Decay of the τ Meson

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The energy spectrum of the π mesons resulting from the decay of the τ meson is calculated on the assumption that the isotopic spin of the τ meson is one.

DALITZ[1] and Fabri[2] have considered the decay of the τ meson into 3 π's:

$$\tau^{\pm} \rightarrow 2\pi^{\pm} + \pi^{\mp}. \quad (1)$$

They computed the energy distribution of the π mesons on the basis of various assumptions about the spin and parity of the τ, and also on the assumption that in the expansion of the matrix element for the process into the orbital angular momenta of the π's, only the lowest momenta need be considered. No assumptions were made about the isotopic spin of the τ. The fact that the τ meson has almost the same mass as the θ particles (966 electron masses for both) suggested that the τ meson and the θ particles form an isotopic spin triplet, i.e., that the τ meson has isotopic spin 1 (see Ref. 3)*

The present paper describes a calculation which takes isotopic spin into account. It turns out that, in certain cases, if the π mesons are emitted with minimal orbital angular momentum, the distributions considered by Dalitz and Fabri are still valid. Furthermore, if the τ meson is a pseudoscalar particle (see Ref. 4) then the ratio of the probability dw_1 for decay into charged π's to the probability dw_2 for decay according to

$$\tau^{\pm} \rightarrow \pi^{\pm} + 2\pi^0 \quad (2)$$

is 4, while if the τ is vector, pseudovector or tensor it is 1. If dw_1/dw_2 is neither 4 nor 1, then higher orbital angular momenta must enter into the matrix element for the process. We shall give below the simplest matrix elements for the decay of a τ into 3 π's such that the ratio dw_1/dw_2 is one if the decaying particle is pseudoscalar and four if it is vector, pseudovector or tensor.

Consider the decay of a particle with spin J_τ, parity P_τ, mass m_τ and isotopic spin $I_\tau = 1$, (which we call the τ meson), into 3π mesons, i.e., into three particles with spin $J_\pi = 0$, parity $P_\pi = -1$, mass m_π and isotopic spin $I_\pi = 1$.

Let the momentum, total energy and kinetic energy of the ith π-meson ($i = 1, 2, 3$) be $\mathbf{p}_i$, $E_i = \sqrt{m_\pi^2 + p_i^2}$, and $\epsilon_i = E_i - m_\pi$ respectively, and $\mathbf{t}_{\alpha_i}$, $\mathbf{t}_\beta$ (α_i, β = +, −, 0) be the isotopic spin vectors, in the canonical basis, for the ith π meson and the τ meson:

$$\mathbf{t}_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{t}_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (3)$$

$$\mathbf{t}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \left.\begin{matrix} \nu = 1 \\ \nu = 0 \\ \nu = -1 \end{matrix}\right.$$

All quantities are referred to the system where the meson is at rest:

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0. \quad (4)$$

Consider now the matrix element $\langle V\rangle$ corresponding to the decay of a τ meson with charge β, and the projection of whose spin in some direction is μ, into 3π mesons with charges $\alpha_1, \alpha_2, \alpha_3$ and momenta $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. In the momentum space of the π mesons, this is the μth component of an irreducible tensor of order J_τ relative to the rotation group, while it is a scalar in the isotopic spin space formed by the vectors $\mathbf{t}_\beta$, $\mathbf{t}^+_{\alpha_1}$, $\mathbf{t}^+_{\alpha_2}$, $\mathbf{t}^+_{\alpha_3}$; + here means the Hermitian conjugate:

$$(t^+)_\nu = (-1)^\nu t_{-\nu}; \quad (5)$$

$$\langle V\rangle = (E_1E_2E_3)^{-1/2} v_\mu^{J_\tau}(\mathbf{p}_1, \mathbf{t}^+_{\alpha_1}; \mathbf{p}_2, \mathbf{t}^+_{\alpha_2}; \mathbf{p}_3, \mathbf{t}^+_{\alpha_3}; \mathbf{t}_\beta) \times \delta(m_\tau - E_1 - E_2 - E_3)\,\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3). \quad (6)$$

* Note added in proof: Several authors[6-8] have recently proposed schemes in which the isotopic spin of various K mesons is half integral. In such schemes the τ and θ mesons do not form an isotopic spin multiplet and the equality of their masses receives no interpretation. However, all the results of the present work will remain valid, no matter what the isotopic spin of the τ meson is, if it decays into 3π mesons which have total isotopic spin one.

[The factor $(E_1 E_2 E_3)^{-1/2}$ in the expression for the matrix element comes from the normalization of the wave functions for the particles].

Up to a constant, the probability per unit time for decay of an unpolarized τ meson is then

$$dw = (1/E_1E_2E_3)\sum_\mu |v_\mu^{J_\tau}|^2\, \delta(m_\tau - E_1 - E_2 - E_3)\,\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)\, d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3. \tag{7}$$

The expression above can be put into the following form, which is more suited to calculation,

$$dw = W(\xi, E_3)\, d\xi\, dE_3, \tag{8}$$

$$W(\xi, E_3) = \sum_\mu |v_\mu^{J_\tau}|^2,$$

where $\xi = E_1 - E_2$. In what follows we take the index 3 in $W(\xi, E_3)$ to refer to the $\pi^\pm$ meson in reaction (1) or to $\pi^\pm$ in reaction (2). Then (8) gives the probability for decay into a state where the difference in energy $E_1 - E_2$ of the identical π mesons lies in the interval $\xi \leq E_1 - E_2 \leq \xi + d\xi$ and the energy of the third π meson is between E_3 and $E_3 + dE_3$. Integrating (8) with respect to ξ over the interval $|\xi| \leq |E_1 - E_2|_{max} = y(E_3)$* will give the energy spectrum of the third π meson.

Since the π mesons are Bose particles, $v_\mu^{J_\tau}(\mathbf{p}_i, \mathbf{t}^+_{\alpha_i}; \mathbf{t}_\beta)^I$ must be symmetric in the indices 1, 2, 3. From (4), it follows that $v_\mu^{J_\tau}(\mathbf{p}_i; \mathbf{t}^+_{\alpha_i}; \mathbf{t}_\beta)$ can be written

$$v_\mu^{J_\tau}(\mathbf{p}_i, \mathbf{t}^+_{\alpha_i}; \mathbf{t}_\beta) = f_\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)\, X_0 + \varphi_\mu(\mathbf{p}_3, \mathbf{p}_1 - \mathbf{p}_2)\, X_{12} + \varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3)\, X_{23} + \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)\, X_{31}. \tag{9}$$

Here

$$X_0(\mathbf{t}^+_{\alpha_i}, \mathbf{t}_\beta) = (\mathbf{t}^+_{\alpha_1}, \mathbf{t}^+_{\alpha_2})(\mathbf{t}^+_{\alpha_3}, \mathbf{t}_\beta) + (\mathbf{t}^+_{\alpha_2}, \mathbf{t}^+_{\alpha_3})(\mathbf{t}^+_{\alpha_1}, \mathbf{t}_\beta) + (\mathbf{t}^+_{\alpha_3}, \mathbf{t}^+_{\alpha_1})(\mathbf{t}^+_{\alpha_2}, \mathbf{t}_\beta); \tag{10}$$

$$X_{12} = (\mathbf{t}^+_{\alpha_1}, \mathbf{t}^+_{\alpha_3})(\mathbf{t}^+_{\alpha_2}, \mathbf{t}_\beta) - (\mathbf{t}^+_{\alpha_2}, \mathbf{t}^+_{\alpha_3})(\mathbf{t}^+_{\alpha_1}, \mathbf{t}_\beta),$$

$$X_{23} = (\mathbf{t}^+_{\alpha_2}, \mathbf{t}^+_{\alpha_1})(\mathbf{t}^+_{\alpha_3}, \mathbf{t}_\beta) - (\mathbf{t}^+_{\alpha_3}, \mathbf{t}^+_{\alpha_1})(\mathbf{t}^+_{\alpha_2}, \mathbf{t}_\beta);$$

$$X_{31} = (\mathbf{t}^+_{\alpha_3}, \mathbf{t}^+_{\alpha_2})(\mathbf{t}^+_{\alpha_1}, \mathbf{t}_\beta) - (\mathbf{t}^+_{\alpha_1}, \mathbf{t}^+_{\alpha_2})(\mathbf{t}^+_{\alpha_3}, \mathbf{t}_\beta) = -X_{12} - X_{23}, \tag{11}$$

$$(\mathbf{t}_\alpha, \mathbf{t}_\beta) = \sum (-1)^\nu (t_\alpha)_\nu (t_\beta)_{-\nu},$$

$f_\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is a symmetric function of the π meson momenta;

$$\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) = -\varphi_\mu(\mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_2).$$

In the case that the τ meson decays into three charged π's (reaction (1)), then $\alpha_1 = \alpha_2 = \beta = \pm$, $\alpha_3 = \pm$ (where 1 and 2 refer to the two identical π mesons). Then

$$W(\xi, E_3) = W_1(\xi, E_3) = 4\sum_\mu |f_\mu|^2 + \sum_\mu |\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) - \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)|^2 - 4\sum_\mu \mathrm{Re}\{f^*_\mu[\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) - \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)]\}. \tag{12}$$

If the τ meson decays according to (2), $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \beta = \pm n$,

$$W(\xi, E_3) = W_2(\xi, E_3) = \sum_\mu |f_\mu|^2 + \sum_\mu |\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) - \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)|^2 + 2\sum_\mu \mathrm{Re}\{f^*_\mu[\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) - \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)]\}. \tag{13}$$

* Using Fabri's results[2] it is easy to show that

$$y = |E_1 - E_2|_{\max} = (m_\tau - 3m_\pi)\sqrt{\frac{1 - x^2 - \alpha(1+x)x^2}{3[1 + \alpha(1 - 3x)]}};$$

$$x = 3\varepsilon_3/(m_\tau - 3m_\pi) - 1;$$

$$\alpha = \frac{(m_\tau - 3m_\pi)/m_\tau}{2[1 - (m_\tau - 3m_\pi)/2m_\tau]^2} = 0.089.$$

Upon integrating (12) and (13) over ξ and E_3, and denoting the total cross sections for (1) and (2) by W_1 and W_2, while

$$\int \sum_\mu |f_\mu|^2 d\xi\, dE_3 = \int F(\xi, E_3)\, d\xi\, dE_3 = F, \tag{14}$$

$$\int \sum_\mu |\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) - \varphi_\mu(\mathbf{p}_2, \mathbf{p}_3 - \mathbf{p}_1)|^3 d\xi\, dE_3 = \int \Phi(\xi, E_3)\, d\xi\, dE_3 = \Phi,$$

we obtain the ratio of the reactions (1) and (2) (cf Refs. 4 and 5)

$$1 \leqslant W_1 / W_2 = (4F + \Phi)/(F + \Phi) \leqslant 4. \tag{15}$$

From (12) and (13) it follows that

$$W_1(\xi, E_3) + 2W_2(\xi, E_3) = 6F(\xi, E_3) + 3\Phi(\xi, E_3). \tag{16}$$

Having (16) and (15), we can consider separately the matrix elements

$$f^\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) X_0 \quad \text{and} \quad \sum_{i \neq j \neq k} \varphi_\mu(\mathbf{p}_i, \mathbf{p}_j - \mathbf{p}_k) X_{jk}.$$

$f_\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and $\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3)$ can be expanded in the following series:

$$f_\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \sum_{i \neq j \neq k} \sum_{ll'} A_{ll'}(p_i, |\mathbf{p}_j - \mathbf{p}_k|)\, Z_{ll'}^{J_\tau, \mu}(\mathbf{p}_i, \mathbf{p}_j - \mathbf{p}_k), \tag{17}$$

$$\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3) = \sum_{ll'} B_{ll'}(\mathbf{p}_1, |\mathbf{p}_2 - \mathbf{p}_3|)\, Z_{ll'}^{J_\tau, \mu}(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3),$$

where

$$Z_{ll'}^{J_\tau, \mu}(\mathbf{p}, \mathbf{p}') = p^l p'^{l'} \sum_{\alpha, \beta} C_{l, \alpha; l' \beta}^{J_\tau, \mu} Y_{l; \alpha}(\mathbf{p}/p)\, Y_{l', \beta}(\mathbf{p}'/p'),$$

$C_{l, \alpha; l', \beta}^{J_\tau, \mu}$ are the Clebsch-Gordon coefficients and $Y_{l\alpha}(\mathbf{p}/p)$ the spherical harmonics of order l.

Matrix elements for the decay of a τ meson into three π mesons which have minimum possible orbital angular momenta. l and l' are the orbital angular momenta entering into (17); $\mu, \nu = 1, 2, 3$. Expressions underlined are considered in Refs. 1 and 2. In $F(p_i)$ and $\Phi(p_i)$ the indices 1, 2 refer to the identical π mesons.

J_τ, P	l, l'	$f_\mu(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$	$F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$	l, l'	$\varphi_\mu(\mathbf{p}_1, \mathbf{p}_2 - \mathbf{p}_3)$	$\Phi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$
0—	0,0	$\underline{1}$	$\underline{1}$	2,2	$2\mathbf{p}_1\mathbf{p}_2 + p_3^2$	$(p_1^2 + p_2^2 - 2p_3^2)^2$
1+	1,0 2,1	$p_1^2\mathbf{p}_1 + p_2^2\mathbf{p}_2 + p_3^2\mathbf{p}_3$	$\{p_1^2\mathbf{p}_1 + p_2^2\mathbf{p}_2 + p_3^2\mathbf{p}_3\}^2$	0,1	$\underline{\mathbf{p}_3}$	$\underline{p_3^2}$
1—	2,2 4,4	$\sum_{i,j=1}^{3} [\mathbf{p}_i\mathbf{p}_j](p_i^2 - p_j^2)^3$	$[\mathbf{p}_1\mathbf{p}_2]^2 \{(p_1^2 - p_2^2)^3 + (p_2^2 - p_3^2)^3 + (p_3^2 - p_1^2)^3\}^2$	2,2	$\underline{[\mathbf{p}_1\mathbf{p}_2](p_1^2 - p_2^2)}$	$\underline{[\mathbf{p}_1\mathbf{p}_2]^2 (p_1^2 - p_2^2)^2}$
2+	2,1 2,3	$\sum_{i \neq j \neq k} p_i^2 [\mathbf{p}_j\mathbf{p}_k]_\mu (\mathbf{p}_j - \mathbf{p}_k)_\nu$	$[\mathbf{p}_1\mathbf{p}_2]^2 \{p_1^2(\mathbf{p}_2 - \mathbf{p}_3) + p_2^2(\mathbf{p}_3 - \mathbf{p}_1)^2 + p_3^2(\mathbf{p}_1 - \mathbf{p}_2)^2\}$	2,1	$\underline{[\mathbf{p}_1\mathbf{p}_2]_\nu (\mathbf{p}_1 - \mathbf{p}_2)_\mu}$	$\underline{[\mathbf{p}_1\mathbf{p}_2]^2 (\mathbf{p}_1 - \mathbf{p}_2)^2}$

Suppose that $A_{ll'}$ and $B_{ll'}$ as functions of p and p' can be written

$$A_{ll'} = a_0 + a_1 p^2 + a_2 p'^2 - \ldots,$$
$$B_{ll'} = b_0 + b_1 p^2 + b_2 p'^2 + \ldots$$

and that only the non vanishing terms of lowest order are important (this is equivalent to neglecting higher momenta l and l'). Certain simple expressions for f_μ and φ_μ then result. These are shown in the table for the cases where the decaying particle is pseudoscalar, pseudovector, vector and tensor. The case that τ is scalar is eliminated by the requirement that parity be conserved (see Ref. 3).

From the above it follows that if the π mesons from τ decay are emitted into states with minimum orbital angular momentum, then

$$W_1(E_3) + 2W_2(E_3) = = aF_0(E_3) + b\Phi_0(E_3), \qquad (18)$$

$$F_0(E_3) = = \int F(\xi, E_3) d\xi / \int F(\xi, E_3) d\xi \, dE_3,$$
$$\Phi_0(E_3) = = \int \Phi(\xi, E_3) d\xi / \int \Phi(\xi, E_3) d\xi dE_3,$$

[$W_1(E_3)$ is the probability that in reaction (1) a $\pi^{\pm}$ meson will appear with energy between E_3 and $E_3 + dE_3$, while $W_2(E_3)$ is defined similarly for a $\pi^{\pm}$ meson in reaction (2)].

The weights a and b in formula (18) are found from(15):

$$a = 2(W_1 - W_2), \qquad b = 4W_2 - W_1. \qquad (19)$$

We note that the Lagrangian for the interaction between the τ and π meson fields can be written

$$\mathscr{L}_{\text{int}} = \Big\{ \sum_{\mu,\lambda} D^{J_\tau,\mu,\lambda} [\psi^+_{\alpha_1}(x_1)\psi^+_{\alpha_2}(x_2)\psi^+_{\alpha_3}(x_3)\Psi^\mu_\beta(x_\tau)] \times X_\lambda(t^+_{\alpha_i}, t_\beta) + \text{c. c.} \Big\},$$

where $\psi^+_{\alpha_i}(x_i)$ is the creation operator for a π^{α_i}-meson at x_i, $\Psi^\mu_\beta(x_\tau)$ is the destruction operator of a τ^β meson at x_τ whose spin projection along some axis is μ and $D^{J_\tau,\mu,\lambda}$ are certain differential operators containing the minimal number of derivatives for given J_τ, p, λ and evaluated at the point $x_1 = x_2 = x_3 = x_\tau = x$. The matrix elements obtained from this agree, up to relativistic corrections, with those in the table.

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