assumptions on the value of Δt .

Thus, even at a temperature very close to T_{12} (from the viewpoint of experimental possibilities), the relaxation time can be considered as practically infinitely large, which supports the basic assumption made in the research.

In conclusion, I am pleased to thank V. L. Ginzburg for supervising the research and for several suggestions made during the final editing.

1 V. L. Ginzburg, J. Exptl. Theoret. Phys. (U.S.S.R.) 19, 36 (1949).
2 A. F. Devonshire, Phil. Mag. 40, 1040 (1949).
3 A. F. Devonshire, Phil. Mag. 42, 1065 (1951).
4 M. Ia. Shirobokov and L. P. Kholodenko, J. Exptl. Theoret. Phys. (U.S.S.R.) 21, 1239 (1951).
5 L. P. Kholodenko and M. Ia. Shirobokov, J. Exptl. Theoret. Phys. (U.S.S.R.) 21, 1250 (1951).
6 G. A. Smoleiskii and R. E. Pasynkov, J. Exptl. Theoret. Phys. (U.S.S.R.) 24, 69 (1953).
7 L. P. Kholodenko , J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 244 (1956); Soviet Phys. JETP 4, 188 (1957).
8 W. J. Merz, Phys. Rev. 91, 513 (1953).
9 W. J. Merz, Phys. Rev. 76, 1221 (1949).
10 H. F. Kay and P. Vousden, Phil. Mag. 40, 1019 (1949).
11 V. L. Ginzburg, Uspekhi Fiz. Nauk 46, 348 (1952).
12 G. B. Dwight, *Tables of Integrals* IIL, 1949 (Russian translation)

Translated by Robert T. Beyer
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The Method of Envelopes for Investigating Free Oscillations in Accelerators*

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We present a derivation of the equation for free oscillations in accelerators with an arbitrary magnetic field having a plane of symmetry. To solve the basic problems of the theory of free oscillations, which arise in the design of accelerators, an envelope method has been developed in which the study of individual orbits is replaced by consideration of the envelope of the trajectory of the particles over a large number of revolutions. The application of the method is illustrated for accelerators with a sector magnet and for strong-focusing accelerators.

1. INTRODUCTION

In cyclic accelerators, the displacement of a particle from some average position is called a free oscillation. This average position of the particle is usually called the instantaneous orbit. The term "free" means that these oscillations are not directly connected with the process of acceleration. We can therefore consider the free oscillations in a constant magnetic field and for constant energy of the particle.

The fact that the free oscillations are independent of the acceleration process was demonstrated in the very first papers on the theory of cyclic accelerators. It was shown that with increasing magnetic field H (for a given configuration) the oscillation amplitude is damped like $H^{-1/4}$

The separation of the particle motion into a motion along the instantaneous orbit and free oscillations can be done uniquely in the absence of resonances. At resonance, the frequency of the free oscillations is integrally related to the frequency of revolution, so that the particle orbit is always closed. In this case the trajectory of the particle over a large number of revolutions does not fill an area, but merely traces out a line. (The plane area filled out by the orbit will be the subject of our investigation .) For practical purposes, because of the presence of all sorts of perturbations in the magnetic field, the acceleration process cannot proceed at resonance, since the amplitude of the free oscillations increases sharply. We shall exclude the resonance case from our further considerations

Free oscillations develop during the process of injection of particles, and also from scattering of particles by the residual gas in the accelerator cham-

*The present paper is based on work[1–5] completed during the period 1950–1953.

*Deceased.

ber. One of the most difficult problems in constructing present–day ring accelerators (synchrotron, synchrophasotron) is the problem of achieving effective injection of particles, i.e., admission of particles into the chamber without colliding with the injector (inflector plates) and the walls of the vacuum chamber.

The essential point is that the efficiency of injection determines the intensity of the accelerated beam, since the loss of particles during the acceleration process is usually small. The geometrical dimensions of the accelerator magnet gap determine all the other geometrical dimensions and the stability of the accelerator. The choice of dimensions is determined to a large extent by the need for achieving conditions for effective injection.

In recent years, various designs have been worked out for strong focusing accelerators. It is of interest to evaluate the efficiency of various types of strong–focusing accelerators. One comparison criterion might be the character of the free oscillation; other well–known criteria are the tolerances in magnet construction and field configuration.

The envelope method developed by us makes it possible, from a unified point of view, to describe the effect of free oscillations on the process of injection and acceleration in cyclic accelerators of any type.

2. EQUATIONS OF FREE OSCILLATIONS

We consider the free oscillations in an arbitrary magnetic field H having a plane of symmetry.

The motion of the particle in the magnetic field is described by the relativistic equation

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right)=\frac{e}{c}\,[\mathbf{v}\times\mathbf{H}] \tag{1}$$

If H does not depend on the time (as we can always assume in considering free oscillations), then $|\mathbf{v}|=$ const. From now on we shall express the velocity in the form $\mathbf{v}=v\mathbf{n}$, where n is a unit vector. Replacing differentiation with respect to the time t by derivatives with respect to σ, the distance along the trajectory, we get

$$\frac{mv}{\sqrt{1-v^2/c^2}}\frac{d\mathbf{n}}{d\sigma}=\frac{e}{c}\,[\mathbf{n}\times\mathbf{H}] \tag{2}$$

We shall assume from now on that the field lines cut some plane at a constant angle. If the upper and lower poles of the magnet are mirror images, the median plane of the magnet will be such a plane. Of all the possible orbits, we select a closed (but otherwise, entirely arbitrary) orbit, lying in this plane. We call this orbit the equilibrium orbit.

Of course, a more general formulation of the problem is possible with an arbitrary space curve as equilibrium orbit, but the simplification made above is sufficient for the purposes of accelerator technology. In this case Eq. (2) becomes very simple:

$$RH=\text{const}, \tag{3}$$

where R is the radius of curvature at a particular point where the magnetic field is H. We shall denote the curvature and magnetic field at each point of the equilibrium orbit by the subscript zero.

To study the deviation of the particular from the equilibrium orbit, we introduce curvilinear (in general, nonorthogonal) coordinates: σ is the length measured along the equilibrium trajectory, ρ is the distance along the normal to this trajectory. In these coordinates, the radius of curvature is:

$$\begin{aligned}\frac{1}{R}=\Big[-\rho''\left(1+\frac{\rho}{R_0}\right)&+2\rho'^2\frac{1}{R_0}-\rho' R_0'\frac{\rho}{R_0^2}\\&+\frac{1}{R_0}\left(1+\frac{\rho}{R_0}\right)^2\Big]\\&\times\left[\left(1+\frac{\rho}{R_0}\right)^2+\rho'^2\right]^{-3/2},\end{aligned} \tag{4}$$

where the primes denote differentiation with respect to σ.

Since the deviations of the particle from the equilibrium orbit are small, we can use a linear approximation for Eq. (4) and for the field strength $H(\sigma,\rho)$ as a function of ρ:

$$\frac{1}{R}=\frac{1}{R_0}-\rho''-\frac{\rho}{R_0}; \tag{5}$$

$$H(\sigma,\rho)=H(\sigma,0)+\frac{\partial H}{\partial\rho}\rho.$$

After substitution of (5), Eq. (3) takes the form

$$\frac{1}{R_0}-\rho''-\frac{\rho}{R_0}=\text{const}\left[H(\sigma,0)+\frac{\partial H}{\partial\rho}\rho\right].$$

By its definition, $1/R_0(\sigma)=$ const $\cdot\, H(\sigma,0)$ along the equilibrium orbit, so that the equation for the free oscillations takes the following simple form:

$$\rho''+q(\sigma)\rho=0,\quad q(\sigma)=R_0^{-2}(\sigma)\,[1-n(\sigma)], \tag{6}$$

where $n(\sigma)$ is the field index

$$n(\sigma) = -(R_0(\sigma)/H(\sigma, 0)) \qquad (7)$$

$$(\partial H(\sigma, \rho)/\partial \rho)_{\rho=0}.$$

3. SOLUTION OF THE EQUATIONS OF FREE OSCILLATIONS

The function $q(\sigma)$ is a periodic function of σ. In weak–focusing accelerators the period is equal to the perimeter Π of the orbit: $q(\sigma + \Pi) = q(\sigma)$. In strong-focusing accelerators and in accelerators with a sector magnet, the period of the function $q(\sigma)$ is $\Pi/N = L$, where N is the number of periodicity elements in the system. The general theory of equations with periodic coefficients can be applied to Eq. (6).

We write the solution of Eq. (6) in the stable region in the following form:

$$\rho(\sigma) = \operatorname{Re} D e^{i\mu\sigma/L} \varphi(\sigma), \qquad (8)$$

where $\varphi(\sigma)$ is a periodic function of σ with period L.

The characteristic exponent μ and the complex Floquet function $\varphi(\sigma)$ can be found for each specific case by an analytic method or by numerical integration over one of the sections of length L.* In addition, we note that the Wronskian of the complex solutions of Eq. (6):

$\Psi = e^{i\mu\sigma/L} \varphi(\sigma)$ and Ψ^*

$= e^{-i\mu\sigma/L} \varphi^*(\sigma)$ is $-2i$.

The solution of Eq. (8) can also be written in another more convenient form:

$$\rho(\sigma) = F(\sigma) \cos[\mu\sigma/L + \alpha(\sigma)], \qquad (9)$$

$$F(\sigma) = |D\varphi(\sigma)|; \ \alpha(\sigma) = \arg(D\varphi(\sigma)). \qquad (10)$$

Obviously $F(\sigma)$ and $\alpha(\sigma)$ are periodic functions with period L.

Thus the free oscillations can always be represented as sinusoidal, with variable amplitude $F(\sigma)$, phase $\alpha(\sigma)$ and frequency μ/L. However, this interpretation gives nothing new, since in most cases the frequency of variation of $F(\sigma)$ and $\alpha(\sigma)$ is greater than or of the same order as μ/L. It is therefore meaningless to regard the free oscillations as modulated harmonic oscillations, as has been done by some authors.[6]

In accelerator theory, when we investigate free oscillations we consider two problems: a) collisions of particles with the injector plates, and b) collision of particles with the walls of the vacuum chamber.

We consider a particle which leaves the injector at an angle γ to the instantaneous orbit and at a distance ρ_0 from it. If the displacement ρ at the injector azimuth σ_i on succeeding turns is greater than $\rho_0 - \Delta$, then the particle will collide with the injector plates. Here Δ is the distance from the point of emergence of the particle to the front plate of the injector (Figure 1).

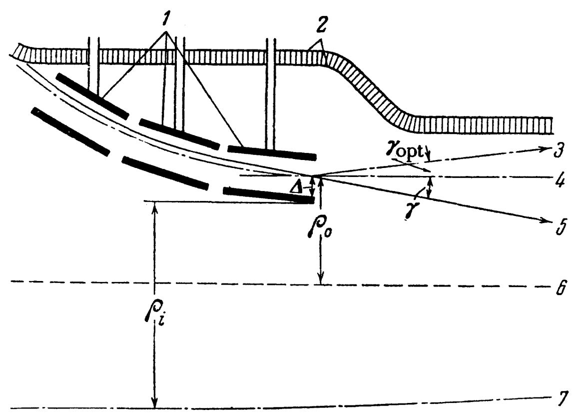

FIG. 1. Diagram of entry of particles into the acclerator. 1—injector plates, 2—chamber walls, 3—optimum direction, 4—direction tangent to the instantaneous orbit, 6—instantaneous equilibrium orbit, 7—mean orbit in the chamber.

For the solution of the problems formulated here, we need only know the deviation of the particle at some definite azimuth σ, in particular at the azimuth of the injector. Obviously the coordinates σ and $\sigma + \Pi k$ correspond to the same azimuth (Π is the perimeter of the orbit, and k is an integer). In this case, (9) can be rewritten as

$$\rho(\sigma) = F(\sigma) \cos(\mu N k + \alpha(\sigma)). \qquad (11)$$

Here σ is a fixed number, and k is the number of revolutions carried out by the particle; $\rho(\sigma)$ is the position of the particle at azimuth σ after the kth revolution. If k were a continuous variable, we could say that the oscillation of the particles at a given azimuth is always harmonic with a constant

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*For example, if we set $\Psi(0) = 1 + 1 \times 0$, then $\cos \mu = \operatorname{Re} \Psi(L)$.

amplitude. When we change from one azimuth to another, only the amplitude and phase of the oscillation changes. Thus we have a situation analogous to that of standing waves, although, in our case, not only the amplitude but also the phase changes in space.

Actually k can only take on integer values. If, for example, $N\mu$ were equal to $2\pi l/g$, where l and g are integers, then the displacement would take on g values at each azimuth. In practice, g must be sufficiently large so that no dangerous resonances arise; we may then assume that at any azimuth ρ takes on practically all values from $-F(\sigma)$ to $+F(\sigma)$.

We have been regarding $F(\sigma)$ as the amplitude of quasi-harmonic vibrations at azimuth σ. We can also consider the curve $\rho = \pm F(\sigma)$ to be the envelope of the particle trajectory. In fact, if $\rho(\sigma) = F(\sigma)$, then according to (9), $d\rho/d\sigma = dF/d\sigma$; i.e., the tangents to the particle trajectory (9) and to the curve $\rho = F(\sigma)$ coincide at azimuths where ρ is a maximum. From now on we shall call the curves $\rho = \pm F(\sigma)$ envelopes.

The trajectory of the particles is contained between the curves $\rho = \pm F(\sigma)$ and, in cases of practical interest, will fill the whole region between the envelopes after a large number of revolutions. Finding the envelope is much simpler than calculating the particle trajectories. At the same time, all the basic problems of accelerator theory which are related to the free oscillations can be solved if the envelope is known.

4. THE ENVELOPE

We shall find the expression for the square modulus of the function $F(\sigma)$ as a function of the initial conditions:

$$\sigma = \sigma_i; \quad \rho(\sigma_i) = \rho_0; \quad (d\rho/d\sigma)_{\sigma_i} = \gamma; \quad k = 0,$$

where σ_i is the azimuth of the injector, ρ_0 is the initial deviation of the particle from the orbit, and γ is the angle between the direction of emergence of the particle and the tangent to the instantaneous orbit (cf. Fig. 1). Substituting the initial conditions in (8), we get the value of the constant D

$$D^* = i\,[\gamma\Psi(\sigma_i) - \rho_0\Psi'(\sigma_i)]. \tag{12}$$

Here the prime denotes differentiation with respect to σ, and the asterisk denotes the complex conjugate:

$$F^2(\sigma) = DD^*\varphi(\sigma)\varphi^*(\sigma) = DD^*\Phi(\sigma), \tag{13}$$

where $\Phi(\sigma)$ is the square modulus of the Floquet function.

Substituting (12) in (13) and simplifying, we get

$$F^2(\sigma) = \frac{\Phi(\sigma)}{\Phi(\sigma_i)}\left[\rho_0^2 + \Phi^2(\sigma_i)\left(\gamma - \frac{\rho_0\Phi'(\sigma_i)}{2\Phi(\sigma_i)}\right)^2\right]. \tag{14}$$

In the usual weak–focusing accelerator, $\Phi(\sigma)$ has the constant value $R_0/\sqrt{1-n}$, where R_0 is the radius of curvature of the orbit and n is the field index. In this case,

$$F^2(\sigma) = \rho_0^2 + \frac{R_0^2\gamma^2}{1-n}. \tag{15}$$

If we are interested in the collisions with the injector plates, then it is essential to know by how much $F(\sigma)$ exceeds ρ_0 for a given angle of inclination γ; the smaller this excess, the less probable is a collision of the particles with the injector. Thus, the smaller $\Phi(\sigma_i)$ the greater the permissible deviation from the optimal angle [$\gamma = 0$, for equation (15)]; the stronger the focusing, the smaller $\Phi(\sigma)$.

The function $\Phi(\sigma)$ has the dimensions of a length, so that we may call the maximum value of $\Phi(\sigma)$ the effective radius of the accelerator

$$\max \Phi(\sigma) = R_{\text{eff}}. \tag{16}$$

We know that the gap dimensions determine the weight of the magnet, which is of prime importance for ultra–high energy accelerators. When comparing accelerators of different types, the ratio of the linear dimensions of the magnet gap to R_{eff} is a parameter which characterizes the maximum value of the focusing forces and the required accuracy of injection of particles into the accelerator chamber, whereas the ratio of these quantities to the radius of curvature R has no physical significance.

The minimum value of the oscillation amplitude at the azimuth of the injector is ρ_0, and occurs when the particle is injected at the optimum angle

$$\gamma = \gamma_{\text{opt.}} = (\rho_0 / 2) (d \ln \Phi (\sigma) / d\sigma)_{\sigma_i}. \tag{17}$$

The direction of optimum entry of particles into the chamber of the accelerator coincides with the direction of the tangent to the envelope. An essential feature is that the optimum entrance angle depends on ρ_0. When injecting into a synchrophasotron, ρ_0 usually varies from zero to some maximum value, so that the optimum angle also varies. If the angular width of the injected beam is less than $\gamma_{\text{opt.}}$, the number of particles captured for acceleration decreases markedly. It is true that one can find an azimuth at which $\Phi'(\sigma_i)=0$, but it is not always convenient for reasons of construction to place the injector plates at this azimuth.

To solve the problem of collision with the chamber walls, we have to study the function $f(\sigma)=\Phi(\sigma)/\Phi(\sigma_i)$ which gives the ratio of the oscillation amplitudes at an arbitrary azimuth σ and at the injector azimuth. The way this is done in practice will be shown in Secs. 5 and 6.

Up to now we have considered the radial oscillations. The vertical oscillations are treated similarly. We need only remember that in this case the quantity $q(\sigma)$ has the value

$$q(\sigma) = n(\sigma) R_0^{-2}(\sigma). \tag{18}$$

5. ACCELERATOR WITH SECTOR MAGNET

Synchrophasotrons with weak focusing are usually built with four straight sections. For example, in the 10 bev synchrophasotron of the Academy of Sciences of the USSR, there are four 8-meter straight sections. The average radius of the four magnet sectors is 28 meters. In Eq. (6), we can in first approximation set

$$q(\sigma) = \begin{cases} 0 \text{ in the straight sections of length } l, \\ \varkappa^2 = (\quad - n) / R_0^2 \quad \text{in the magnet sectors of length } \nu. \end{cases}$$

To find the envelope, it is sufficient to know the solution over a single periodicity element of length L

$$\Psi(\sigma) = \begin{cases} D(\sin \varkappa\sigma + d \cos \varkappa\sigma) \text{ in the magnet sectors} \\ D(e^{i\mu} \varkappa\sigma + s + dc_i) \text{ in the straight sections} \end{cases}$$

Here we have used the following abbreviations:

$$s = \sin \varkappa\nu; \quad c = \cos \varkappa\nu; \quad d = (c - e^{i\mu}) / s; \quad \cos\mu = c - \varkappa ls / 2. \tag{19}$$

In the formulas given above, thelength σ is measured from thebeginning of the sector, and from the beginning of the straight section, respectively.

As already noted, the constant D must be chosen* so that the Wronskian of the functions Ψ and Ψ^* is equal to $-2i$, so we require

$$DD^* = s / \varkappa \sin\mu. \tag{20}$$

From this it is easy to find the expression for the envelope

$$\Phi(\sigma) = \begin{cases} [s + \varkappa l \cos \varkappa\sigma \cdot \cos \varkappa(\nu - \sigma)] / \varkappa \sin\mu \quad \text{for magnet sectors} \\ [s + \varkappa lc + s\varkappa^2 (\sigma - l) \sigma] / \varkappa \sin\mu \quad \text{for straight sections} \end{cases} \tag{21}$$

For the data of the synchrophasotron of the Academy of Sciences of the USSR, the envelope is shown in Fig. 2. The function $\Phi(\sigma)$ is a maximum in the middle of the sectors and a minimum in the middle of the straight sections. From (21), it is easy to find the optimum angle of emergence of the particles from the acclerator chamber. For the parameters of the Soviet synchrophasotron, γ_{opt} is shown in Fig. 3.

For small lengths of straight sections, $\varkappa\Phi(\sigma)$ is of course not very different from unity, but this difference already affects the computation of the beam intensity[3] andthe scattering by residual gas. The envelope is extremely important in the design of strong-focussing accelerators.

6. STRONG–FOCUSING ACCELERATORS

As an example, we consider a standard type of strong–focusing accelerators. To simplify the formulas, we shall treat an acclerator without straight sections. Of course, thegeneral theory developed here can be applied to any type of accelerator with a constant or variable magnetic field.

Suppose that the accelerator consists of $2N$ magnets with the same lengths ν of arc with radius R_0 and with field index $n_1 < 0$ in the even sectors and $n_2 > 0$ in the odd sectors. At the end of the sectors, n changes abruptly from n_1 to n_2. If we are not designing a specific accelerator, but are interested in thecharacteristics of the motion, this assumption is not important. There is no difficulty in including straight sections or the transition region between n_1 and n_2. In our case,

*If $D = 1/d$, then $\Psi(0) = 1$ and $\text{Re}\,\Psi(L) = \cos\mu$ (cf. footnote on p.).

$$q(\sigma)=\begin{cases}\varkappa_1^2=(1-n_1)/R_0^2 & \text{for even sectors,}\\ -\varkappa_2^2=(n_2-1)/R_0^2 & \text{for odd sectors.}\end{cases}$$

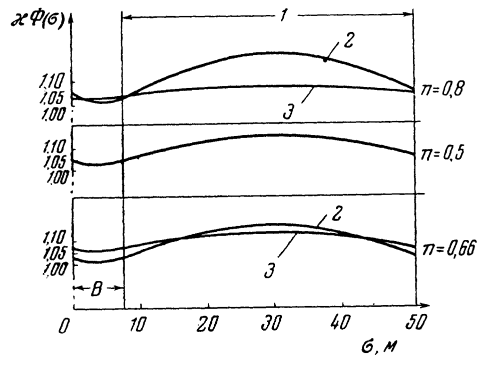

FIG. 2. Modulus of the Floquet function of the equation of free (vertical and radial) oscillations for the data of the synchrophasotron of the Academy of Sciences, USSR. *1*—circular sector, *2*—vertical oscillations, *3*—radial oscillations, straight section.

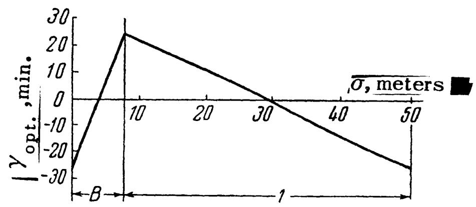

FIG. 3. Optimum angle γ_{opt} for the data of the synchrophasotron of the Academy of Sciences, USSR. $\rho_0/R = 1/56$; $l/R = 2/7$; *1*—circular sector, *B*—straight section.

Solving Eq. (1) by matching solutions, we find the complex solutions over a periodicity element of length $L=2\nu$:

$$\Psi(\sigma)=\sqrt{\frac{2}{\varkappa_1(f-f^*)}}$$

$$\times\begin{cases}\sin\varkappa_1\sigma+f\cos\varkappa_1\sigma & \text{for even sectors.}\\ p(c-fs)\,\mathrm{sh}\,\varkappa_2(\sigma-\nu)+(s+fc)\,\mathrm{ch}\,\varkappa_2(\sigma-\nu) & \text{for odd sectors.}\end{cases}$$

The lengths are measured from the beginning of the focusing sector. Here we have used the following notation:

$$p=\varkappa_1/\varkappa_2;\quad s_1=\sin\varkappa_1\nu;\quad c_1=\cos\varkappa_1\nu;$$

$$c_2=\mathrm{ch}\,\varkappa_2\nu;\quad s_2=\mathrm{sh}\,\varkappa_2\nu,$$

while the function f is

$$f=f_1-f_2e^{i\mu},\quad f_1=(s_1s_2+pc_1c_2)/(ps_1c_2-c_1s_2);$$

$$f_2=p/(ps_1c_2-c_1s_2).$$

It is easily verified that $\Psi(\sigma)$ satisfies Eq. (1) and the conditions given above if

$$\cos\mu=c_1c_2+(1-p^2)\,s_1s_2/2p.$$

We first find the expression for $\Phi(\sigma)$ in a focusing (even) sector

$$\Phi_1(\sigma)=\Phi(0)+[(1+p^2)\,s_2/2\varkappa_1p\sin\mu]\times[\cos\varkappa_1(\nu-2\sigma)-c_1],\tag{22}$$

$$\Phi(0)=(s_1c_2+pc_1s_2)/\varkappa_1\sin\mu.\tag{23}$$

The function $\Phi_1(\sigma)$ is a maximum in the middle of the focusing sector

$$\Phi_1(\nu/2)=\Phi(0)+(1-c_1)\times(1+p^2)\,s_2/2\varkappa_1p\sin\mu.\tag{24}$$

We note that the function $\Phi(\sigma)$ is symmetric with respect to the point $\sigma=\nu/2$. For the defocusing sector

$$\Phi_2(\sigma)=\Phi(0)-[s_1(1+p^2)/2\varkappa_1\sin\mu]\times[c_2-\mathrm{ch}\,\varkappa_2(3\nu-2\sigma)].\tag{25}$$

As we see from this formula, $\Phi_2(\sigma)$ is a minimum at the middle of the defocusing sector.

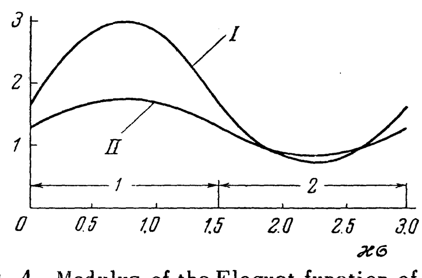

FIG. 4. Modulus of the Floquet function of the equation of free oscillations for a strong-focusing accelerator, at the center of the stability region ($\varkappa\nu=1/5$; $n_1+n_2=0$). *1* – focusing sector, *2* – defocusing sector; $\varkappa\sigma$ is the length along the orbit in dimensionless units. $\varkappa\nu=1.5$; I – $\Phi(\sigma)/\nu$; II – $\sqrt{\Phi(\sigma)/\nu}$.

A graph of the dimensionless quantity $\Phi(\sigma)/\nu$ as a function of $\varkappa\sigma$ is shown in Fig. 4 for the case when $n_1 + n_2 = 0$ and $\varkappa\nu = 1.5$. With these parameters, the stability of the motion is a maximum. As abscissa in Fig. 4, we have used the dimensionless quantity $\varkappa\sigma$ in place of the length σ. The choice of such a scale enables one to use the graphs for different values of $\varkappa$.

In order to get a picture of the function $\Phi(\sigma)$ for values of $\varkappa\nu$ other than those corresponding to the center of the stability region, we show in Fig. 5 the dependence of the function $\Phi(\sigma)/\nu$ on $\varkappa^2\nu^2$ for three azimuths: the function $\Phi_1(\nu/2)/\nu$ at the middle of the focusing sector, the function $\Phi_2(3\nu/2)/\nu$ at the middle of the defocusing sector, and the function $\Phi(0)/\nu$ at the junction of the two sectors. As we see from Fig. 5, the stability remains practically constant over a wide range of variation of $\varkappa\nu$. As is well-known, rigid limits of tolerance on the value of $\varkappa\nu$ are not determined by the problem of stability in a given magnetic field, but rather by the presence of a large number of resonances with imperfections in the magnetic field. Obviously, for a strong-focusing accelerator

$$R_{\mathrm{eff}} = \Phi_1(\nu/2).$$

For example, for $n_1 + n_2 = 0$,

$$R_{\mathrm{eff}} = \nu 2\sqrt{3}\left(1 + \frac{\varkappa^2\nu^2}{4} - \frac{\varkappa^4\nu^4}{80} + \ldots\right) \tag{26}$$

$$\doteq \varkappa^2\nu^2\sqrt{1 - \varkappa^4\nu^4/12 + \ldots}\,.$$

The minimum value of $\Phi_1(\nu/2)/\nu$ (as a function of $\varkappa\nu$) can be found from Fig. 5 or Eq. (26). It is equal to 3.026 (for $\varkappa\nu \approx 1.5$). Thus the optimum value $R_{\mathrm{eff}}^{\mathrm{opt}}$ is

$$R_{\mathrm{eff}}^{\mathrm{opt}} \approx 3\nu = 3R_0\delta, \tag{27}$$

where δ is the length of a sector in radians.

The two figures (6 and 7) are constructed similarly to Figs. 4 and 5, for the logarithmic derivative of $\Phi(\sigma)$, which is proportional to $\gamma_{\mathrm{opt.}}$, the optimum angle of emergence of the particle from the injector.

Figure 6 shows a graph of the function

$$(R/\varkappa\nu)\,d\ln\Phi(\sigma)/d\sigma = 2R\gamma_{\mathrm{opt.}}/\varkappa\nu\rho_0$$

for $\varkappa\nu = 1.5$, as a function of $\varkappa\sigma$. (ρ_0 is the distance from the injector to the instantaneous orbit.)

In Fig. 7 the same function is shown plotted against $\varkappa^2\nu^2$ for a fixed value of $\varkappa\sigma$ (at the junction of focusing and defocusing sectors, where γ_{opt} takes on its maximum value). At the center of the stability region, $\gamma_{\mathrm{opt}} \approx 0.6\,\rho_0/R$. The change of the optimum angle with time (which accompanies the change of ρ_0) leads to additional difficulties for multi-turn injection into strong-focusing accelerators.

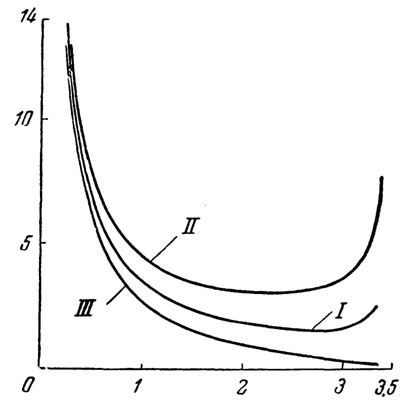

FIG. 5. Values of the modulus of the Floquet function at the centers of focusing and defocusing sectors ($\Phi_1(\nu/2)$ and $\Phi_2(3\nu/2)$, and between sectors ($\Phi(0)$, as a function of the parameter of focusing strength $\varkappa^2\nu^2$. ($n_1 + n_1 = 0$). The ordinates are I—$\Phi(0)/\nu$; II—$\Phi_1(\nu/2)/\nu$; III—$\Phi_2(3\nu/2)/\nu$. The abscissa is $\varkappa^2\nu^2$.

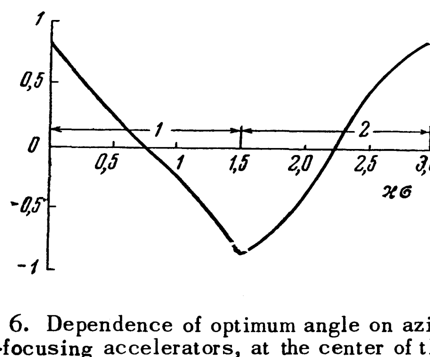

FIG. 6. Dependence of optimum angle on azimuth for strong-focusing accelerators, at the center of the stability region ($\varkappa\nu = 1.5$, $n_1 + n_2 = 0$). 1—focusing sector, 2—defocusing sector, $\varkappa\sigma$ is the length along the orbit in dimensionless units. The solid curve is $(R/\varkappa\nu)\,d \times \ln\Phi(\sigma)/d\sigma = (2R/\varkappa\nu\rho_0)\gamma_{\mathrm{opt}}$.

The general theory developed here and the examples considered show that the method of envelopes is entirely adequate for the problems arising in the computation and design of different types of

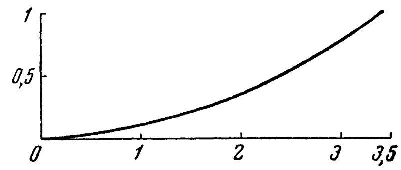

FIG. 7. Dependence of optimum angle on accelerator parameter $\varkappa^2 \nu^2$ ($n_1 + n_2 = 0$) for injector located between sectors. The ordinates are $(2R/\varkappa\nu\rho_0)\,\gamma_{\text{opt}}$. The abscissa is $\varkappa^2 \nu^2$.

accelerators. By means of this method one can easily compare the focusing of different types of accelerators and solve the problems associated with the achievement of maximum efficiency of injection.

1 Baldin, Mikhailov and Rabinovich, *Investigation of particle motion in a synchrophasotron with straight sections;* Reports of Phys. Inst. Acad. Sci. USSR, 1949.

2 Baldin, Mikhailov and Rabinovich, *On the theory of free oscillations;* Reports of Phys. Inst. Acad. Sci. USSR, 1950.

3 M. S. Rabinovich, *Fundamentals of the theory of the synchrophasotron;* Dissertation, Phys. Inst. Acad. Sci. USSR 1952.

4 A. M. Baldin and V. V. Mikhailov. *Almost-periodic motion of charged particles in an arbitrary constant magnetic field;* Reports of Phys. Inst. Acad. Sci. USSR, 1952.

5 M. S. Rabinovich, *Theoretical Investigation of accelerators with variable field index;* Reports of Phys. Inst. Acad. Sci. USSR, 1953.

6 D. M. Dennison and T. H. Berlin, Phys. Rev. 70, 764 (1946).

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Interaction of 230–250 mev Negative π-Mesons with Carbon and Lead Nuclei

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The interaction of 230–250 mev negative pions with carbon and lead nuclei was investigated by the method of the Wilson chamber in a magnetic field. The total and differential cross sections for both elastic and inelastic scattering were determined, as well as the total cross section for all the inelastic scattering processes. Within the experimental errors, the elastic scattering is in agreement with the diffraction pattern of an opaque nucleus. The energy spectrum of the scattered pions shows that the major part of the inelastic scattering between 60° and 180° is due to the collisions of the incoming pions with single nucleons in the nucleus.

THE most complete data on the different interaction processes of pions with complex nuclei have been obtained in experiments with thick layer photo-emulsions and Wilson chambers. Most data correspond to an energy range from 30 to 150 mev; there have been only a few experiments done for nuclei in emulsion[1–3] at higher energies, and a single experiment for the helium nucleus[4] at higher energy.

The present work has been carried out on the synchrocyclotron of the Institute of Nuclear Problems of the USSR Academy of Sciences. The interaction of 230–250 mev pions with carbon and lead nuclei was investigated by the method of a Wilson chamber in a magnetic field.

EXPERIMENTAL ARRANGEMENT AND METHOD OF PROCESSING THE PHOTOGRAPHS*

The experimental arrangement is shown schematically on Fig. 1. The source of negative pions was a graphite target placed inside the accelerator in the circulating beam of 670 mev protons. The 230–250 mev pions ejected from the target in the forward direction were directed by a four meter collimator and by a deflecting magnet on the Wilson chamber; the Wilson chamber had a diameter of 400 mm and was placed behind a concrete

*A detailed description of the experimental arrangement is given in Ref. 5.