Polarization Effects in the Scattering of Electrons and Positrons by Electrons

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Polarization effects arising in the scattering of electrons or positrons by electrons are considered. A possibility is found for using Möller scattering in magnetized foils to analyze polarized beams of electrons or positrons.

1. INTRODUCTION

THE approach to several important physical experiments involves the production and study of polarized electron and positron beams. For example, a determination of the polarization of β-particles from oriented nuclei could provide information about the interaction responsible for β-decay[1] experiments with polarized high energy electrons would be of undoubted interest; and so on. At the present time the main method for producing and analyzing polarized electron beams is scattering in the Coulomb fields of heavy nuclei[2]: a) if the incident beam is not polarized, the scattered electrons are found to be partially polarized; b) if the incident beam is polarized at a certain angle with the direction of motion, then an azimuthal asymmetry is observed in the angular distribution of the scattered electrons. We note that these effects are due to terms of the order $(Z/137)^2$ in the scattering amplitude, and accordingly are completely absent in the first Born approximation.

As an analyzer for polarized electron and positron beams, the scattering by heavy nuclei has the following disadvantages: 1) the polarization effects disappear in the limiting cases of nonrelativistic and extreme relativistic energies, i.e., in the high-energy region this method fails completely; 2) the measurement of the azimuthal asymmetry in the scattering of a polarized beam gives the projection of the polarization vector on the normal to the plane of scattering; the presence of a longitudinal polarization can be detected by first deflecting the beam with an electric field[3], but this is not always convenient to do; 3) for positrons the polarization effects are manifested considerably more weakly than for electrons; 4) double scattering in the foil with oblique incidence of the beam can strongly distort the result at small energies; just this effect held up the experimental verification of the theory for more than 10 years.[4]

In the present note, we consider the polarization effects occurring in the scattering of an electron (positron) by an electron*. It can easily be shown that in the second approximation of the perturbation theory, which leads to the well-known Möller formula (cf., for example, Ref. 5) the scattering of an unpolarized beam by an unpolarized target leaves the beam unpolarized, and in the scattering of a polarized beam by an unpolarized target (or an unpolarized beam by a polarized target) no azimuthal asymmetry appears. These effects show up first in the third approximation of the perturbation theory, which gives the radiation corrections to the Möller formula[6].

But also in the second approximation of the perturbation theory, the following effects occur: 1) in the scattering by a polarized target (a magnetized ferromagnetic substance) the electron beam becomes polarized; 2) in the scattering of a polarized beam by a polarized target the angular distribution departs from that given by the Möller formula (a different dependence on the angle ϑ and an azimuthal asymmetry). Unlike those in the case of scattering of electrons in the Coulomb field of a nucleus, these polarization effects do not disappear in the limiting cases of nonrelativistic and extreme relativistic energies; they are of comparable magnitude for positrons and electrons (at nonrelativistic energies, however, they are absent for positrons); the presence of a longitudinal polarization in the incident beam is directly manifested in the scattering; and the registration of coincidences produced by the scattered electron (positron) and the recoil electron makes it possible to eliminate altogether the effect of double scattering in the foil.

These properties of the Möller scattering in magnetized foils lead to the idea of a new method for the analysis of polarized electron and positron beams, which for certain cases turns out to be useful.

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* This problem was suggested to us by A. I. Akhiezer, to whom we express our indebtedness.

2. SCATTERING OF A POLARIZED BEAM BY A POLARIZED TARGET

The state of polarization of a beam of nonrelativistic particles with spin ½ is described by a density matrix $\rho = \frac{1}{2}(1 + \zeta\sigma)$, where ζ is the polarization vector ($|\zeta| \leq 1$; for $|\zeta| = 1$ the beam is completely polarized). For relativistic particles with spin ½ and momentum p,the form of the four-rowed density matrix is given in Ref. 1 (cf. also the added note in another paper[7]); but the writers of Ref. 1 did not notice that the cumbersome expression found by them for ρ can be brought into a compact form by separating out factors $\eta^{(+)}(p)$ on the right and left:

$$\rho(\zeta^0, \mathbf{p}) = \eta^{(+)}(p)\, {}^1/_2 (1 + \zeta\Sigma\gamma_4)\, \eta^{(+)}(p). \quad (1)$$

Here

$$\Sigma\gamma_4 = -i\gamma\gamma_5 = \begin{pmatrix}\sigma & 0\\ 0 & -\sigma\end{pmatrix},$$

and $\eta^{(+)}(p)$ is the projection operator onto states of positive energy $p_0 = \varepsilon = \sqrt{m^2 + \mathbf{p}^2} > 0$,

$$\eta^{(+)}(p) = (1/2\varepsilon)(m - i\hat{p})\gamma_4 \quad (\hat{p} = p_\mu\gamma_\mu). \quad (2)$$

The polarization vector ζ is given by

$$\zeta = \zeta^0_\perp + (\varepsilon/m)\zeta^0_\parallel, \quad (3)$$

where ζ^0 is the polarization vector in the system of reference in which the electrons are at rest ($|\zeta^0| \leq 1$), and $\zeta^0_\perp$ and $\zeta^0_\parallel$ are the components of ζ^0 perpendicular and parallel to the vector **p**. We note that in the relativistic case an unpolarized beam ($\zeta = 0$) is described by a projection operator and not by the unit matrix.

The analogous formulas for positrons with momentum p and energy $\epsilon > 0$ are:

$$\rho^{(p)}(\zeta^0, \mathbf{p}) = \eta^{(p)}(p)\, {}^1/_2 (1 - \zeta\Sigma\gamma_4)\, \eta^{(p)}(p), \quad (4)$$

$$\eta^{(p)}(p) = -(1/2\varepsilon)(m + i\hat{p})\gamma_4. \quad (5)$$

In the case of $e - e$ scattering the density matrix ρ_i for the initial state is a "direct product" of matrices ρ_1 and $\rho_{1'}$ which describe respectively the incident beam (momentum $\mathbf{p}_1$) and the electrons of the target (momentum $\mathbf{p}_{1'}$):

$$\rho_i = \rho(\zeta^0_1, \mathbf{p}_1) \times \rho(\zeta^0_{1'}, \mathbf{p}_{1'}). \quad (6)$$

Similarly, for the final state,

$$\rho_f = \rho(\zeta^0_2, \mathbf{p}_2) \times \rho(\zeta^0_{2'}, \mathbf{p}_{2'}). \quad (7)$$

The cross section for the scattering of a polarized electron beam by a polarized target is given by the "trace"

$$\sigma(\vartheta, \varphi) \sim \mathrm{Sp}\,[S\rho_i S^+ \eta^{(+)}(p_2)\, \eta^{(+)}(p_{2'})], \quad (8)$$

where S is the scattering matrix, whose opposite element is given, apart from a factor, by

$$S \approx \frac{(\bar{u}_2\gamma_\mu u_1)(\bar{u}_{2'}\gamma_\mu u_{1'})}{(p_2 - p_1)^2} \quad (9)$$

$$- \frac{(\bar{u}_{2'}\gamma_\mu u_1)(\bar{u}_2\gamma_\mu u_{1'})}{(p_{2'} - p_1)^2}.$$

Substituting Eqs. (1) and (6) into the "trace" (8), we obtain four terms: one of them, independent of ζ_1 and $\zeta_{1'}$ agrees, to within a constant factor, with the Möller cross section; two others, depending linearly on the components of ζ_1 or $\zeta_{1'}$, are identically equal to zero; the fourth term contains the components of the vectors ζ_1 and $\zeta_{1'}$ bilinearly, and is given by

$$(p_2 - p_1)^{-4}\mathrm{Sp}\,\Gamma^-_{\mu\nu}(p_1, p_2, \zeta_1)\,\mathrm{Sp}\,\Gamma_{\mu\nu}(p_{1'}, p_{2'}, \zeta_{1'})$$

$$+ (p_{2'} - p_1)^{-4}\,\mathrm{Sp}\,\Gamma_{\mu\nu}(p_{1'}, p_2, \zeta_{1'})$$

$$\times \mathrm{Sp}\,\Gamma_{\mu\nu}(p_1, p_{2'}, \zeta_1) - 2(p_2 - p_1)^{-2}$$

$$\times (p_{2'} - p_1)^{-2}\,\mathrm{Sp}\,[\Gamma_{\mu\nu}(p_1, p_{2'}, \zeta_1)\,\Gamma_{\mu\nu}(p_{1'}, p_2, \zeta_{1'})],$$

$$\Gamma_{\mu\nu}(p_1, p_2, \zeta_1) = (1/16\varepsilon_1^2\varepsilon_2)\,\gamma_\mu (m - i\hat{p}_1)$$

$$\times (\zeta_1\Sigma)\,(m - i\hat{p}_1)\,\gamma_\nu (m - i\hat{p}_2).$$

Introducing a rectangular system of coordinates with the unit vectors **k**, **l**, **n**, where **k** is a unit vector in the direction of the incident beam, **n** is a unit vector normal to the plane of scattering, and $\mathbf{l} = [\mathbf{n} \times \mathbf{k}]$, and going over from ζ_1, $\zeta_{1'}$ to ζ^0_1, $\zeta^0_{1'}$, we write the cross-section for scattering of electrons by electrons in the form

$$\sigma(\vartheta, \varphi) = \sigma_0(\vartheta)\;[1 + T_{ik}(\xi, \vartheta)\,\zeta^0_{1i}\zeta^0_{1'k}]; \quad (10)$$

Here $\sigma_0(\vartheta)$ is the Möller cross section, ϑ is the scattering angle in the center-of-mass system, and $\xi = \epsilon_{1\,\mathrm{lab}}/m$ is the energy of the incident electron in the laboratory reference system, expressed in units m. As the result of rather cumbersome developments we obtain the following formulas for the coefficients T_{ik}:

$$T_{11}=\tau(\xi,\ \vartheta)\,[4\xi\,(2\xi-1)-(\xi-1)\,(\xi+3)\sin^2\vartheta],$$

$$T_{22}=\tau(\xi,\ \vartheta)\,[4\xi+(\xi-1)\,(\xi+3)\sin^2\vartheta],\qquad(11)$$

$$T_{33}=\tau(\xi,\ \vartheta)\,[4\,(2\xi-1)-(\xi-1)^2\sin^2\vartheta],$$

$$T_{12}=T_{21}=\tau(\xi,\ \vartheta)\,(\xi-1)\sqrt{2\,(\xi+1)}\sin 2\vartheta,$$

$$T_{31}=T_{13}=T_{23}=T_{32}=0;$$

$$-\tau(\xi,\vartheta)=\sin^2\vartheta\,[4\xi^2\,(1+3\cos^2\vartheta)\qquad(12)$$

$$+(\xi-1)^2\,(4+\sin^2\vartheta)\sin^2\vartheta]^{-1}.$$

The values of T_{ik} at $\vartheta=\pi/2$ are shown in Fig. 1.

In limiting cases, the cross-section (10) can be represented by relatively simple formulas.

a) The nonrelativistic case ($\xi-1\ll 1$)

$$\sigma(\vartheta)\approx\sigma_0(\vartheta)\left[1-\frac{\sin^2\vartheta}{1+3\cos^2\vartheta}\,(\zeta_{1}^{0}\zeta_{1'}^{0})\right];\qquad(13)$$

There is no azimuthal asymmetry, but the dependence on ϑ differs from Möller's result.

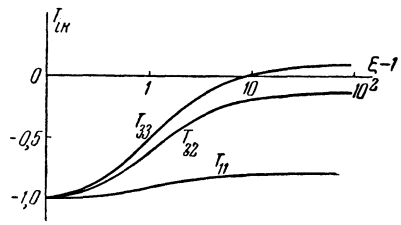

FIG. 1. The coefficients T_{ik} in Eq. (11), at $\vartheta=\pi/2$

b) The extreme relativistic case ($\xi\gg 1$)

$$\sigma(\vartheta,\ \varphi)\approx\sigma_0(\vartheta)\left\{1-\frac{\sin^4\vartheta}{(3+\cos^2\vartheta)^2}\right.\qquad(14)$$

$$\times\left[\left(\frac{8}{\sin^2\vartheta}-1\right)(\mathbf{k}\zeta_{1}^{0})\,(\mathbf{k}\zeta_{1'}^{0})+(\mathbf{l}\zeta_{1}^{0})\,(\mathbf{l}\zeta_{1'}^{0})\right.$$

$$\left.\left.-(\mathbf{n}\zeta_{1}^{0})\,(\mathbf{n}\zeta_{1'}^{0})\right]\right\}.$$

The scattering cross section for positrons by electrons is found in a similar way:

$$\sigma^{(p)}(\vartheta,\ \varphi)=\sigma_0^{(p)}(\vartheta)\,[1+T_{ik}^{(p)}(\xi,\vartheta)\,\zeta_{1i}^{0}\zeta_{1'k}^{0}];\qquad(15)$$

$$T_{11}^{(p)}=\tau^{(p)}(\xi,\vartheta)\,[(\xi+1)^2\,(7\xi+1)+(\xi+1)\,(7\xi^2-4\xi+5)\cos\vartheta$$
$$+(\xi-1)\,(\xi+3)^2\cos^2\vartheta+(\xi-1)^2\,(\xi+3)\cos^3\vartheta],$$

$$T_{22}^{(p)}=\tau^{(p)}(\xi,\vartheta)\,[(\xi+1)^2\,(\xi+7)+(\xi+1)\,(\xi^2+4\xi-13)\cos\vartheta$$
$$-(\xi-1)\,(\xi+3)^2\cos^2\vartheta-(\xi-1)^2\,(\xi+3)\cos^3\vartheta],$$

$$T_{33}^{(p)}=\tau^{(p)}(\xi,\vartheta)\,[-(\xi+1)\,(\xi^2-8\xi-17)-(\xi^2-1)\,(\xi-11)\cos\vartheta\qquad(16)$$
$$+(\xi-1)^2\,(\xi+3)\cos^2\vartheta+(\xi-1)^3\cos^3\vartheta],$$

$$T_{12}^{(p)}=T_{21}^{(p)}=\tau^{(p)}(\xi,\vartheta)\,2\sqrt{2\,(\xi+1)}\sin\vartheta\,[2\,(\xi+1)+(\xi-1)\,(\xi+3)\cos\vartheta+$$
$$+(\xi-1)^2\cos^2\vartheta],$$

$$T_{31}^{(p)}=T_{13}^{(p)}=T_{23}^{(p)}=T_{32}^{(p)}=0;$$

$$\tau^{(p)}(\xi,\vartheta)=(\xi-1)\,(1-\cos\vartheta)\,[(\xi+1)\,(9\xi^3-\xi^2-\xi+25)+$$
$$+4\,(\xi^2-1)\,(3\xi+11)\cos\vartheta+2\,(\xi-1)^2\,(3\xi^2+12\xi+11)\cos^2\vartheta+\qquad(17)$$
$$+4\,(\xi-1)^3\cos^3\vartheta+(\xi-1)^4\cos^4\vartheta]^{-1}.$$

The dependence of $T_{ik}^{(p)}$ on ξ at $\vartheta=\pi/2$ is given in Fig. 2.

The limiting cases are: a) $\xi-1\ll 1$

$$\sigma^{(p)}(\vartheta)\approx\sigma_0^{(p)}(\vartheta),\qquad(18)$$

i.e., in the nonrelativistic case, the polarization has no effect on the positron electron scattering; b) $\xi \gg 1$

$$\sigma^{(n)}(\vartheta, \varphi) \cong \sigma_0^{(n)}(\vartheta)\Big\{1 + \frac{\sin^4\vartheta}{(3+\cos^2\vartheta)^2}\Big[\Big(\frac{8}{\sin^2\vartheta} - 1\Big)(\mathbf{k}\zeta_1^0)(\mathbf{k}\zeta_{1'}^0) + (\mathbf{l}\zeta_1^0)(\mathbf{l}\zeta_{1'}^0) - (\mathbf{n}\zeta_1^0)(\mathbf{n}\zeta_{1'}^0)\Big]\Big\} \quad (19)$$

We shall show how the relations obtained here can be used for the experimental determination of the polarization of an electron beam. All measurements are to be carried out in such a way that the normal N to the plane of the scattering

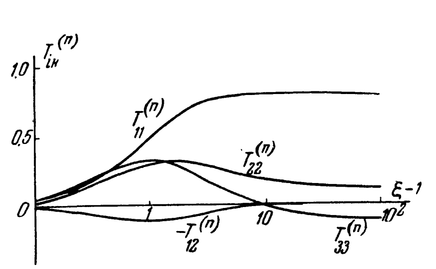

FIG. 2. The coefficients $T_{ik}^{(p)}$ in Eq. (15), at $\vartheta = \pi/2$

foil and the vector k form a fixed angle $\alpha \sim 45°$ (cf. Fig. 3). The electron counters 1 and 2 are set in the plane (k, N) at the angles θ_1 and θ_2 corresponding to the angles ϑ and $\pi - \vartheta$ in the

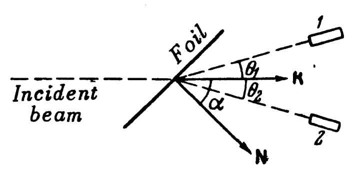

FIG. 3

center-of-mass system ($\vartheta \sim \pi/2$), and connected to a coincidence circuit. The polarization of the incident beam is determined in three steps.

1) The polarization $\zeta_{1'}^0$ of the foil is directed along the normal n to the plane of the scattering. We determine the number of coincidences D_1, referred to unit current in the incident beam. Then we turn the foil and the counters through 180° around the axis **k**, so that $\zeta_{1'}^0$ is in the direction −n, and determine the number of coincidences D_2. The ratio $D_1 : D_2$ gives the projection $(\zeta_1^0\mathbf{n})$ by the formula

$$D_1/D_2 = [1 + T_{33}(\xi, \vartheta)\zeta_{1'}^0(\zeta_1^0\mathbf{n})] / [1 - T_{33}(\xi, \vartheta)\zeta_{1'}^0(\zeta_1^0\mathbf{n})].$$

2) We carry out analogous measurements after turning the foil and counters through 90° and 180° around the axis **k**, so that the new normal **n**′ coincides with our former vector l. Obviously,

$$D_1'/D_2' = [1 + T_{33}(\xi, \vartheta)\zeta_{1'}^0(\zeta_1^0\mathbf{l})] / [1 - T_{33}(\xi, \vartheta)\zeta_{1'}^0(\zeta_1^0\mathbf{l})].$$

3) Having thus determined $\zeta_{1\perp}^0$, we turn the counters and foil around the axis k so as to bring them into the plane perpendicular to the vector $\zeta_{1\perp}^0$, then turn the foil around the axis N so that the vector $\zeta_{1'}^0$ lies in the plane of the scattering, and again carry out measurements. Then

$$D_1''/D_2'' = [1 + T_{11}(\xi, \vartheta)(\zeta_{1'}^0\mathbf{k})(\zeta_1^0\mathbf{k})] / [1 - T_{11}(\xi, \vartheta)(\zeta_{1'}^0\mathbf{k})(\zeta_1^0\mathbf{k})].$$

The procedure set forth here is, of course, not a universal one: at energies $\xi \sim 10$, where T_{33} changes sign (but $T_{22} \neq 0$), it must be changed. In the case of scattering of positrons, it is assumed that counter 1 registers only positrons, and counter 2, only electrons.

A weak aspect of this method is that it gives not ζ_1^0, but the product $\zeta_{1'}^0\zeta_1^0$. If we take the number of "polarized" electrons per iron atom equal to 2.06[8] and assume that all of the atomic electrons scatter like free electrons (which is correct for sufficiently large energies ϵ_1), then for $\zeta_{1'}^0$ we get the value $\zeta_{1'}^0 = 0.08$, i.e., the asymmetry is small even for $\zeta_1^0 = 1$. The magnetized foil used as an analyzer must be calibrated in a corresponding way, i.e., the value of $\zeta_{1'}^0$ for it must be determined.

3. POLARIZATION IN THE SCATTERING OF AN UNPOLARIZED ELECTRON BEAM BY A POLARIZED TARGET

By the use of Eq. (1) it is not difficult to show that the polarization of the scattered beam (momentum $\mathbf{p}_2$) is given by

$$\zeta_2 = (\varepsilon_2/m)\,\mathrm{Sp}\left(\Sigma\rho_f\right), \qquad (20)$$

where ρ_f is obtained from ρ_i by means of the scattering matrix S:

$$\rho_f = \eta^{(+)}(p_2)\,\eta^{(+)}(p_{2'})\,S\rho_i S^+\eta^{(+)}(p_2)\,\eta^{(+)}(p_{2'})\,/\,\mathrm{Sp}\,[S\rho_i S^+\eta^{(+)}(p_2)\,\eta^{(+)}(p_{2'})]. \qquad (21)$$

If the incident beam is not polarized (momentum $\mathbf{p}_1$, $\zeta_1 = 0$), and the electrons of the target are polarized $(\mathbf{p}_{1'} = 0,\ \zeta_{1'} = \zeta^0_{1'} \neq 0)$, then the polarization of the scattered beam

$$\zeta^0_2 = \zeta_{2\perp} + (m/\varepsilon_2)\,\zeta_{2\parallel} \qquad (22)$$

can be written in the form

$$\zeta^0_{2i} = M_{ik}\zeta^0_{1'k}; \qquad (23)$$

The coefficients M_{ik} are given by

$$M_{11} = \mu(\xi, \vartheta)\{8\xi^2 - \xi(1 + \cos\vartheta)[2(\xi + 1)^2 + (\xi - 1)(\xi + 3)\cos\vartheta + (\xi^2 - 1)\cos^2\vartheta]\},$$
$$M_{22} = \mu(\xi, \vartheta)\{8\xi^2 - \xi(1 + \cos\vartheta)[\xi^2 + 6\xi + 1 - (\xi - 1)^2\cos\vartheta - 2(\xi - 1)\cos^2\vartheta]\}, \qquad (24)$$

$$M_{33} = \mu(\xi, \vartheta)\{8\xi^2 + (1 + \cos\vartheta)[(\xi + 1)(\xi^2 - 8\xi + 3) - (\xi - 1)(\xi^2 + 2\xi - 1)\cos\vartheta]\},$$
$$M_{12} = \mu(\xi, \vartheta)\,\xi(\xi - 1)\sqrt{2(\xi + 1)}\sin\vartheta\,(2 - 3\cos\vartheta - \cos^2\vartheta),$$
$$M_{21} = -\mu(\xi, \vartheta)\,\xi(\xi - 1)\sqrt{2(\xi + 1)}\sin\vartheta\,(4 + \cos\vartheta + \cos^2\vartheta), \qquad (25)$$
$$M_{31} = M_{13} = M_{23} = M_{32} = 0,$$
$$\mu(\xi, \vartheta) = (1 - \cos\vartheta)\{[\xi + 3 + (\xi - 1)\cos\vartheta][4\xi^2(1 + 3\cos^2\vartheta) + (\xi - 1)^2(4 + \sin^2\vartheta)\sin^2\vartheta]\}^{-1}.$$

The values of M_{ik} for $\vartheta = \pi/2$ and various energies ξ are shown in Fig. 4. (We recall that the vectors k, l, n are taken as coordinate axes).

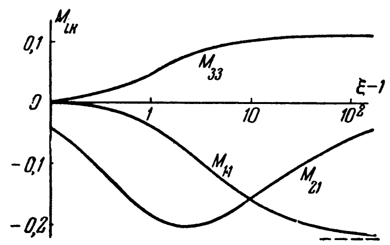

FIG. 4. The coefficients M_{ik} in Eq. (23) for $\vartheta = \pi/2$; $M_{22} = {}^1\!/_2 M_{11}$; $M_{12} = -{}^1\!/_2 M_{21}$

Equation (23) is decidedly simplified in the limiting cases:

a) the nonrelativistic case

$$\zeta^0_2 \approx -\zeta^0_{1'}(1 - \cos\vartheta)\cos\vartheta\,/\,2(1 + 3\cos^2\vartheta); \qquad (26)$$

b) the extreme relativistic case

$$\zeta^0_2 \approx -\frac{(1 - \cos\vartheta)}{(3 + \cos^2\vartheta)^2}\{(2 + \cos\vartheta + \cos^2\vartheta) \qquad (27)$$
$$\times(\zeta^0_{1'}\mathbf{k})\,\mathbf{k} + (1 - \cos\vartheta)[(\zeta^0_{1'}\mathbf{l})\,\mathbf{l} - (\zeta^0_{1'}\mathbf{n})\,\mathbf{n}]\}.$$

We note that the polarization of nonrelativistic electrons is a purely exchange effect and is due to the second term in the expression (9) for the matrix element of the scattering. It is natural that in the case of positron-electron scattering the polarization effects vanish in the nonrelativistic approximation.

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