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On the Oscillations of an Electron Plasma in a Magnetic Field

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The kinetic theory of the oscillations of an electron plasma in a constant magnetic field is examined. An investigation is made of plasma oscillations of frequencies which are integral multiples of the gyromagnetic frequency. The indices of refraction are determined for the ordinary, the extraordinary and the plasma waves which are propagated at an arbitrary angle θ with respect to the magnetic field. It is shown that at frequencies which are integral multiples of the gyromagnetic frequency the plasma wave is highly damped if $\theta < \pi/2$. If $\theta \approx \pi/2$ then the plasma waves corresponding to these frequencies cannot be propagated at all. In this paper the width of the "gaps" in the frequency spectrum of the plasma oscillations is determined.

THE study of electromagnetic processes in electron plasma in an external magnetic field is of interest for a number of problems in radiophysics and in astrophysics.

In the absence of external fields the oscillatory properties of the plasma have been studied in the papers of Vlasov[1] and Landau[2]. A characteristic feature which distinguishes plasma from other media from the point of view of the propagation of electromagnetic waves is the possibility of the existence in the plasma of weakly damped longitudinal electromagnetic waves (plasma oscillations).

The presence of a magnetic field leads to an anisotropy of the properties of the plasma and also gives rise to a number of characteristic resonance effects. The properties of an electron plasma situated in a magnetic field have been investigated on the basis of the kinetic theory first of all by Akhiezer and Pargamanik[3], and later by Gross[4] who showed the existence of bands of forbidden frequencies, which are integral multiples of the gyromagnetic frequency, when the plasma wave is propagated in a direction perpendicular to the direction of the magnetic field. Subsequently, the properties of plasma in a magnetic field have been studies in Refs. 5-7. In particular, Gershman[6] investigated the influence of thermal motion on the propagation of electromagnetic waves in the plasma. The theory of the propagation of electromagnetic waves in plasma is given in the hydrodynamic approximation in the monograph by Al'pert, Ginzburg and Feinberg[8].

The present article is devoted to the investigation of the oscillations of an electron plasma in a magnetic field on the basis of kinetic theory.

1. THE DISPERSION EQUATION

We shall examine the free oscillations of a plasma in a constant and uniform magnetic field **H**. Small oscillations of such a plasma are described by the linearized kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}\frac{\partial f_0}{\partial \mathbf{v}} + \frac{e}{mc}[\mathbf{vH}]\frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

where $f(\mathbf{r}, \mathbf{v}, t)$ is a small deviation of the electron distribution function from the Maxwellian function

$$f_0 = n_0 (m/2\pi T)^{3/2} e^{-mv^2/2T},$$

n_0 is the equilibrium electron density, e and m are the charge and mass of the electron, T is the plasma temperature, **E** is the self-consistent electric field, determined by the equation

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} - c^{-2}\partial^2\mathbf{E}/\partial t^2 = 4\pi c^{-2}\partial\mathbf{j}/\partial t, \quad (2)$$

where **j** is the electron current density

$$\mathbf{j} = e\int \mathbf{v} f d\mathbf{v}. \quad (3)$$

The properties of electromagnetic oscillations propagated in an unbounded electron plasma after a sufficiently long interval of time after the introduction of the initial disturbance are described by the dispersion equation which relates the frequency of the oscillations ω to the propagation vector **k**. In order to find the dispersion equation we shall look for the solutions of Eqs. (1) and (2) of the form

$$f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, \mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)};$$

$$E(\mathbf{r}, t) = E(\mathbf{k}, \omega)\, e^{i(\mathbf{kr}-\omega t)}. \quad (4)$$

Substituting (4) into (1) and (2) we shall obtain a system of equations for the amplitudes $f(\mathbf{v}, \mathbf{k}, \omega)$ and $E(\mathbf{k}, \omega)$:

$$i(\mathbf{kv} - \omega) f + e(\mathbf{Ev}) f'_0 - \omega_H \partial f / \partial \vartheta = 0, \quad (5)$$

$$-k^2 \mathbf{E} + \mathbf{k}(\mathbf{kE}) + (\omega^2 / c^2) \mathbf{E} = -i \frac{4\pi}{c^2} \omega \int \mathbf{v} f d\mathbf{v}. \quad (6)$$

Here $f'_0 = \partial f_0 / \partial \epsilon$, $\epsilon = mv^2/2$, $\omega_H = eH/mc$ is the gyromagnetic frequency of the electrons and ϑ is the polar angle in the velocity space (the Z axis is directed along the magnetic field $\mathbf{H}$, and the angle ϑ is measured from the plane containing the vectors $\mathbf{H}$ and $\mathbf{k}$).

The integration of Eq. (5) yields

$$f(\mathbf{v}) = \frac{e}{\omega_H} f'_0 \mathbf{E} \exp\left\{\frac{i}{\omega_H} \int_0^{\vartheta} (\mathbf{kv} - \omega) d\vartheta\right\}$$

$$\times \left\{\int_0^{\vartheta} \mathbf{v} \exp\left\{-\frac{i}{\omega_H} \int_0^{\psi} (\mathbf{kv} - \omega) d\psi\right\} d\psi + \mathbf{C}\right\}, \quad (7)$$

where the constant of integration $\mathbf{C}$ is determined from the condition of periodicity $f(\vartheta + 2\pi) = f(\vartheta)$:

$$\mathbf{C} = \int_0^{2\pi} \mathbf{v} \exp\left\{\frac{i}{\omega_H} \int_0^{\psi} (\mathbf{kv} - \omega) d\psi\right\}$$

$$\times d\psi \Big/ \left[1 - \exp\left\{\frac{i}{\omega_H} \int_0^{2\pi} (\mathbf{kv} - \omega) d\vartheta\right\}\right].$$

Substituting (7) into (6) we shall obtain a system of equations which determines the electric field of the plasma waves

$$\sum_{k=1}^{3} \{n^2 (\varkappa_i \varkappa_k - \delta_{ik}) + \varepsilon_{ik}\} E_k = 0 \qquad (i = 1,2,3). \quad (8)$$

Here $n = kc/\omega$ is the index of refraction for the wave of frequency ω, $\varkappa_i = k_i/k$ and ϵ_{ik} is the dielectric permittivity tensor of the plasma in the magnetic field which is determined by the expression (see Ref. 7),

$$\varepsilon_{ik}(\omega, \mathbf{k}) = \delta_{ik} + i \frac{4\pi e^2}{\omega \omega_H} \int v_i f'_0 \exp(i\alpha \sin\vartheta + i\beta\vartheta) \quad (9)$$

$$\times \left\{\int_0^{\vartheta} v_k \exp(-i\alpha \sin\psi - i\beta\psi) d\psi + C_k\right\} d\mathbf{v},$$

where

$$\alpha = k_x v_r / \omega_H, \quad \beta = (k_z v_z - \omega) / \omega_H.$$

The dielectric permittivity tensor introduced above depends not only on the frequency ω, but also on the propagation vector $\mathbf{k}$, i.e., the plasma is a medium in which the dispersion depends both on space and time[9]. In such media the value of the vector of the electric displacement $\mathbf{D}(\mathbf{r}, t)$ at the point $\mathbf{r}$ and at the instant of time t is determined by the values of the field $\mathbf{E}(\mathbf{r}', t')$ over all space and at all instants of time.

Taking into account the fact that

$$e^{-i\alpha \sin\psi} = \sum_{n=-\infty}^{\infty} J_n(\alpha) e^{-in\psi};$$

$$\int_0^{2\pi} e^{i\alpha \sin\psi - in\psi} d\psi = 2\pi J_n(\alpha), \quad (10)$$

and also using the well known recurrence relations for the Bessel functions $J_n(\alpha)$, we shall write the components of the tensor ϵ_{ik} in the following form:

$$\varepsilon_{11} = 1 - \frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}} \quad (11)$$

$$\times \sum_{n=-\infty}^{\infty} \frac{n^2}{\lambda^2} \int_0^{\infty} t e^{-t^2} J_n^2(\lambda t) dt) \int_C \frac{e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{12} = -i \frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{n}{\lambda} \int_0^{\infty} t^2 e^{-t^2} J_n(\lambda t) J'_n(\lambda t) dt \int_C \frac{e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{13} = -\frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{n}{\lambda} \int_0^{\infty} t e^{-t^2} J_n^2(\lambda t) dt \int_C \frac{y e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{22} = 1 - \frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \int_0^{\infty} t^3 e^{-t^2} J'^2_n(\lambda t) dt \int_C \frac{e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{23} = i \frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}}$$

$$\times \sum_{n=-\infty}^{\infty} \int_0^{\infty} t^2 e^{-t^2} J_n(\lambda t) J'_n(\lambda t) dt \int_C \frac{y e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{33} = 1 - \frac{\Omega^2}{\omega^2} \frac{4z_0}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \int_0^{\infty} t e^{-t^2} J_n^2(\lambda t) dt \int_C \frac{y^2 e^{-y^2}}{z_n - y} dy;$$

$$\varepsilon_{21} = -\varepsilon_{12}; \quad \varepsilon_{31} = \varepsilon_{13}; \quad \varepsilon_{32} = -\varepsilon_{23},$$

where

$$\lambda = \sqrt{\frac{2}{3}}\frac{k_x s}{\omega_H},$$

$$z_n = \sqrt{\frac{3}{2}}\frac{(\omega - n\omega_H)}{k_z s}, \quad \Omega = \sqrt{\frac{4\pi n_0 e^2}{m}}$$

$$s = \sqrt{3T/m}, \quad a = \sqrt{T/4\pi n_0 e^2}$$

(a is the Debye radius, Ω is the Langmuir frequency). Integration over y in (11) is carried out along the contour C along the real axis going around the singular point $y = z_n$ on the lower side[2].

The system of equations (8) has a solution different from zero provided its determinant is equal to zero. This dispersion equation, which connects the frequency ω and the propagation vector **k** of the electromagnetic waves in the plasma, can be written in the form

$$An^4 + Bn^2 + C = 0, \tag{12}$$

$$A = \varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + 2\varepsilon_{13}\cos\theta\sin\theta, \tag{13}$$

$$B = 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13})\cos\theta\sin\theta - (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2)\cos^2\theta + \varepsilon_{13}^2 - \varepsilon_{11}\varepsilon_{22} - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2)\sin^2\theta, \quad C = |\varepsilon_{ik}|.$$

θ is the angle between the directions of **H** and **k** ($\varkappa_1 = \sin\theta$, $\varkappa_2 = 0$, $\varkappa_3 = \cos\theta$).

In the general case, the dispersion equation (12) is quite complicated, and therefore we shall restrict ourselves to the examination of the limiting cases of a weak magnetic field ($\omega_H \ll \Omega$) and "low" temperatures ($\omega_H \gg ks$).

2. WEAK MAGNETIC FIELD

In the case of a weak magnetic field ($\omega_H \ll \Omega$) it is convenient for the calculation of ϵ_{ik} to start not with Eq. (11), but directly with Eq. (9). Integrating over ψ in (9) by parts and noting that for $|z| = \sqrt{(3/2)}\,|(\omega/ks)| \gg 1$ and $|\operatorname{Im} z| \ll 1$

$$\frac{1}{\sqrt{\pi}}\int\limits_C \frac{e^{-y^2}}{z-y}dy \approx \frac{1}{z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots\right), \tag{14}$$

we obtain for the components of the dielectric permittivity tensor the following expressions

$$\varepsilon_{11} = \varepsilon_0 - k^2a^2v^2(\cos^2\theta + 3\sin^2\theta) - uv; \tag{15}$$

$$\varepsilon_{12} = -\varepsilon_{21} = -i\sqrt{u}\,v;$$

$$\varepsilon_{13} = \varepsilon_{31} = -2k^2a^2\cos\theta\sin\theta v^2;$$

$$\varepsilon_{22} = \varepsilon_0 - k^2a^2v^2 - uv;$$

$$\varepsilon_{33} = \varepsilon_0 - k^2a^2v^2(3\cos^2\theta + \sin^2\theta);$$

$$\varepsilon_{32} = -\varepsilon_{23} \approx 0.$$

Here $\epsilon_0 = 1 - \Omega^2/\omega^2$ is the dielectric permittivity of the plasma in the absence of a magnetic field, $v = \Omega^2/\omega^2$, $u = \omega_H^2/\omega^2$. Using (15) we write the dispersion equation (12) in the form

$$\frac{s^2}{c^2}vn^6 - \left(\varepsilon_0 - uv\sin^2\theta + \frac{8}{3}\frac{s^2}{c^2}v\varepsilon_0\right)n^4 + \left[2\varepsilon_0^2 - uv(2\varepsilon_0 + \sin^2\theta) + \frac{5}{3}\frac{s^2}{c^2}v\varepsilon_0^2\right]n^2 - \varepsilon_0^3 + \varepsilon_0 uv(1 + \varepsilon_0) = 0. \tag{16}$$

Neglecting terms in (16) which are proportional to $s^2/c^2 \ll 1$, we find the indices of refraction for the ordinary and the extraordinary waves

$$n_{1,2}^2 = \frac{2\varepsilon_0 - uv(2\varepsilon_0 + \sin^2\theta) \pm \{[2\varepsilon_0 - uv(2\varepsilon_0 + \sin^2\theta)]^2 - 4(\varepsilon_0 - uv\sin^2\theta)[\varepsilon_0^3 - \varepsilon_0 uv(1+\varepsilon_0)]\}^{1/2}}{2(\varepsilon_0 - uv\sin^2\theta)}. \tag{17}$$

For the index of refraction of the plasma wave n_3 we obtain the following expression in the case $\epsilon_0 \ll 1$:

$$n_3^2 = (\varepsilon_0 - u\sin^2\theta)c^2/s^2. \tag{18}$$

If $\epsilon_0 \approx 1$, then the third solution of (16) does not satisfy the condition $|z| \gg 1$, under which Eq. (16) itself has been obtained.

Expressions (17) and (18) hold if the terms in (16) neglected in obtaining (17) and (18) are small compared to the terms which have been retained. For this to hold it is necessary that the following inequalities should be satisfied

$$|\varepsilon_0 - u\sin^2\theta| \gg (s/c)\varepsilon_0,$$

$$(s/c)\sqrt{u\varepsilon_0}, \quad (s/c)\sqrt{u}\,|\sin\theta|.$$

In the opposite case, when

$$(\varepsilon_0 - u\sin^2\theta) \ll (s/c)\sqrt{u}\,|\sin\theta|$$

we have

$$n_1^2 = u, \qquad (19)$$

$$n_2^2 = \sqrt{u}\,|\sin\theta|\,c/s \quad (|\sin\theta| \gg s/c\sqrt{u}).$$

In the case $\theta = 0$ and $\epsilon_0 = 0$ all three solutions of (16) reduce to zero.

3. STRONG MAGNETIC FIELD (LOW TEMPERATURES)

We shall now investigate the dispersion equation in the case of low temperatures ($\lambda \ll 1$). Far from the resonance frequencies ($|Z_n| \gg 1$) we may use the asymptotic expression (14) for the integrals along the contour C in (11). Expanding the functions $J_n(\lambda t)$ and $J_n'(\lambda t)$ in series in powers of λ, we obtain the following expressions for the components ϵ_{ik}:

$$\varepsilon_{11} = 1 - v/(1-u) - k^2a^2v^2\left[\frac{1+3u}{(1-u)^3}\cos^2\theta + 3(1-u)^{-1}(1-4u)^{-1}\sin^2\theta\right];$$

$$\varepsilon_{12} = -iv\sqrt{u}/(1-u) - ik^2a^2v^2\sqrt{u}\,[(3+u)(1-u)^{-3}\cos^2\theta + 6(1-u)^{-1}(1-4u)^{-1}\sin^2\theta];$$

$$\varepsilon_{13} = -2k^2a^2v^2(1-u)^{-2}\sin\theta\cos\theta;$$

$$\varepsilon_{22} = 1 - v/(1-u) - k^2a^2v^2[(1+3u)(1-u)^{-3}\cos^2\theta + (1+8u)(1-u)^{-1}(1-4u)^{-1}\sin^2\theta];$$

$$\varepsilon_{23} = ik^2a^2v^2\sqrt{u}\,(3-u)(1-u)^{-2}\sin\theta\cos\theta;$$

$$\varepsilon_{33} = 1 - v - k^2a^2v^2[3\cos^2\theta + (1-u)^{-1}\sin^2\theta), \qquad (20)$$

The terms in (20) which are proportional to $(ka)^2$ take into account the thermal motion of the electrons in the plasma which determines the spatial dispersion of the medium.

Making use of (20) we evaluate the coefficients A, B, C of Eq. (12)

$$A = A_0 + A_1n^2; \qquad (21)$$

$$B = B_0 + B_1n^2; \quad C = C_0 + C_1n^2,$$

$$A_0 = \frac{1-u-v+uv\cos^2\theta}{1-u}, \qquad (22)$$

$$B_0 = \frac{(2-v)u - 2(1-v)^2 - vu\cos^2\theta}{1-u},$$

$$C_0 = \frac{(1-v)[(1-v)^2-u]}{1-u};$$

$$A_1 = -\frac{s^2}{3c^2}v\left\{3\cos^4\theta + \frac{6-3u+u^2}{(1-u)^3}\cos^2\theta\sin^2\theta + \frac{3}{(1-u)(1-4u)}\sin^4\theta\right\};$$

$$B_1 = \frac{s^2}{3c^2}v\left\{\frac{2(1+u-v)}{(1-u)^2}\cos^2\theta\sin^2\theta + \frac{1+\cos^2\theta}{1-u}\left[(1-u-v)\left(3\cos^2\theta + \frac{\sin^2\theta}{1-u}\right) + (1-v)\left(\frac{1+3u}{(1-u)^2}\cos^2\theta + \frac{3\sin^2\theta}{1-4u}\right)\right] + \frac{2\sin^2\theta}{(1-u)^2}\left[\frac{1+3u-v-uv}{1-u}\cos^2\theta + \sin^2\theta\,\frac{2(1-u)(1+2u-v)}{1-4u}\right]\right\};$$

$$C_1 = -\frac{s^2}{3c^2}v\left\{\frac{2(1-v)}{(1-u)^2}\left[\frac{1+3u-v-uv}{1-u}\cos^2\theta + \frac{(1-u)(1+2u-v)}{1-4u}2\sin^2\theta\right] + \frac{(1-v)^2-u}{1-u}\left(3\cos^2\theta + \frac{\sin^2\theta}{1-u}\right)\right\}. \qquad (23)$$

The coefficients A_1, B_1, C_1, which are proportional to s^2/c^2, represent corrections to the hydrodynamic approximation $A \approx A_0$, $B \approx B_0$, $C \approx C_0$.

Thus the dispersion equation (12) takes on the form:

$$A_1n^6 + (A_0 + B_1) n^4 + (B_0 + C_1) n^2 + C_0 = 0. \tag{24}$$

The three roots of this equation n_1^2, n_2^2 and n_3^2 determine, respectively, the indices of refraction of the ordinary, the extraordinary and the plasma waves.

Neglecting the terms in (24) proportional to s^2/c^2, we obtain the indices of refraction of the ordinary and the extraordinary waves in the hydrodynamic approximation[8]

$$n_{1,2}^2 = n_\pm^2 = (-B_0 \pm \sqrt{B_0^2 - 4A_0C_0}) / 2A_0. \tag{25}$$

Taking into account the thermal motion of the electrons we obtain

$$n_{1,2}^2 = (1 + \varepsilon_\pm) n_\pm^2, \tag{26}$$

$$\varepsilon_\pm = -(A_1n_\pm^4 + B_1n_\pm^2 + C_1) / (2A_0n_\pm^2 + B_0),$$

$$|\varepsilon_\pm| \ll 1.$$

Since under the usual conditions $s^2/c^2 \ll 1$, the thermal corrections to $n_{1,2}^2$ are very small.

If the absolute value of $A_0 = (\omega^2 - \omega_+^2) \times (\omega^2 - \omega_-^2)/\omega^2(\omega^2 - \omega_H^2)$ is small compared to unity, i.e., if ω^2 is close to ω_+^2 or to ω_-^2,

$$\omega_\pm^2 = \tfrac{1}{2}(\Omega^2 + \omega_H^2 \pm \sqrt{(\Omega^2 + \omega_H^2)^2 - 4\Omega^2\omega_H^2 \cos^2\theta}), \tag{27}$$

then from (25) we obtain approximately,

$$n_1^2 = -C_0/B_0, \quad n_2^2 = -B_0/A_0. \tag{28}$$

Since $B_0(\omega) > 0$ for $\omega^2 \approx \omega_\pm^2$, then $n_2^2 \to +\infty$ as $\omega^2 \to \omega_+^2$ (or ω_-^2) from the direction $\omega^2 < \omega_+^2$ (or $\omega^2 < \omega_-^2$) and $n_2^2 \to -\infty$ as $\omega^2 \to \omega_+^2$ (or ω_-^2) from the direction $\omega^2 > \omega_+^2$ (or $\omega^2 > \omega_-^2$). However, for very small values of A_0 the expression (28) for n_2^2 no longer holds, since it was obtained under the condition

$$s^2/c^2 \ll |A_0(\omega)| \ll 1. \tag{29}$$

The condition (29) means that the terms which were discarded in (24) in order to obtain (28) are small compared to the terms which were retained.

For the index of refraction of the plasma wave we obtain by the condition (29)

$$n_3^2 = -A_0/A_1. \tag{30}$$

Expression (30) coincides with the expression for n_3^2, which was obtained by Gershman.[6] If $|A_0| \gtrsim 1$, then the third solution of the cubic equation (24) for n^2 does not satisfy the condition $sn/c \ll 1$, which must hold if the expressions (20) for ϵ_{ik} are to be valid.

Let us now find the solution of (24) for $|A_0| \ll 1$. The index of refraction for the ordinary wave is determined as before by (29). We obtain the indices of refraction for the extraordinary and the plasma waves by assuming that for $|A_0| \ll 1$ $n_{2,3}^2 \gg 1$. Retaining the largest terms in (24) we obtain

$$n_{2,3}^2 = A_0\left(-1 \pm \sqrt{1 - \frac{4s^2A_1B_0}{c^2A_0^2}}\right) \Big/ 2A_1s^2/c^2. \tag{31}$$

In the limiting case $|A_0| \gg s/c$ (31) leads to (28) and (30). For $|A_0| \ll s/c$ we obtain* from (31)

$$n^2 = (c/s)\sqrt{-B_0/A_1}. \tag{32}$$

The second solution given by (31) will be negative in this case.

4. THE CASE OF RESONANCE

Let us now examine the case of resonance $\omega \approx \omega_H$. Assuming that in the integrals occurring in the expressions for ϵ_{ik} in (10)

$$|\lambda| \ll 1, \quad |z_1| = \sqrt{3/2}\,|\omega - \omega_H| / k_z s \ll 1$$

and consequently that

$$\int_C \frac{e^{-y^2}}{z_1 - y} dy \approx -i\pi,$$

we obtain, neglecting terms proportional to $(ka)^2$ or to z_1,

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* For $\theta = \pi/2$, (32) gives Gershman's result[6].

$$\varepsilon_{11} = \varepsilon_{22} = 1 - \frac{v}{4} + i\sqrt{\frac{3\pi}{8}}\frac{v}{(ns/c)\cos\theta}; \quad (33)$$

$$\varepsilon_{12} = -\varepsilon_{21} = i\frac{v}{4} - \sqrt{\frac{3\pi}{8}}\frac{v}{(ns/c)\cos\theta}; \quad \varepsilon_{33} = 1 - v;$$

$$\varepsilon_{23} = \varepsilon_{32} = \varepsilon_{13} = \varepsilon_{31} = 0.$$

We now substitute these expressions into the dispersion Eq. (12) under the assumption that n is small, and retain terms proportional to $c/s \gg 1$. We then obtain

$$n_{1,2}^2 = n_\pm^2 = \sin^{-2}\theta\,[1 + {}^1/_2\sin^2\theta - v \pm \pm\sqrt{(1 + {}^1/_2\sin^2\theta - v)^2 - \sin^2\theta(1-v)(2-v)}]. \quad (34)$$

In the next approximation we find

$$n_{1,2}^2 = (1 + \Delta_\pm)\,n_\pm^2, \quad (35)$$

$$\Delta_\pm = i\sqrt{\frac{8}{3\pi}}\frac{s\cos\theta}{cn_\pm v} \times \frac{[1-({}^1/_4\sin^2\theta+\cos^2\theta)v]n_\pm^4 - [(1-v)(1-{}^1/_4v)(1+\cos^2\theta)+(1-{}^1/_2v)\sin^2\theta]n_\pm^2 + (1-v)(1-{}^1/_2v)}{2\sin^2\theta\, n_\pm^2 + 2v - 2 - \sin^2\theta}. \quad (36)$$

Thus, the electromagnetic waves are damped when $\omega \approx \omega_H$. The order of magnitude of the damping coefficient is equal to s/c, i.e., it is appreciably larger than the usual thermal corrections to the indices of refraction of the ordinary and the extraordinary waves which are proportional to s^2/c^2.

As the angle θ is decreased $n_1 \approx n_+$ increases $(n_+^2 \to 2(1-v)\theta^{-2}]$; however, for small values of θ one cannot use expression (34) for n_1^2, since it was obtained under the condition

$$\theta \gg v^{-1/4}\sqrt{|1-v|}\,(s/c)^{1/3},$$

which means that the terms in the dispersion equation which are proportional to c/s are the largest ones.

5. LONGITUDINAL PLASMA OSCILLATIONS

Let us now consider in greater detail the problem of the longitudinal plasma oscillations. As is well known[4], in the presence of a magnetic field the electromagnetic waves in a plasma cannot be separated into strictly longitudinal and transverse ones. However, in the limiting case $n \gg 1$ we can distinguish a longitudinal plasma wave, the dispersion equation for which may be written approximately in the form: $A(\omega, \mathbf{k}) = 0$.

Substituting (9) into (13) we reduce this equation to the form*:

$$k^2a^2 + 1 - i\frac{\omega'}{\omega_H}\int\frac{f_0}{n_0}e^{i\alpha\sin\vartheta + i\beta\vartheta}\left\{\int_0^\vartheta e^{-i\alpha\sin\psi - i\beta\psi}d\psi + (1 - e^{2\pi i\beta})^{-1}\int_0^{2\pi} e^{i\alpha\sin\psi + i\beta\psi}d\psi\right\}d\mathbf{v} = 0, \quad (37)$$

where

$$\alpha = k_x v_z/\omega_H, \quad \beta = (k_z v_z - \omega')/\omega_H, \quad \omega' = \omega - i\gamma,$$

ω' is the complex frequency [we have replaced ω by ω' in (9)]. In the future we shall take the propagation vector $\mathbf{k}$ to be real. Equation (37) then determines the frequency ω and the damping γ as functions of $\mathbf{k}$.

Making use of relation (10), and carrying out the integration over the angles in (37), and then over α, we finally obtain[10]

$$k^2a^2 + 1 - e^{-\mu}\sum_{n=-\infty}^{\infty} I_{|n|}(\mu)\frac{z_0}{\sqrt{\pi}}\int_C\frac{e^{-y^2}}{z_n - y}dy = 0,$$

$$\mu = k_x^2 s^2/3\omega_H^2, \quad z_n = \sqrt{3/2}\cdot(\omega' - n\omega_H)/k_z s, \quad (38)$$

where $I_n(\mu)$ is the modified Bessel function.

For $k_x = 0$ ($\mu = 0$) the dispersion equation (38) has the same form as in the absence of magnetic field[5], i.e., the magnetic field has no effect on

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* We note that (37) may be obtained directly by starting with the kinetic equation and with the equation $\operatorname{div}\mathbf{E} = 4\pi e\int f d\mathbf{v}$.

plasma oscillations being propagated parallel to it.

In the case of "low" temperatures of the plasma, when $\mu \ll 1$ by expanding the functions $I_n(\mu)$ and $e^{-\mu}$ in powers of μ and by using the asymptotic expansion of the integral

$$\frac{1}{\sqrt{\pi}}\int\limits_C \frac{e^{-y^2}}{z-y}dy \approx \frac{1}{z}\left(1+\frac{1}{2z^2}+\frac{3}{4z^4}+\ldots\right) - i\sqrt{\pi}e^{-z^2}, (|z|\gg 1, |\operatorname{Im} z| \ll 1), \tag{39}$$

we obtain

$$k^2a^2 - \left(\frac{1}{2z_0^2}+\frac{3}{4z_0^4}+\ldots\right) - \mu\left[\frac{z_0}{2}\left(\frac{1}{z_1}+\frac{1}{z_{-1}}+\frac{1}{2z_1^3}+\frac{1}{2z_{-1}^3}\right)-1-\frac{1}{2z_0^2}\right] - \mu^2\left[\frac{z_0}{8}\left(\frac{1}{z_2}+\frac{1}{z_{-2}}-\frac{4}{z_1}-\frac{4}{z_{-1}}\right)+\frac{3}{4}\right] + \ldots + i\sqrt{\pi}z_0\left[\left(1-\mu+\frac{3}{4}\mu^2\right)e^{-z_0^2} + \frac{\mu}{2}(1-\mu)\left(e^{-z_1^2}+e^{-z_{-1}^2}\right) + \frac{\mu^2}{8}\left(e^{-z_2^2}+e^{-z_{-2}^2}\right)+\ldots\right] = 0. \tag{40}$$

If we neglect the thermal motion of the electrons then (40) reduces to the dispersion equation of the hydrodynamic approximation

$$1-\frac{\Omega^2}{\omega^2}\cos^2\theta - \frac{\Omega^2}{\omega^2-\omega_H^2}\sin^2\theta = 0,$$

from which we obtain the characteristic frequency of plasma oscillations in the hydrodynamic approximation[3]:

$$\omega^2 = \omega_\pm^2 = \tfrac{1}{2}(\Omega^2+\omega_H^2) \pm \tfrac{1}{2}\sqrt{(\Omega^2+\omega_H^2)^2 - 4\Omega^2\omega_H^2\cos^2\theta}. \tag{41}$$

Taking into account that $ka \ll 1$, we look for the solution of the dispersion equation (40) in the form

$$\omega_{1,2}^2 = (1+\varepsilon_\pm)\omega_\pm^2, \quad |\varepsilon_\pm| \ll 1. \tag{42}$$

For the corrections $\epsilon_\pm$ to the characteristic frequencies $\omega_{1,2}$, we obtain the expression

$$\varepsilon_\pm = \frac{k^2s^2}{\omega_\pm^2}\,\frac{v_\pm}{[1+v_\pm u_\pm(1-u_\pm)^{-2}\sin^2\theta]} \times \left\{\cos^4\theta + \frac{\left(2-u_\pm+\frac{1}{3}u_\pm^2\right)\cos^2\theta\sin^2\theta}{(1-u_\pm)^3} + \frac{\sin^4\theta}{(1-u_\pm)(1-4u_\pm)}\right\}, \tag{43}$$

$$v_\pm = \Omega^2/\omega_\pm^2, \quad u_\pm = \omega_H^2/\omega_\pm^2.$$

Thus, in the case of low plasma temperatures ($\omega_H \gg ks$, "strong" magnetic field), there exist two characteristic frequencies of plasma oscillation, which are determined by Eqs. (41)-(43). We obtain the damping which corresponds to these frequencies by taking into account in Eq. (40) terms which are exponentially small:

$$\gamma_{1,2} = \sqrt{\frac{\pi}{8}}\,\frac{\omega_\pm^2}{\Omega(ka)^3\cos\theta}\,\frac{1}{[1+\Omega^2\omega_H^2\sin^2\theta/(\omega_\pm^2-\omega_H^2)^2]} \times\{\exp\{-\omega_{1,2}^2/2\Omega^2k^2a^2\cos^2\theta\} + \frac{k^2a^2\Omega^2}{2\omega_H^2} \times\sin^2\theta\,[\exp\{-(\omega_{1,2}-\omega_H)^2/2\Omega^2k^2a^2\cos^2\theta\} + \exp\{-(\omega_{1,2}+\omega_H)^2/2\Omega^2k^2a^2\cos^2\theta\}]+\ldots\}. \tag{44}$$

Expression (42) was obtained under the condition $|z_n| \gg 1$ ($|\epsilon_+| \ll 1$). If $\theta \to 0$, then $\omega_1 \approx \omega_+ \to \omega_H$ (for $\Omega < \omega_H$), and $\omega_2 \approx \omega_- \to \omega_H$ (for $\Omega > \omega_H$). In this case the inequality $|z_1| \gg 1$ is not fulfilled, and the expressions (42) no longer hold. From the condition $|z_1| \gg 1$ (or from the condition $|\epsilon_\pm| \ll 1$), we find that the applicability of the Eq. (42) for ω_1 with $\omega_H > \Omega$ and for ω_2 with $\omega_H < \Omega$ is restricted by the condition

$$\theta \gg \sqrt{2\sqrt{2}ka\,|\Omega^2-\omega_H^2|/\Omega\omega_H}. \tag{45}$$

A unique solution $\omega \approx \Omega$ exists, as may be seen from the exact dispersion equation (38) for $\theta = 0$ and $ka \ll 1$.

If $\omega_H < \Omega$, then for certain ω_H and Ω there exists an angle $\theta = \theta_m$, for which the frequency

ω_1, determined by (41), turns out to be an integral multiple of ω_H:

$$\omega_1 \approx m\omega_H \quad (m = 2,\ 3,\ \ldots). \tag{46}$$

However, the dispersion equation (40) was itself obtained under the assumption $|z_n| \gg 1$. Therefore, one cannot use expression (42) at angles close to θ_m, if (46) holds for these angles. In order to obtain a dispersion equation which is valid for $\theta \approx \theta_m$, one should retain the mth integral in (38), and one should use for the other integrals the asymptotic expansion (39), as was done earlier:

$$k^2a^2 - \frac{1}{2z_0^2} - \mu\left[\frac{z_0}{2}\left(\frac{1}{z_1} + \frac{1}{z_{-1}}\right) - 1\right] + \ldots - \left(\frac{\mu}{2}\right)^m \frac{z_0}{m!\,\sqrt{\pi}} \int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy = 0. \tag{47}$$

If θ_m is not close to $\pi/2$, then assuming that

$$|z_m| \ll 1, \quad \int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy \simeq -i\pi,$$

we find $\omega' = m\omega_H - i\gamma_m$, where

$$\gamma_m = \frac{\sqrt{\pi}\, m^4 \sin^{2m}\theta}{2^{m+3/2}\, 3^{m-3/2}\, m!\, \cos^3\theta\, \left(1 + m^4 (m^2 - 1)^{-2}\, \mathrm{tg}^2\,\theta\right)} \times \left(\frac{ks}{\omega_H}\right)^{2m-4} ks \quad (m = 2,\ 3,\ \ldots). \tag{48}$$

Thus the waves with frequencies which are integral multiples of ω_H are strongly damped for θ not close to $\pi/2$. The damping coefficient γ_m is proportional to $(ks/\omega_H)^{2m-4}$ and decreases as m increases. For $m = 2$ the damping exceeds in order of magnitude by a factor $(ka)^{-1}$ the usual thermal corrections to the frequency.

If $\theta_m \approx \pi/2$, then assuming that

$$\int\limits_C \frac{e^{-y^2}}{z_m - y}\,dy \approx \frac{\sqrt{\pi}}{z_m}, \quad |z_m| \gg 1,$$

we obtain

$$\omega = m\omega_H \pm \varepsilon_m, \quad \varepsilon_m = (m^2 - 1)\left(2^{m+1} 3^{m-1} m!\right)^{-1/2} (ks/\omega_H)^{m-2} ks. \tag{49}$$

Thus, for $\theta = \pi/2$ and for a given magnetic field, longitudinal waves with frequencies in the range $m\omega_H - \epsilon_m < \omega < m\omega_H + \epsilon_m$ cannot be propagated in the plasma. The width of the "gap" $2\epsilon_m$ decreases as m increases. Gross[4] has pointed out the existence of such "gaps" and has computed the value of ϵ_2.

Finally, if $\omega_H \approx \Omega$, Eqs. (42) are not applicable for small θ, since in such a case the condition $|z_1| \gg 1$ is not fulfilled. In this case the exact dispersion equation (38) takes on the form

$$1 - \frac{\Omega^2}{\omega^2} - \frac{\theta^2\omega}{2\sqrt{2}\,ka\Omega\sqrt{\pi}} \int\limits_C \frac{e^{-y^2}}{z_1 - y}\,dy = 0. \tag{50}$$

Assuming that $|z_1| \ll 1$ we find that

$$\omega = \Omega \approx \omega_H; \quad \gamma = {}^1\!/_4 \sqrt{\pi/2}\,(\theta^2/ka)\,\Omega; \quad \theta \ll 2\sqrt{ka}, \tag{51}$$

i.e., for $\omega_H \approx \Omega$ the plasma wave of frequency $\omega = \Omega \approx \omega_H$ is strongly damped.

In the case of a weak magnetic field ($\omega_H \ll \Omega$) Eq. (37) can be brought to the form

$$k^2a^2 - \frac{1}{\sqrt{\pi}} \int\limits_C \frac{ye^{-y^2}}{z - y}\,dy - \frac{\omega_H^2 \sin^2\theta\, z^3}{4\Omega^2\sqrt{\pi}} \int\limits_C \frac{ye^{-y^2}}{(z - y)^4}\,dy - \ldots = 0, \quad z = \sqrt{\frac{3}{2}}\,\frac{\omega'}{ks}. \tag{52}$$

by means of integration by parts.

Solving this equation by successive approximations ($|z| \gg 1$), we find the frequency ω and the damping γ of the plasma oscillations in a weak magnetic field

$$\omega = \Omega + k^2s^2/2\Omega + \omega_H^2 \sin^2\theta/2\Omega, \tag{53}$$

$$\gamma = \sqrt{\frac{\pi}{8}}\,\frac{\Omega}{(ka)^3}\left(1 + \frac{\omega_H^2 \sin^2\theta}{4!\,(ka)^{\text{[illegible]}}\,\Omega^2}\right) e^{-\omega^2/2k^2a^2\Omega^2}. \tag{54}$$

Expressions (53) and (54) agree with the results obtained by Gordeev[5]. For $\omega_H = 0$ we obtain from this Vlasov's formula for the frequency[1] and Landau's formula for the damping*

$$\gamma = \sqrt{\pi/8}\Omega\,(ka)^{-3}e^{-{}^1/_2k^2a^2}e^{-{}^3/_2}.$$

In conclusion we express our sincere thanks to Prof. A. I. Akhiezer for his attention, for his assistance, and for the detailed discussion of the results of this work.

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* We note that in the expression for γ obtained by Landau[2] the factor $e^{-3/2}$ is missing.

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Investigation of the $Be^9(dn)B^{10}$ Nuclear Reaction

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An investigation is carried out on the reaction between the nucleus of beryllium and a deuteron in which the latter is captured by the nucleus and the unpaired neutron is ejected. The effective cross section of the process and the angular distribution of the freed neutrons were found. The comparison of the angular distribution with experimental data results in a satisfactory agreement for small angles up to 70°.

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1. THE model of the Be^9 nucleus, according to which the unpaired neutron moves in the field of the nuclear remainder Be^8, was applied by many investigators to the problem of the electron and photoelectric disintegration of this nucleus[1-3]. The success of this model is determined first by the weak binding of the unpaired neutron in the Be^9 nucleus, considerably smaller than the mean binding energy per particle , and second, by the relatively long life of the Be^8 nuclear remainder relative to the decay into two α-particles. The current research is dedicated to the investigation of the $Be^9(dn)B^{10}$ reaction on the basis of this model[1].

It is customarily assumed that the (dn) reaction can proceed by the formation of a compound nucleus and stripping of a proton by a nucleus from a deuteron passing nearby. Calculations on the basis of the compound nucleus model are often not feasible in actual cases because the line widths of the corresponding processes are unknown; therefore, most of the theoretical investigations of the (dn) reaction are made from the point of view of the stripping process. The corresponding angular distributions of neutrons are then determined on the basis of Butler's theory[4]. When there is no agreement between this theory and experiment it is pointed out that in such cases the reaction does not proceed by stripping, but by the formation of a compound nucleus, which then undergoes various cascade transitions.

Other Errata

Page	Column	Line	Reads	Should Read
Volume 4				
38	1	Eq. (3)	$\ldots \frac{\pi r^2 \rho^2 \rho^2{}_n}{\rho_s^2},$	$\ldots \frac{\pi r^2 \rho^2 \rho_n}{\rho_s^2},$
196		Date of submittal	May 7, 1956	May 7, 1955
377	1	Caption for Fig. 1	$\delta_{35} = \eta - 21 \cdot \eta^5$	$\delta_{35} = -21^\circ \eta^5.$
377	2	Caption for Fig. 2	$\alpha_3 = 6.3^\circ\ \eta$	$\alpha_3 = -6.3^\circ \eta$
516	1	Eq. (29)	$s^2/c^2 \ldots$	s/c
516	2	Eqs. (31) and (32)	Replace $A_1 s^2/c^2$ by A_1	
497		Date of submittal	July 26, 1956	July 26, 1955
900	1	Eq. (7)	$\ldots \frac{i}{4\pi} \sum_{c,\,\alpha} \frac{\partial w_a(t, P)}{\partial P^\alpha} \ldots$	$\ldots\ 2\pi^2 i \sum_{c,\,\alpha} \ldots$
			(This causes a corresponding change in the numerical coefficients in the expressions that result from the calculation of the effects of the plasma particles on each other).	
804	2	Eq. (1)	$\ldots \exp\{-(\overline{T} - V')\}$	$\ldots \exp\{-(\overline{T} - V')\tau^{-1}\}$
Volume 5				
59	1	Eq. (6)	$v_l\,(l\partial F_0/\partial x) + \ldots$ where E_l is the projection of the electric field E on the direction l	$\overline{(v\partial F_0/\partial x)} + \ldots$ where the bar indicates averaging over the angle θ and E_l is the projection of the electric field E along the direction l
91	2	Eq. (26)	$\Lambda = 0.84\,(1+22/A)$	$\Lambda = 0.84/(1+22/A)$
253		First line of summary	$Tl^{204,\ 206}$	$Tl^{203,\ 205}$
318	1	Figure caption	$\ldots\ e^2 mc^2 = 2.8 \cdot 10^{-23}$ cm,	$\ldots e^2/mc^2 = 2.8 \cdot 10^{-13}$ cm,
398		Figure caption	... to a cubic relation. A series of points etc.	...to a cubic relation, and in the region 10 – 20°K to a quadratic relation. A series of points ●, coinciding with points O, have been omitted in the region above 10°K.