On the Possibility of Formulating a Meson Theory with Several Fields

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The following cases of interaction of several fields are considered: 1) N fermion and n boson fields with the same coupling constant, 2) two fermion and two boson fields, interacting with different constants, 3) fermion and boson fields with different isotopic properties. In all cases considered the physical charge tends to zero as the extended interaction approaches a point one.

IN articles by Pomeranchuk[1] it has been shown that in pseudoscalar meson theory the interaction constant tends to zero as the transition from an extended interaction to the limit of a point one is carried out. It was assumed that the result does not depend upon the particular way of carrying out the limiting procedure, and therefore a special form proposed by Abrikosov and Khalatnikov[2] was employed.

In the several examples, we consider this conclusion is not changed by the introduction of several interacting fields.

1. As a first example we consider N fermion fields (all with isotopic spin ½) and n boson fields (all scalar or pseudoscalar with isotopic spin 1) interacting with each other with the same coupling constant g. We do not introduce any selection rules, so that all types of fermions can transform into each other, as can all types of bosons (the problem presents itself as a purely methodological one and the relation of the fields introduced to real particles is not considered).

It is necessary to take into account the fact that virtual bosons (or fermions) can be transformed from one type into another. Therefore the Green's function of the bosons D_{ab} is a matrix, in which the indices a and b can take on n values. The diagonal elements D_{aa} correspond to the propagation of a boson of type a without transformation, and the non-diagonal elements D_{ab} to the preparation of a boson of type a with transformation into one of type b.

Neglecting the mass of the bosons, we express

$$D_{ab} = d_{ab} / k^2.$$

In view of the fact that all bosons are equivalent (have the same charge, differences in mass being unimportant in the study of the asymptotic Green's function), all diagonal elements d_{ab} are equal ($d_{aa} \equiv d_1$), as well as all non-diagonal elements ($d_{ab} \equiv d_2$, $a \neq b$). The matrix d_{ab} is connected with the polarization operator in the following way (P is the polarization operator, divided by k^2):

$$d_{ab} + d_{ac} P_{cb} = \delta_{ab}. \tag{1}$$

From the equivalence of all bosons it follows that all elements of P_{ab} are equal ($P_{ab} \equiv P$). Then Eq. (1) gives

$$d_1 + d_1 P + (n-1) d_2 P = 1,$$

$$d_2 + d_1 P + (n-1) d_2 P = 0,$$

from which

$$d_1 = \frac{1+(n-1)P}{1+nP}, \qquad d_2 = -\frac{P}{1+nP}. \tag{2}$$

We consider now a given boson line of a given diagram. To each such line there will correspond a factor

$$nd_1 + n(n-1) d_2 \equiv n\Delta$$

(n cases of propagation of a boson without transformation, $n(n-1)$ with transformation). From Eq. (2) follows

$$\Delta = 1/(1+nP). \tag{3}$$

Analogously, for each fermion line it is necessary to write

$$N\beta_1 + N(N-1)\beta_2 \equiv NB,$$

where β_1 corresponds to propagation of a fermion without transformation, and β_2, with transformation (the fermion Green's function G is, for large momenta, equal to $G = \beta/\mathbf{p}$). Analogous to Eq. (3) we have

$$B = 1/(1 + NM), \tag{4}$$

where M is the mass operator (divided by $\mathbf{p}$).

We note that Δ and B satisfy the same initial conditions as d and β in the theory with one boson and fermion, that is $\Delta(L) = B(L) = 1$, where L is the momentum at which the interaction is cut off. In fact, when their interaction is excluded, the particles cannot undergo mutual transformations and therefore $d_2(L) = \beta_2(L) = 0$.

Now if we turn our attention to the fact that the polarization operator arises from two fermion lines, the mass operator from one fermion and one boson line, and the vertex part from two fermion lines and one boson line, and take into account that all possible transitions of one fermion and boson into another are allowed, then it is immediately obvious that the equations for the Green's function and vertex parts in our case can be obtained from the corresponding equations of the theory with one boson and fermion, using the simple substitutions

$$d \to \Delta, \ \beta \to B, \ \Gamma \to \Gamma, \ g \to N^2 ng.$$

From this it follows that all conclusions which are valid for the theory with one boson and one nucleon can be automatically carried over to the case of many interacting fields considered.

2. Now we consider the case of different (in magnitude or in sign) charges. The general form of the interaction Lagrangian is as follows:

$$\mathscr{L} = \sum_{i,k,l} a_{ik,l} \bar{\psi}_i \psi_k \varphi_l.$$

By using a linear transformation of the fields ψ_1 it is possible to bring the quadratic form $\sum_{ik} a_{ik,l} \bar{\psi}_i \psi_k$ into diagonal form, leaving the Lagrangian of the free fermion field unchanged.

Then

$$\mathscr{L} = \sum_{il} a_{i,l} \bar{\psi}_i \psi_i \varphi_l.$$

The further transformation of the Lagrangian using the substitution $\Phi_i = \sum_i a_{i,l} \varphi_l$ leads — in so far as this transformation is not, in general, orthogonal — to complicated free equations for the boson fields and does not simplify the problem.

We consider the following Lagrangian

$$\mathscr{L} = g_{11}\bar{\psi}_1\psi_1\varphi_1 + g_{22}\bar{\psi}_2\psi_2\varphi_2 + g_{12}\bar{\psi}_1\psi_1\varphi_2 + g_{21}\bar{\psi}_2\psi_2\varphi_1. \tag{5}$$

Looking at the equations for the vertex parts we see that it is sufficient to introduce two vertex parts

$$\Gamma_{\psi_1\psi_1\varphi_1} = \Gamma_{\psi_1\psi_1\varphi_2} \equiv \Gamma_1; \ \Gamma_{\psi_2\psi_2\varphi_1} = \Gamma_{\psi_2\psi_2\varphi_2} \equiv \Gamma_2.$$

From these equalities it follows that the polarization operator (and also the boson Green's function) are symmetrical: $P_{ik} = P_{ki}$. From the equality

$$d_{ij} + d_{ik}P_{kj} = \delta_{ij} \tag{6}$$

we obtain

$$d_{11} = (1 + P_{22})\Delta_1^{-1}; \ d_{22} = (1 + P_{11})\Delta_1^{-1}; \quad d_{12} = -P_{12}\Delta_1^{-1}, \tag{7}$$

$$P_{11} = -1 + d_{22}\Delta^{-1}; \ P_{22} = -1 + d_{11}\Delta^{-1}; \quad P_{12} = -d_{12}\Delta^{-1}, \tag{8}$$

where

$$\Delta = d_{11}d_{22} - d_{12}^2; \quad \Delta_1 = (1 + P_{22})(1 + P_{11}) - P_{12}^2, \ \Delta\Delta_1 = 1. \tag{9}$$

Symbolically, the equation for Γ_1 has the following form (we always consider the problem in the same approximation as in Ref. 3):

$$\Gamma_1 = 1 + \Gamma_1 G_1 \Gamma_1 G_1 \Gamma_1 (g_{11}^2 D_{11} + g_{22}^2 D_{22} + 2g_{11}g_{22}D_{12}), \tag{10}$$

where G_1 is the Green's function for the field ψ_1. Just as in the theory with a single field, we obtain for high momenta: [The case where the boson field has isotopic spin 1 is considered; the arguments of all functions are $\xi = \ln(-p^2/m^2)$. The prime indicates differentiation with respect to ξ.]

$$\alpha_1'/\alpha_1 = -\lambda_1; \ \alpha_2'/\alpha_2 = -\lambda_2; \tag{11}$$

$$\lambda_1 = (\alpha_1^2\beta_1^2/4\pi)(g_{11}^2 d_{11} + g_{12}^2 d_{22} + 2g_{11}g_{12}d_{12}),$$
$$\lambda_2 = (\alpha_2^2\beta_2^2/4\pi)(g_{21}^2 d_{11} + g_{22}^2 d_{22} + 2g_{22}g_{21}d_{12}). \tag{12}$$

For the fermion Green's function we have (the calculation is completely analogous to the theory with a single field[4]):

$$\beta_1'/\beta_1 = 3\lambda_1/2; \ \beta_2'/\beta_2 = 3\lambda_2/2. \tag{13}$$

The polarization operator P_{ij} satisfies equations which can be written symbolically

$$k^2 P_{ij} = \sum_{l=1}^{2} g_{il} g_{jl} G_l \Gamma_l G_l. \tag{14}$$

From this we obtain*

$$P'_{11} = -\frac{1}{\pi}(g_{11}^2\beta_1^2\alpha_1^2 + g_{21}^2\alpha_2^2\beta_2^2), \tag{15}$$
$$P'_{22} = -\frac{1}{\pi}(g_{12}^2\alpha_1^2\beta_1^2 + g_{22}^2\alpha_2^2\beta_2^2),$$
$$P'_{12} = -\frac{1}{\pi}(g_{11}g_{12}\alpha_1^2\beta_1^2 + g_{22}g_{21}\alpha_2^2\beta_2^2).$$

Using Eqs. (7)–(9), it is easy to show that

$$\Delta'/\Delta = 4(\lambda_1 + \lambda_2). \tag{16}$$

From (12) we obtain

$$\frac{\lambda_1'}{\lambda_1} = 2\frac{\alpha_1'}{\alpha_1} + 2\frac{\beta_1'}{\beta_1} + \frac{g_{11}^2 d'_{11} + g_{12}^2 d'_{22} + 2g_{11}g_{12}d'_{12}}{g_{11}^2 d_{11} + g_{12}^2 d_{22} + 2g_{11}g_{12}d_{12}},$$

or, using Eqs. (11), (13), (7)–(9), we have

$$\lambda_1'/\lambda_1 = 5\lambda_1 + 4\lambda_2 - 4\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2(g_{11}g_{22} - g_{21}g_{12})^2\Delta/(4\pi)^2\lambda_1. \tag{17}$$

The quantities λ_1 and λ_2 characterize the effective interaction of two fermions ψ_1, ψ_1 or ψ_2, ψ_2. In addition, the interaction of ψ_1 and ψ_2 can be considered. We denote by λ_3 the corresponding effective charge

$$\lambda_3 = (\alpha_1\alpha_2\beta_1\beta_2/4\pi)[g_{11}g_{21}d_{11} + g_{12}g_{22}d_{22} + (g_{11}g_{22} + g_{12}g_{21})d_{12}]. \tag{18}$$

It is easy to verify that

$$\lambda_3^2 = \lambda_1\lambda_2 - \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2(g_{11}g_{22} - g_{21}g_{12})^2\Delta/(4\pi)^2,$$

and therefore Eq. (17) can be written as

$$\lambda_1'/\lambda_1 = 5\lambda_1 + 4\lambda_3^2/\lambda_1 \tag{19}$$

and analogously

$$\lambda_2'/\lambda_2 = 5\lambda_2 + 4\lambda_3^2/\lambda_2, \quad \lambda_3'/\lambda_3 = 9(\lambda_1 + \lambda_2)/2. \tag{19'}$$

We note that although four constants occur in the Lagrangian (5), only three combinations of these constants have physical meaning, as one can show by making the substitutions

$$\sqrt{g_{11}^2 + g_{12}^2}\,\Phi_1 = g_{11}\varphi_1 + g_{12}\varphi_2,$$
$$\sqrt{g_{22}^2 + g_{21}^2}\,\Phi_2 = g_{21}\varphi_1 + g_{22}\varphi_2.$$

Also, effective charges λ_i correspond to these three combinations of constants.

From Eq. (19) it is seen that all derivatives of λ_i are positive, *i.e.*, λ_i decreases with decreasing momentum.

Following Abrikosov and Khalatnikov,[2] we introduce two limiting momenta: Λ_k for the integration over virtual bosons, and Λ_p for the integration over virtual fermions, with $L_p - L_k = \ln\Lambda_p/\Lambda_k \gg 1$. The Eqs. (19) are valid for $\xi < L_k$ and the values of $\lambda_i(L_k)$ are initial conditions for them. Just as in the theory with a single field, for sufficiently large $L_p - L_k$ the values of $\lambda_i(L_k)$ must be small.

In fact, following Ref. 2, we obtain from the equations for the polarization operator the following boundary conditions for P_{ij} at $\xi = L_k$:

$$P_{11} = \frac{1}{\pi}(g_{11}^2 + g_{21}^2)(L_p - L_k),$$
$$P_{22} = \frac{1}{\pi}(g_{12}^2 + g_{22}^2)(L_p - L_k),$$
$$P_{12} = \frac{1}{\pi}(g_{11}g_{12} + g_{22}g_{21})(L_p - L_k).$$

From this we have, for example, for $\lambda_i(L_k)$:

$$\lambda_1(L_k) = \frac{1}{4\pi}\frac{g_{11}^2 + g_{12}^2 + \frac{1}{\pi}(L_p - L_k)(g_{11}g_{22} - g_{12}g_{21})^2}{1 + \frac{1}{\pi}(L_p - L_k)(g_{11}^2 + g_{22}^2 + g_{12}^2 + g_{21}^2) + \frac{1}{\pi^2}(L_p - L_k)^2(g_{11}g_{22} - g_{21}g_{12})^2},$$

from which it is clear, for sufficiently large $L_p - L_k$, that indeed $\lambda_1(L_k) \ll 1$.

If $\lambda_i(L_k) \ll 1$, then Eqs. (19) are exact in so far as the more complicated vertex parts play no role.[1] In order that the physical charge (*i.e.*, the solution

*We do not give the calculation in so far as it coincides completely with that in Ref. 4. The only question arising concerns "small additions" to the vertex part. One can verify directly that, just as in the theory with a single field, their contribution is two powers of α higher in the expressions for β' and P'_{ij}.

of Eqs. (19) for small momenta) tends to zero as $L_k \to \infty$, it is sufficient in these conditions that the right-hand sides of Eqs. (19) are positive. We see that this condition is fulfilled.

Thus, the conclusion about the physical charge being zero still holds in the case considered.

3. We consider now a mixture of fields with different isotopic spins. The most characteristic peculiarities of this mixture of fields will be made clear by an example of four interacting fields: the ψ field, which is a spinor in ordinary space and a spinor in isotopic space; the Y_i field, which is a spinor in ordinary space and a vector in isotopic space; the φ_i field, pseudoscalar in ordinary space and vector in isotopic space; the θ field, scalar in ordinary space (it can also be taken as pseudoscalar) and spinor in isotopic space.

The interaction Lagrangian has the form ($\bar\psi = \beta\psi^*$; $\bar Y = \beta Y^*$, ϵ_{ijk} is the completely antisymmetrical tensor):

$$\mathscr{L} = ig_1\bar\psi\gamma_5\tau_j\psi\varphi_j + ig_2\bar Y_i\gamma_5 Y_k\varphi_j i\varepsilon_{ikj} + g_3\bar\psi\tau_i Y_j\theta. \quad (20)$$

We denote the Green's functions of the fields ψ, Y, φ and θ as follows:

$$G_\psi = \beta_1/\mathrm{p};\ G_Y = \beta_2/\mathrm{p};\ D_\varphi = d_1/k^2,\ D_0 = d_2/k^2. \quad (21)$$

It is necessary to consider three vertex parts

$$\Gamma_{\psi\psi\varphi_j} = \tau_j\gamma_5\alpha_1;\ \Gamma_{Y_iY_k\varphi_l} = i\varepsilon_{ikl}\gamma_5\alpha_2;\ \Gamma_{\psi Y_j\theta} = \tau_j\alpha_3. \quad (22)$$

The equations for the mass and polarization operators and the vertex parts are shown schematically on the figure.

The structure of the equations for α, β and d do not differ at all from the theory with one pseudoscalar. If we introduce the following three effective charges

$$\lambda_1 = (g_1^2/4\pi)\,\alpha_1^2\beta_1^2 d_1; \quad (23)$$
$$\lambda_2 = (g_2^2/4\pi)\,\alpha_2^2\beta_2^2 d_1; \qquad \lambda_3 = (g_3^2/4\pi)\,\alpha_3^2\beta_1\beta_2 d_2,$$

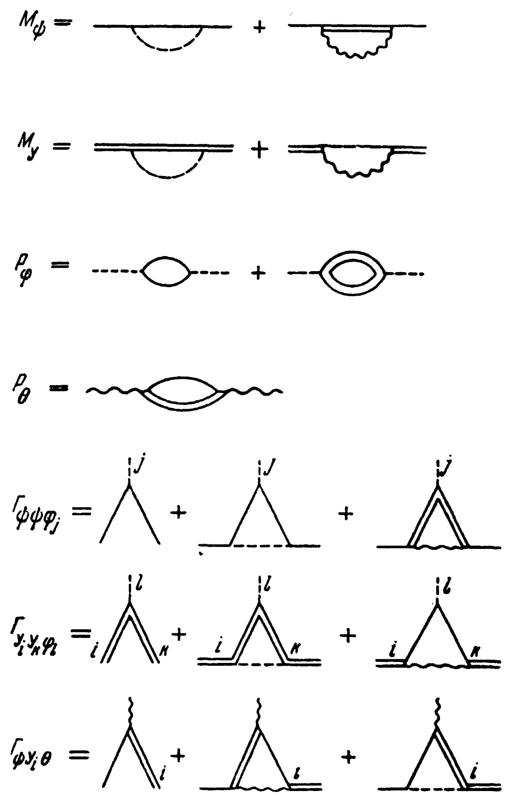
then the equations for the functions α_i, β_i, d_i have the form

$$\beta_1'/\beta_1 = 3\lambda_1/2 + 3\lambda_3/2;\ \beta_2'/\beta_2 = \lambda_2 + \lambda_3;$$

$$d_1'/d_1 = 4(\lambda_1 + \lambda_2);\ d_2'/d_2 = 6\lambda_3,$$

$$\alpha_1'/\alpha_1 = -\lambda_1 - 2\lambda_3\sqrt{\lambda_2/\lambda_1};$$
$$\alpha_2'/\alpha_2 = \lambda_2 - 2\lambda_3\sqrt{\lambda_1/\lambda_2};$$
$$\alpha_3'/\alpha_3 = 3\lambda_3 - 2\sqrt{\lambda_1\lambda_2}. \quad (24)$$

The negative sign of the second terms in the equations for α comes from the choice of sign for the charges g_1 and g_2 (if θ is the pseudoscalar field, then the equations have the same form under a different choice of the relative sign of g_1 and g_2).



Schematical description of the equations for mass and charge operators and vertex parts. Conventions: —— ψ field, ---- φ field, ~~~ θ field, ═══ Y field.

Employing Eqs. (23) and (24), we obtain the following equations for the effective charges

$$\lambda_1'/\lambda_1 = 5\lambda_1 + 4\lambda_2 + 3\lambda_3 - 4\lambda_3\sqrt{\lambda_2/\lambda_1},$$
$$\lambda_2'/\lambda_2 = 4\lambda_1 + 8\lambda_2 + 2\lambda_3 - 4\lambda_3\sqrt{\lambda_1/\lambda_2}, \quad (25)$$
$$\lambda_3'/\lambda_3 = 3\lambda_1/2 + \lambda_2 + 29\lambda_3/2 - 4\sqrt{\lambda_1\lambda_2}.$$

Again using two limiting momenta, it is possible to show that the initial conditions for the system of equations (25) are $\lambda_i(L_k) \ll 1$. To guarantee the validity of this conclusion, it is only necessary that the d_i'/d_i in Eqs. (24) are positive.

From Eqs. (25) it can be seen that if λ_1, λ_2 and λ_3 are of the same order, then all derivatives λ_1', λ_2', and λ_3' are positive and all charges λ_i decrease with diminishing momentum. If initial values of λ_i of different order are given at the limiting momenta Λ_k, for example, $\lambda_1(L_k) \gg \lambda_2(L_k)$; $\lambda_3(L_k) \sim \lambda_1(L_k)$, then as the momentum decreases, λ_2 will grow and λ_1 will diminish. Therefore, the point will come where the derivative of λ_2 changes sign and λ_2 also starts to decrease. Finally, if $\lambda_1(L_k)$ and $\lambda_2(L_k)$ are of the same order and $\lambda_3(L_k)$ is large, then λ_1 and λ_2 grow and λ_3 diminishes (with decreasing momentum). Therefore, the growth of λ_1 and λ_2 will become slower, and finally all three charges will decrease.

From this it is clear that qualitatively the situation does not differ from the theory with one pseudoscalar field, *i.e.*, all charges tend to zero with unlimited increase of L_k.

4. In all cases considered, the physical charge was equal to zero after the limiting momentum tended to infinity. For this two characteristics are important: (1) the positive sign of the derivative d'/d, which leads to a small effective charge at the momentum Λ_k and allows restriction to the approximation considered. This sign is a consequence of general features of the theory (see Ref. 5) and should be positive (the d-function grows) in all variants of the theory; (2) the further decrease of charges, for momenta less than Λ_k. This decrease of charges occurs in all examples considered, although the derivative λ'/λ, in general, contains terms of different signs. The case more complicated than that considered in Sec. 3, namely, the interaction of three fermion and three boson fields with different isotopic spins (0, ½, 1) does not lead to a different result. One might think that also the second characteristic is connected with general features of all variants of contemporary field theory.

We note that we required $L_p - L_k = \ln(\Lambda_p/\Lambda_k)$ to be large compared to unity, but finite. If we require the stronger condition, namely, $\ln(\Lambda_p/\Lambda_k) \to \infty$ as $\Lambda_p \to \infty$ and $\Lambda_k \to \infty$, then the physical charge will go to zero also for $\lambda' = 0(\xi < L_k)$ and even for $\lambda' < 0(\xi < L_k)$, provided only that $\ln(\Lambda_p/\Lambda_k)$ increases sufficiently rapidly as $\Lambda_p \to \infty$ and $\Lambda_k \to \infty$. If, however, this case actually occurs in some variant of the theory, then this signifies that there is ambiguity in the limiting transition from an extended interaction to a point one. In fact, setting $\Lambda_k = \Lambda_p$ in this case, we come to the conclusion that the physical charge can differ from zero.

The analysis of the different variants of the theory given above shows that the case $\lambda' \leq 0$ does not occur.

Thus, the construction of a meson theory with several nucleon and meson fields is apparently just as impossible as the construction, in the framework of modern quantum field theory, of a non-contradictory theory with one meson field.

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