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Contribution to Field Theory Involving a Cut-off Factor

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A study is made of the question of the uniqueness of quantum field theory involving a cut-off factor, and it is found that even finite (renormalized) expressions depend on the form of the cut-off factor. Examples are given in which the renormalized Green's function of a boson has no pole for finite momenta, but the critical momentum in the charge renormalization can be made arbitrarily large. In this connection difficulties with the vanishing of the charge and the existence of a pole in the Green's function are considered, and also the question of the domain of applicability of meson theory.

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1. BECAUSE of the divergence difficulties inherent in quantum field theory, use is often made of cut-off factors, which have the effect of reducing the part played by high-frequency virtual quanta, so as to secure the convergence of the expressions occurring in the theory[1,2]. Upon completion of the intermediate calculations, the cut-off parameters (which play the part of effective limiting momenta) are let go to infinity, the cut-off factor (CF) approaches unity, and, at least formally, the original "not cut-off" theory is recovered. This procedure corresponds to regarding a point interaction as the limit of a smeared-out interaction.

In such an approach to the problems of quantum field theory there inevitably arises the question as to its uniqueness, i.e., as to the dependence of the results obtained on the form of the CF used. The most important aspect of this question is considered in the present paper: do the finite (renormalized) expressions depend on the CF?

This question has been given partial consideration in Refs. 2 and 3, where it was answered in the negative; but it will be shown that this conclusion was essentially based on the use of CF's that approached unity sufficiently rapidly with increase of the cut-off parameter. We consider below a wider class of CF's, for which the problem of lack of uniqueness takes on primary significance.

The study will be carried out in the framework of the asymptotic theory of Landau, Abrikosov, and Khalatnikov[2] (cf. also Ref. 4), with a single modification — replacement of the plateau-shaped cut-off factor by a CF of more general type (but still very close to the plateau-shaped). The basic relationships of the theory are the integral equations connecting the Green's functions G and D and the vertex part Γ, the integrands being certain combinations of G, Γ, D, and the CF. In the calculation of the asymptotic forms at large momenta it turns out that essential parts are played not only by the principal parts of G and Γ, but also by terms of the next order, which are given by increments of the corresponding integrands. For the CF's under consideration, contributions are made to these increments not only by the increments of the functions G and Γ, but also by that of the CF itself; this fact is indeed the cause of the lack of uniqueness, which turns out to be intrinsic only in the integrals that diverge quadratically, i.e., for the boson Green's function D. As for the equations for G and Γ, they give the same results as in Ref. 2.

In Secs. 3 and 4 the analysis is given for a number of cases that occur with various choices of the cut-off factors in electrodynamics and in meson theory with pseudoscalar coupling. Section 5 is devoted to a discussion of these cases from the point of view of the well-known difficulties associated with the vanishing of the charge and the existence of a pole of the D-function.

2. The introduction of the CF into the theory is accomplished by smearing out the interaction,

$$S \sim e_1 \int F(p, k, p-k, \Lambda)\, \bar{\psi}(p) \hat{A}(k)\, \psi(p-k)\, d^4p\, d^4k + \text{charge conj.} \quad (1)$$

for electrodynamics, and similarly for mesodynamics. Here F is the CF, and Λ is the cut-off parameter. The CF must satisfy the conditions:

$$F \to 1 \quad \text{for} \quad \Lambda \to \infty \quad (1a)$$

$$\text{and } F(p, k, q) = F^*(q, k, p).$$

The second condition assures the preservation of Hermiticity under the smearing-out[5].

It is not difficult to convince oneself that when the smearing-out is introduced there must be associated with each vertex of the Feynman diagram (or with each vertex-part operator) a factor F depending on the corresponding momenta.

Concretely, the CF is taken in the factored form (but cf. footnote *):

$$F(p^2, k^2, (p-k)^2) = f(k^2)\, F(p^2)\, F((p-k)^2), \quad (2)$$

where all the functions are real. The cut-off parameters involved are in general taken to be unequal for the boson CF, $f(k^2, \Lambda_k^2)$, and the fermion CF, $F(p^2, \Lambda_p^2)$, where, following Pomeranchuk, one takes $\Lambda_p^2/\Lambda_k^2 = V > 1$. The fermion factor is chosen plateau-shaped: $F(p^2) = 0$ for $p^2 > \Lambda_p^2$ and $F(k^2) = 1$ for $p^2 < \Lambda_p^2$, and the boson function is nearly plateau-shaped: $f(k\ )\ f(k^2) = g(k^2) f_0(k^2)$, where $f_0(k^2) = 0$ for $k^2 > \Lambda_k^2$, $f_0(k^2) = 1$ for $k^2 < \Lambda_k^2$, and $g(k^2)$ is a function nearly equal to unity, with $g = 1$ for $\Lambda_k \to \infty$ or $k \to 0$.

3. Considering the asymptotic case in electrodynamics for $p \to \infty$ and introducing the notations used in Ref. 2, we have

$$G(p) = \frac{\beta(p)}{\hat{p}}, \quad \Gamma_\mu = \gamma_\mu \alpha, \quad (3)$$

$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[d(k^2)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_l(k^2)\frac{k_\mu k_\nu}{k^2}\right].$$

Here (with the choice $d_l = 0$) we shall have $\alpha = \beta = 1$.*

The remaining equation for D (here D^0 is the unperturbed Green's function)

$$D_{\mu\nu} = D^0_{\mu\nu} - D_{\mu\sigma} P_{\sigma\tau} D^0_{\tau\nu} \quad (4)$$

already turns out to depend essentially on the CF. The polarization operator involved,

$$P_{\mu\nu}(k) = f^2(k) \frac{e_1^2}{\pi i} \mathrm{Sp} \int G(p) \Gamma_\mu (p, p-k, k)\, G(p-k)\, \gamma_\nu F^2(p)\, F^2(p-k)\, d^4p \quad (5)$$

diverges quadratically, which corresponds to the appearance of a photon mass. Since this last cannot be removed by any choice of a CF of the type (1)[6], it is necessary to destroy it in a formal way — either by introducing a supplementary CF of the Pauli-Villars type[1] or simply by subtracting from the expression (5) the quantity $P_{\mu\nu}(0)$, as is done in Ref. 2. Both ways give the same result, but we shall concern ourselves with the second, as the simpler. This gives $P_{\mu\nu} = 0$ for $k^2 > \Lambda_k^2$ and for $k^2 < \Lambda_k^2$

$$\tilde{P}_{\mu\nu}(k) = P_{\mu\nu}(k) - P_{\mu\nu}(0) = \varphi(k) \frac{e_1^2}{\pi i} \mathrm{Sp} \int G(p) \Gamma_\mu(p, p, 0)\, G(p)\, \gamma_\nu F^4(p)\, d^4p$$
$$+ g^2(k) \frac{e_1^2}{\pi i} \mathrm{Sp} \int G(p) \cdot \Delta [\Gamma_\mu (p, p-k, k)\, G(p-k)\, F^2(p-k)]\, F^2(p)\, d^4p, \quad (6)$$
$$\varphi(k) = g^2(k) - g^2(0) = g^2(k) - 1.$$

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*It can be shown without difficulty that in the case $d_l \neq 0$, with plateau-shaped fermion CF and arbitrary boson CF, as in Ref. 2, α and β have the form of phase factors, with $\alpha\beta = 1$.

The first term corresponds to the contribution of the increment of the CF to $\widetilde{P}_{\mu\nu}$, the second to those of the increments of Γ and G (of order k^2). Equation (6) agrees with the corresponding Eq. (5) of Ref. 3, except for the first term of the right-hand member of Eq. (6), which was not taken into account in Ref. 3.

The expression (6) must be transverse,* *i.e.*,

$$\widetilde{P}_{\mu\nu}(k) = P(k)(\delta_{\mu\nu} - k_\mu k_\nu / k^2);$$

then from Eq. (4) it will follow that

$$d^{-1}(k) = 1 + P(k)/k^2, \qquad (7)$$
$$P(k) = (e_1^2/2\pi)\varphi(k)\Lambda_p^2 + (e_1^2/3\pi)k^2 \ln(\Lambda_p^2/k^2).$$

Here the factor $g^2(k)$ before the second integral in Eq. (6) can be dropped, with accuracy up to terms that vanish for $\Lambda \to \infty$.

For d we now obtain

$$d^{-1}(k) = 1 + \varphi(k)\frac{e_1^2}{2\pi}\frac{\Lambda_p^2}{k^2} + \frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{k^2}. \qquad (8)$$

The cause of the lack of uniqueness of the result lies in the second term of Eq. (8); by choosing different expressions for φ, one can obtain different forms for d. On the other hand, it is more convenient in practice to prescribe d and find the corresponding CF.

A) Let us require, for example, that d agree with the usual expression in the regions

$$k^2 < Nm^2, \quad N < \exp(3\pi/e^2)$$

and $\varepsilon\Lambda_k^2 < k^2 < \Lambda_k^2$,

where ϵ is a small quantity (it can go to zero for $\Lambda_k \to \infty$, but more weakly than Λ_k^{-2}). In the regions indicated one has the usual expressions[2,4]

$$d^{-1}(k) = 1 + \frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{k^2}, \quad \frac{e_1^2}{e^2} = 1 + \frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{m^2},$$
$$d_c^{-1}(k) = 1 - \frac{e^2}{3\pi}\ln\frac{k^2}{m^2}. \qquad (9)$$

Here $d_c = (e_1/e)^2 d$ is the renormalized d-function. In this connection, the charge renormalization remains as before with all its inherent difficulties.[7,4]

In the intermediate region

$$Nm^2 < k^2 < \varepsilon\Lambda_k^2$$

we take d constant and equal to its value at the left-hand end:

$$d^{-1}(k) = 1 + \frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{Nm^2}, \qquad (10)$$
$$d_c^{-1}(k) = 1 - \frac{e^2}{3\pi}\ln N.$$

The corresponding CF's are found from Eq. (8)

$$\varphi(k) = \frac{2\pi}{e_1^2}\frac{k^2}{\Lambda_p^2}\left[d^{-1}(k) - 1 - \frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{k^2}\right], \qquad (11)$$

from which we have

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*Strictly speaking, the first integral in Eq. (6) is a longitudinal quantity; this leads to a dependence of the function d_l on the field. But this can easily be avoided by taking a somewhat more complicated CF:

$$F(p, k, p-k) = f(p, k)F(p)F(p-k),$$
$$f^2(p, k) = 1 + 3\frac{(pk)^2}{p^2k^2}\varphi(k).$$

Then the first term of Eq. (6) becomes equal to

$$(e_1^2/2\pi)\varphi(k)\Lambda_p^2[\delta_{\mu\nu} - k_\mu k_\nu/k^2]$$

instead of the value $(e_1^2/2\pi)\Lambda_p^2\varphi(k)\delta_{\mu\nu}$ for the case of the factored CF. Therefore no attention can be given to the nontransversality of Eq. (6), if we take $\varphi(k)$ to mean not $g^2(k) - 1$, but

$$\frac{p^2k^2}{3(pk)^2}[f^2(p, k) - 1].$$

The remark about the nontransversality of Eq. (6) is due to A. A. Abrikosov.

$$\varphi(k) = \begin{cases} 0, & k^2 < Nm^2, \quad \varepsilon\Lambda_k^2 < k^2 < \Lambda_k^2 \\ (2k^2/3\Lambda_p^2)\ln(k^2/Nm^2), & Nm^2 < k^2 < \varepsilon\Lambda_k^2 . \end{cases}$$

This example gives d_c that has no pole at any finite value of k.

B) Another example relates to the change of the limiting momentum, under which, in the theory of Ref. 2, there appears the difficulty of the vanishing of the charge. We consider a d of the form

$$d^{-1}(k) = \tag{12}$$

$$\begin{cases} 1 + q\,(e_1^2/3\pi)\ln(\Lambda_p^2/k^2), & k^2 < \varepsilon\Lambda_k^2, \\ 1 + (e_1^2/3\pi)\ln(\Lambda_p^2/k^2), & \varepsilon\Lambda_k^2 < k^2 < \Lambda_k^2 . \end{cases}$$

Here

$$\frac{e_1^2}{e^2} = 1 + q\frac{e_1^2}{3\pi}\ln\frac{\Lambda_p^2}{m^2} = \left[1 - q\frac{e^2}{3\pi}\ln\frac{\Lambda_p^2}{m^2}\right]^{-1};$$

$$d_c^{-1}(k) = \begin{cases} 1 - q\,(e^2/3\pi)\ln(k^2/m^2), & k^2 < \varepsilon\Lambda_k^2, \\ 1 - (e^2/3\pi)\ln(k^2/m^2) + (e^2/3\pi)(1-q)\ln(\Lambda_p^2/m^2), & \varepsilon\Lambda_k^2 < k^2 < \Lambda_k^2 . \end{cases}$$

We find the corresponding CF from Eq. (11)

$$\varphi(k) = \tag{13}$$

$$\begin{cases} 0, & \varepsilon\Lambda_k^2 < k^2 < \Lambda_k^2 \\ -{}^2/_3\,(1-q)(k^2/\Lambda_p^2)\ln(\Lambda_p^2/k^2), & k^2 < \varepsilon\Lambda_k^2 . \end{cases}$$

From this it can be seen that for sufficiently small q the critical value of the momentum can be shifted as far out as one wishes.

In both examples the CF satisfy the general requirements of Sec. 2; the discontinuities appearing in them can easily be removed by taking a smoothed form of the function d.

We shall now show that in logarithmically divergent integrals there is no need to take into account the fact that the CF is not plateau-shaped. Indeed, for $\Lambda_k \to \infty$

$$\max\left(\varphi(k)\ln\frac{\Lambda_p^2}{m^2}\right) \sim \varepsilon\frac{\Lambda_k^2}{\Lambda_p^2}\ln\frac{\Lambda_p^2}{m^2} \to 0,$$

in virtue of

$$\varepsilon\ln(\Lambda_p^2/m^2) \to 0.$$

We can now determine for what sorts of functions $\varphi(k)$ the results of Refs. 2 and 3 remain valid. From Eq. (8) it can be seen that this is so if for finite k the quantity

$$\varphi(k)(\Lambda_p^2/k^2) = \varphi(k^2/\Lambda_k^2, k^2)(\Lambda_k^2/k^2)V$$

goes to zero as Λ_k increases. If we introduce $x = \Lambda_k^{-2}$, then for fixed k this condition will mean that $\varphi(x)/x = d\varphi/dx$ must vanish for $x \to 0$. Thus the CF will make a contribution to d if $df^2/d(\Lambda_k^{-2})$ either approaches a finite limit (as in the first example) or increases (as in the second example). The remaining details of the structure of the CF, besides the angle of inclination of the tangent at $k^2/\Lambda_k^2 = 0$, such as the curvature, *etc.*, have no residual effect on d because of the quadratic divergence of $P_{\mu\nu}$, since their contribution vanishes for $\Lambda \to \infty$.

As for the calculation of the complicated vertex-part diagrams not considered in Ref. 2, in the first example, d is equal to or smaller than the corresponding d in Ref. 3; therefore the estimates carried out in Ref. 3 can be used. The same applies to the second example.

4. Going over to mesodynamics with weak pseudoscalar coupling and using the notations of Ref. 8,

$$G(p) = \beta/\hat{p}, \quad \Gamma_\mu = \gamma_\mu\gamma_5\alpha,$$

$$D_{\mu\nu}(k) = k^{-2}d(k^2)\delta_{\mu\nu},$$

we obtain for G and Γ the well known results:[8,4]

$$\alpha = \beta^{-2/3}, \tag{14}$$

$$d\beta/d\xi = (3g_1^2/8\pi)[\beta(\xi)]^{-5/3}d(\xi), \quad \xi = \ln(k^2/m^2).$$

The equation for d, after renormalization of the meson mass, gives:

$$d^{-1}(\xi) = 1 + \Phi(\xi) + \frac{g_1^2}{\pi}\int_{\xi}^{L_p} \beta^{2/5}(z)\,dz;$$

$$\Phi(\xi) = 2I\,\frac{g_1^2}{\pi}\,\frac{\Lambda_p^2}{k^2}\,[g^2(k) - 1],$$

$$I = \int_0^1 \beta^{4/5}(z)\,dz, \qquad L = \ln\frac{\Lambda^2}{m^2}. \tag{15}$$

The CF is connected with the relationship, following from Eqs. (14) and (15),

$$\Phi(\xi) = d^{-1}(\xi) - \left[1 + \frac{g_1^2}{\pi}(L_p - L_k)\right] \tag{16}$$

$$-\frac{g_1^2}{\pi}\int_{\xi}^{L_k} dz \Big/ \left[1 - \frac{g_1^2}{4\pi}\int_z^{L_k} d(\omega)\,d\omega\right].$$

A) Let us consider a problem like the first example in electrodynamics — the destruction of a pole in the ordinary charge renormalization. We take the function d of the form given by $[N < \exp(4\pi/5g^2)]$:

$$d^{-1}(\xi) = \begin{cases} \left[1 + \frac{g_1^2}{\pi}(L_p - L_k)\right] Q^{4/5}(L_k - \xi), \\ \qquad \xi < \ln N, \quad L_k - \ln\frac{1}{\varepsilon} < \xi < L_k, \\ \left[1 + \frac{g_1^2}{\pi}(L_p - L_k)\right] Q^{4/5}(L_k - \ln N), \\ \qquad \ln N < \xi < L_k - \ln\frac{1}{\varepsilon}. \end{cases} \tag{17}$$

Here we have introduced the notation

$$Q(x) = \left(1 + \frac{5g_1^2}{4\pi}x\right) \Big/ \left[1 + \frac{g_1^2}{\pi}(L_p - L_k)\right].$$

The corresponding CF is given by Eq. (16)

$$g^2(k) - 1 \tag{18}$$

$$= \begin{cases} 0, & L_k - \ln(1/\varepsilon) < \xi < L_k, \\ (k^2/\Lambda_p^2)\,A(\xi), & \ln N < \xi < L_k - \ln(1/\varepsilon), \\ (k^2/\Lambda_p^2)\,B(\xi), & \xi < \ln N, \end{cases}$$

where A and B are slowly varying functions of ξ, whose forms we shall not present because of their cumbersomeness.

B) We go on to the second example — the displacement of the critical momentum in the charge renormalization. We take the function

$$d^{-1}(\xi) = \tag{19}$$

$$\begin{cases} \left[1 + \frac{g_1^2}{\pi}(L_p - L_k)\right] Q^{4/5}(L_k - \xi), \\ \qquad L_k - \ln\frac{1}{\varepsilon} < \xi < L_k, \\ d^{-1}(0)\left[1 - \frac{5}{4\pi}g^2\xi\right]^{4/5}, \quad \xi < L_k - \ln\frac{1}{\varepsilon}. \end{cases}$$

Here we have used the renormalized charge

$$g^2 = d(0)\,\alpha^2(0)\,\beta^2(0)\,g_1^2. \tag{20}$$

We prescribe also the charge renormalization

$$g^2 = g_1^2\left[1 + \frac{g_1^2}{4\pi}q(4L_p + L_k)\right]^{-1} \tag{21}$$

and determine by means of Eqs. (20) and (21) the quantity $d(0)$, after which we find from Eq. (16) the corresponding CF:

$$g^2(k) - 1 = \begin{cases} 0, \quad L_k - \ln\frac{1}{\varepsilon} < \xi < L_k, \\ -\frac{k^2}{2I\Lambda_p^2}\left(L_p + \frac{L_k}{4}\right) \\ \times(1 - q)\,Q^{-1/5}\left(\ln\frac{1}{\varepsilon}\right), \\ \qquad \xi < L_k - \ln\frac{1}{\varepsilon}. \end{cases} \tag{22}$$

In this case, just as in electrodynamics, one can find a CF that simultaneously fulfills the requirements for the CF of both of the problems considered, *i.e.*, displaces the pole in the charge renormalization and removes the pole in the d-function.

5. In passing on to the discussion of the examples analyzed above, it is first of all necessary to emphasize the fact of the nonuniqueness of the theory based on the limiting process. Generally speaking, the lack of uniqueness of diverging expressions with respect to the form of the limiting process has been repeatedly noted in the literature.[1,9] Examples are given above in which also the finite (renormalized) expressions depend on the type of CF used. This circumstance forces us to proceed with caution to the conclusions of a theory that makes use of a limiting process.

In all of the examples considered above, there appeared a difficulty with the vanishing of the renormalized charge [Eqs. (9), (12), (21)],[4,7] which provides evidence of the fact that the present form of quantum field theory is not a closed theory.* It is, however, also possible to take the view that the vanishing of the charge is in itself not an essential difficulty that paralyzes the development of the theory in its present form. The quantum field theory includes in itself the renormalized theory and the relation between the priming and renormalized constants. In view of the fact that only the renormalized theory is compared with experiment, it would be possible to pay no attention to the relation between the constants, leaving the difficulties connected with this to a future theory. In other words, the expressions that play the role of the renormalized quantities would in this procedure have assigned to them the values that follow from experiment, despite the fact that in the present theory they actually vanish. Such an approach does not lead to its goal when a plateau-shaped CF is used, because of the appearance of difficulties in the renormalized theory itself — besides the vanishing of the charge, a pole at a finite momentum appears in the renormalized d-function.

The situation is essentially altered, as is shown by the examples given above, when one uses CF's that are not plateau-shaped. On one hand, the appearance of a pole and the vanishing of the charge cease to be facts that are coupled to each other. Concrete examples have been presented of CF's for which the d-function does not have a pole at any finite momentum. On the other hand, the value of the critical momentum at which the renormalized charge becomes small is also essentially not uniquely determined by the theory of Landau, Abrikosov, and Khalatnikov.[2] Therefore, for example, the conclusions about the supposed inapplicability of mesodynamics already at energies $\sim Mc^2$, which are often drawn on the basis of the theory of Ref. 2 and which rest on the fact that with a plateau-shaped CF the critical momentum $\sim Mc$, cannot be regarded as entirely convincing.**

* In this connection it is of course assumed that the approximate treatment of this question actually reflects, even if only qualitatively, the true situation.

** In electrodynamics, because of the smallness of the coupling constant, the critical momentum is very large, and there this question appears less sharply.

We note that in the examples considered above the well known theorem of Lehmann and Källen about the increase of the renormalized d-function with the momentum[10] is satisfied. In a certain sense this provides evidence of the existence in the renormalized theory of a complete orthogonal system of eigenfunctions of the energy and momentum.* One can without difficulty assume a CF such that the d-function has no pole and at the same time the renormalized charge is finite. For this it suffices to consider example (B), where one must take q negative (and large absolute value). Then the priming charge e_1 and the renormalized charge e will satisfy

$$\frac{e_1^2}{e^2} = 1 - \frac{e_1^2}{3\pi}|q|\ln\frac{\Lambda_p^2}{m^2} = \left[1 + \frac{e^2}{3\pi}|q|\ln\frac{\Lambda_p^2}{m^2}\right]^{-1},$$

i.e., e_1 vanishes for $\Lambda \to \infty$, and e remains finite.

The renormalized function d_c takes the form

$$d_c^{-1} = \begin{cases} 1 + |q|\frac{e^2}{3\pi}\ln\frac{k^2}{m^2}, & k^2 < \varepsilon\Lambda_k^2 \\ 1 + |q|\frac{e^2}{3\pi}\ln\frac{\Lambda_p^2}{k^2} + \frac{e^2}{3\pi}\ln\frac{\Lambda_p^2}{k^2}, & \\ \qquad \varepsilon\Lambda_p^2 < k^2 < \Lambda_k^2. & \end{cases}$$

Moreover, one can choose the CF so that in the region of small momenta d_c agrees with the perturbation theory (*cf.* example A). At the same time the correction diagrams of the vertex part are small for this case. But in this example the Lehmann-Källen theorem[10] does not hold. The question of the superiority of a plateau-shaped CF in comparison with that just considered nevertheless remains unclear, since the existence of a complete orthogonal system is only a sufficient condition for the Lehmann-Källen theorem. Therefore, even if one uses a CF that does not contradict this theorem there is no assurance of the existence of a complete orthogonal system (except for $e = 0$).

The question arises of the possibility of finding a CF such that e would be finite and at the same time the Lehmann-Källen theorem[10] would hold. Within the framework of the approximate theory[2] such a CF can easily be found. For this it suffices to let q approach zero in the example (B) just discussed, in such a way that the critical momentum

*Attention was called to the importance of this fact by I. Ia. Pomeranchuk.

will increase with Λ: $q > 0$, $q(e_1^2/3\pi)\ln(\Lambda^2/m^2) < 1$ for all Λ above a certain value. Here e_1 can be either finite or infinite. Together with this one can choose a function d_c with no pole.

But in the evaluation of the overlapping diagrams of a vertex part it is found that in this case the "three-gamma" approximation[2] is inadequate. This fact is of a general character. Indeed, if we begin with a theory with $e \neq 0$ and d_c increasing with the momentum, then for the diagrams in question we shall have the expression [with $\alpha = \beta = 1$ as in Eq. (3)]:

$$e_1^4 \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} [d(k)]^2 = e^4 \int_{p^2}^{\Lambda^2} \frac{dk^2}{k^2} [d_c(k)]^2 > e^4 \ln \frac{\Lambda^2}{p^2}$$

$[d_c(0) = 1]$, which is large in comparison with the zeroth approximation. The estimates of Pomeranchuk,[3] which indicated that the part played by these chains is small, rested essentially on having $e^2 \to 0$ for $\Lambda \to \infty$. Thus the applicability of the "three-gamma" approximation is fundamentally connected with the condition $e^2 \to 0$.

Therefore the question of the possibility of choosing a CF such that the charge does not go to zero remains essentially open, because of the lack of a simple scheme of calculation in which this question could be studied.

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