathbf{p}'(\mathbf{v}_n\mathbf{v}_s') + \mathbf{p}''(\mathbf{v}_n\mathbf{v}_s''). \quad (33)$$

For low velocities, we find the dependence of the chemical potentials on the relative velocities from Eq. (32):

$$\mu_1 = \mu_{10} + \frac{\rho_s'}{2\rho}(\mathbf{v}_s' - \mathbf{v}_n)^2,$$
$$\mu_2 = \mu_{20} + \frac{\rho_s''}{2\rho}(\mathbf{v}_s'' - \mathbf{v}_n)^2. \quad (34)$$

Here μ_{10} and μ_{20} are the velocity-independent parts of the chemical potentials.

The presence of three motions in solutions of superconducting liquids can lead to a series of singular phenomena. Thus, for example, in such solutions propagation of sound vibrations of three types is possible, with different velocities. In addition to ordinary sound waves, propagation of two types of waves is possible, in which vibrations of temperature and solution concentration occur.

In conclusion, we note that we know of only two superconducting liquids: liquid He^4 and liquid He^6. Unfortunately, the isotope of He^6 is short-lived (half life = 0.8 sec), and this circumstance naturally makes difficult the possibility of experiments with solutions of these liquids.

I consider it my duty to express my deep gratitude to Academician L. D. Landau for valuable discussions.

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[2] L. Landau, and I. Pomeranchuk, Dokl. Akad. Nauk SSSR **59**, 669 (1948).

Translated by R. T. Beyer
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