which diminishes as $x \to \infty$. The asymptotic behavior of such a solution for large values of x may be represented in the form

$$u = -g\sqrt{\lambda}\, e^{-x}, \qquad (3)$$

where g is an arbitrary constant.

For small distances, Eq. (2) can be replaced by the asymptotic souation

$$d^2u / dx^2 - u^3 / x^2 = 0. \qquad (4)$$

Equation (4) is an analog of the Emden-Fowler equation, and it can be reduced to an equation of the first order.[10] Indeed, as a result of making the substitution $x = e^{-5}$ and of the introduction of $y = du/dt$, we shall obtain

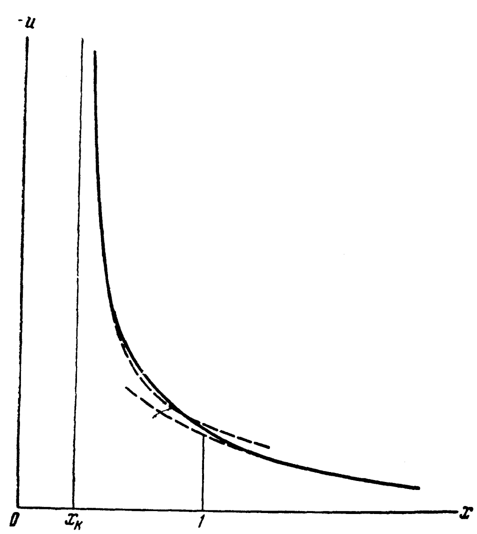
$$dy / du = u^3 / y - 1. \qquad (5)$$

A qualitative investigation of the behavior of phase trajectories in the (y, u) plane shows that a characteristic feature of all the solutions of Eq. (4) is the existence of a singular point whose position is not fixed, but depends on the constant of integration.

Applying Hardy's theorem to Eq. (5), we obtain for the asymptotic solution (for small distances)

$$u = -\sqrt{2} / \ln(x / x_k), \qquad (6)$$

where x_k is an arbitrary constant.



In addition, Eq. (2), was integrated numerically. The integration was carried out by starting with the asymptotic solution at large distances. To each value of the quantity g corresponds a definite value of the quantity x_k.

As may be seen from the diagram the solution obtained by numerical integration (solid curve) may be roughly approximated by the functions (3) and (6) matched at the point $x = 1$ (dotted curves).

The author hopes to give the interpretation of the result obtained above and its application to the description of the properties of a system of two nucleons at low energies in a subsequent article.

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On the Hydrodynamic Description of Plasma Oscillations

B. B. KADOMTSEV

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SOMETIMES for the theoretical study of plasma one uses in place of the kinetic equation for the distribution function of the particles the simpler "transport" equations for the moments of this function (see, for example, Refs. 1,2). In so doing, in order to obtain a closed system of equations, one usually assumes that the distribution function is Maxwellian. Such an approximation cannot be applied to the description of high frequency plasma oscillations since in this case the electron distribution deviates appreciably from the Maxwellian distribution. However, in this case also, one can obtain a closed system of transport equations by using the fact that the only reason for the deviation of the distribution from the equilibrium distribution is the presence of a high frequency electrical field.

By restricting ourselves to the investigation of electron oscillations only we regard the ions and the molecules as infinitely heavy. Moreover, we assume that the amplitude of oscillations is small, and that consequently the electron distribution function deviates but little from the Maxwellian distribution $f_0(v)$. As a result of being acted on by the electric field, the electrons acquire a velocity increment $d\mathbf{v} = -(eE/m)\,dt$, as a result of which the distribution function changes at the initial instant by an amount $df' = (\partial f_0/\partial \mathbf{v}) = (e\mathbf{E}/m\,dt.)$ At later times, the distribution df' will vary in accordance with Boltzmann's equation, and during the first stages, this variation will be determined by the free acceleration of the electrons accompanied by their removal from the beam as a result of collisions. By denoting the frequency of collisions of a given electron with other electrons by α, and the frequency of collisions with ions (and molecules) by β and by assuming that α and β do not depend on the velocity, we shall obtain the following expression for the distribution of those electrons which have experienced no collision at all:

$$f'(\mathbf{r}, \mathbf{v}, t) = -\int_{-\infty}^{t}\int \frac{f_0 e\mathbf{v}\mathbf{E}(r', t')}{\varkappa T_0} e^{-(\alpha+\beta)(t-t')} \times \delta(\mathbf{r}-\mathbf{r}'-\mathbf{v}(t-t'))\,d\mathbf{r}'dt' \tag{1}$$

($\varkappa$ is the Boltzman constant).

Electrons which have been removed from the initial beam, i.e., which have experienced at least one collision, also lead to a certain change in the equilibrium distribution function. By assuming that one collision is already sufficient to establish a Maxwellian distribution, we represent the above change as a change in the parameters of the Maxwellian distribution—density, velocity and temperature. This process can be described hydrodynamically. If we denote by n, T the deviations of the electron density and temperature from their equilibrium values n_0, T_0, and by u their macroscopic velocity, we can write the hydrodynamic equations in the form

$$\partial n/\partial t + n_0 \operatorname{div}\mathbf{u} = (\alpha+\beta)\, n_0 I_1; \tag{2}$$

$$\partial \mathbf{u}/\partial t + \beta\mathbf{u} + (\varkappa T_0/mn_0)\nabla n + (\varkappa/m)\nabla T = \alpha \mathbf{I}_2,$$

$$\partial T/\partial t + {}^2/_3 T_0 \operatorname{div}\mathbf{u} - \chi\Delta T = (\alpha+\beta)(m/3\varkappa)\, I_3.$$

Here the term $\beta\, u$ takes into account the slowing down of electrons by the heavy particles, χ is the coefficient of thermal conductivity, and the expressions on the righthand side arise as a result of the removal of electrons from the distribution (1).

In accordance with (1),

$$I_1 = \frac{1}{n_0}\int f' d\mathbf{v} = -\frac{e}{\varkappa T_0}\left(\frac{m}{2\pi\varkappa T_0}\right)^{3/2}\int_{-\infty}^{t}\int \frac{(\mathbf{r}-\mathbf{r}')\,\mathbf{E}(\mathbf{r}', \mathbf{t}')}{(t-t')^4} \times \exp\left\{-(\alpha+\beta)(t-t') - \frac{m(\mathbf{r}-\mathbf{r}')^2}{2\varkappa T_0 (t-t')^2}\right\} dr'\,dt',$$

while I_2 and I_3 differ from I_1 only by the additional appearance in the integrand of the factors

$$(\mathbf{r}-\mathbf{r}')/(t-t')$$

and

$$(\mathbf{r}-\mathbf{r}')^2/(t-t')^2.$$

respectively. One must also add to the system (2) the equation for the electric field

$$\operatorname{div}\mathbf{E} = -4\pi e(n+n'), \tag{3}$$

where

$$n' = n_0 I_1.$$

Equations (2), (3) are the desired transport equations. For $\alpha \to \infty$ they reduce to the usual hydrodynamic equations, while for $\alpha = \beta = 0$ Eq. (3) coincides with the dispersion equation obtained from the linearized Vlasov equation.[2] A characteristic feature of the system (2), (3) is the fact that it takes into account roughly, but consistently, the effect of electron collisions.

If one assumes that all the quantities vary in accordance with $e^{i\omega t + ikr}$ then it is not difficult to obtain from (2), (3) the dispersion equation. In the general case it has a rather complicated appearance, but if one restricts the discussion to small k, then an expansion in powers of k can be carried out in the integrals I_i and a simpler equation can be obtained. Up to terms in k^3 it has the form

$$k\left\{\omega\left[\omega_0^2-\omega^2+i\omega\beta+\frac{5}{3}\frac{\varkappa T_0}{m}k^2+\chi k^2(i\omega+\beta)\right]+k^2\omega_0^2\frac{\varkappa T_0}{m}\frac{3i(\alpha+\beta)-4\omega}{(\alpha+\beta+i\omega)^2}-i\omega_0^2\chi k^2\right\}=0, \qquad (4)$$

where $\omega_0=\sqrt{4\pi e^2 n_0/m}$.

This equation has the following solutions:
1. $k=0$,
With

$$n=T=0,\ (i\omega+\beta)\,u=-e\alpha E/m\,(\alpha+\beta+i\omega).$$

For $i\omega \neq -\beta$ this gives the current under the influence of the external field

$$j=-en_0u-en_0I_2=n_0e^2E/m\,(i\omega+\beta).$$

2. $$\omega^2\approx\omega_0^2+i\omega_0\beta+\frac{3\varkappa T_0}{m}k^2$$

(taking into account that α, $\beta \ll \omega_0$); this is the usual expression for the frequency of plasma oscillations with "friction" taken into account.

3. $i\omega\approx-[\chi-\varkappa T_0/m\,(\alpha+\beta)]\,k^2,\ u=eE/m\,(\alpha+\beta),$

$$T=-T_0n/n_0+i\,(e/\varkappa k)\,E,\quad n=-(ik/4\pi e)\,[1+\omega_0^2(\alpha+\beta)^{-2}]\,E.$$

This solution can be referred to as the electro-entropy wave. For $e\to 0$ it reduces to the usual entropy wave.[3].

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The Variational Principle and the Virial Theorem for the Continuous Dirac Spectrum

IU. V. NOVOZHILOV

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THE variational principle for the phase shifts and scattering amplitudes has been investigated for the nonrelativistic case in several works.[1,2] Parzen[3] has formulated the variational principle for the phase shifts in the relativistic case. The present note gives a generalization of the variational principle for all scattering amplitudes to the case of the Dirac equation, and a derivation of the virial theorem for the continuous Dirac spectrum. The results can be applied to the theory of high-energy electron scattering by nuclei.

Let us consider the functional

$$I\{\Psi_1,\Psi_2\}=\int\Psi_2^+(\mathbf{r})\,[\alpha\mathbf{p}+\beta m+V(\mathbf{r})-E]\,\Psi_1(\mathbf{r})\,d\mathbf{r}, \qquad (1)$$

where the functions Ψ_i are not in general solutions ψ_i of the Dirac equation

$$[\alpha\mathbf{p}+\beta m+V(\mathbf{r})-E]\,\psi_i=0.$$

For the exact solutions, $I\{\psi_1,\psi_2\}=0$. Let us restrict ourselves initially to potentials V which decrease faster than $1/r$ as $r\to\infty$. Then, in order that the functional (1) converge at the upper limit, the asymptotic form of the trial functions ψ_1 and ψ_2 should be

$$\Psi_i=u_i\exp[ip\boldsymbol{\nu}_i\mathbf{r}]+G(\boldsymbol{\nu}_i,\mathbf{n})\,(pr)^{-1}\exp[\pm ipr], \qquad (2)$$

where u_i is a unit spinor, $G(\nu,\mathbf{n})$ is a single-valued function of direction, and $\mathbf{n}=\mathbf{r}/r$; the upper sign in the second term of Eq. (2) refers to ψ_1 and the lower one to ψ_2. The function ψ_1 contains a plane wave propagating in the direction ν_1 and an outgoing wave; the function ψ_2, on the otherhand, is seen from Eq. (2) to contain, in addition to a plane wave with propagation direction ν_2, an incoming wave. The asymptotic forms of the exact solutions ψ_1 and ψ_2 are similar to Equation (2), but instead of the function $G(\nu,\mathbf{n})$, they contain scattering amplitudes $G^\circ(\nu,\mathbf{n})$. Thus in the asymptotic form, variations $\delta\psi_1$ are due to variations $\delta G(\nu_1,\mathbf{n})$.

The first variation of the functional (1), caused by variations $\delta\psi_1=\psi_i-\psi_i$ of the functions ψ_i about the exact solutions ψ_i, is given by

$$\delta I=\int\psi_2^+\{\alpha\mathbf{p}+\beta m+V-E\}\,\delta\Psi_1 d\mathbf{r}=-i\int\psi_2^+\alpha\mathbf{n}\,\delta\Psi_1 dS, \qquad (3)$$