Isentropic Relativistic Gas Flows

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General baroclinic isentropic relativistic gas flows are analyzed. Equations of vorticity, and a nonlinear equation of propagation of sound waves are derived. In the case of barotropic flow, a relativistic generalization of Thompson's theorem is found.

IN classical hydrodynamics, one can prove that for a barotropic isentropic gas flow* the circulation of the velocity around any closed curve moving with the fluid remains constant in time. In such a flow, vorticity of the velocity field can neither be created nor destroyed. If the flow is at one time described by a velocity potential, it retains this property for all time.

We shall prove that analogous theorems** hold in relativistic hydrodynamics. The situation differs only in that the ordinary 3-velocity of classical hydrodynamics must now be replaced not by the relativistic 4-velocity

$$u_i = g_{ik}u^k = g_{ik}dx^k/ds$$

(s being the proper time), but by the "pseudovelocity"

$$v_i = Ju_i, \tag{1}$$

where $J = (w/\rho)$ is the relativistic heat content per unit of rest-energy. Here w is the relativistic heat-content per unit of proper volume, and ρ is the rest-energy per unit proper volume, i.e., the rest-energy which the gas would have at absolute zero temperature. The dimensionless state-parameter J is always greater than unity.

By S we denote entropy per unit of rest-energy. In relativistic thermodynamics[1], S satisfies the equation

$$TdS = dJ - dp/\rho \tag{2}$$

where p is the pressure. We make no assumptions about the properties of the gas. We assume an equation of state $p = p(\rho, T)$, and a dependence of the internal energy-density e on ρ and T, these relations being completely arbitrary, subject only to the laws of relativistic thermodynamics and to the identity $w = e + p$. We further introduce the dimensionless relativistic sound-velocity

$$\bar{a} = a/c = \sqrt{(\partial p/de)_S} \tag{3}$$

where c is the velocity of light.

We shall prove that, when the ordinary velocity is replaced by the pseudovelocity v_i, the relativistic theory gives a system of equations for v_i and S completely analogous to the equations of classical hydrodynamics. We carry through the analysis for the case of rectilinear coordinates in special relativity, i.e., assuming g_{ik} constant. The transition to general relativity can be made in the usual way, by rewriting the equations in a form which is invariant under general coordinate transformations.

The energy-momentum equations are*

$$\partial T_i^k / dx_k = 0, \quad T_i^k = wu_iu^k - \delta_i^k p \tag{4}$$

and the equation of conservation of mass is

$$\partial(\rho u^k)/\partial x_k = 0. \tag{5}$$

Using Eq. (5) we reduce Eq. (4) to the form

$$\rho u^k dv_i / \partial x_k = \partial p / \partial x_i \tag{6}$$

(the relativistic Euler equations), and hence, by means of Eq. (2) to the form

$$u^k \partial v_i / \partial x_k = \partial J / \partial x_i - T\,\partial S / \partial x_i. \tag{7}$$

* By isentropic we mean a flow in which the entropy of each small element of the gas remains constant in time; by barotropic we mean a flow in which the entropy per unit mass is the same for all elements.

** Khalatnikov[2] was the first to investigate relativistic potential flows. We shall here discuss in greater detail flows possessing vorticity.

* The sign of the tensor g_{ik} is chosen so that the differential of proper times is $ds = (g_{ik}dx^i dx^k)^{1/2}$.

Because of the identity

$$g^{kl}u_k u_l = 1$$

the expression (1) for the pseudovelocity implies

$$g^{kl}v_k v_l = J^2. \quad (8)$$

Differentiation with respect to x_i gives

$$g^{kl}v_l\, \partial v_k / \partial x_i = J\, \partial J / \partial x_i$$

or

$$u^k\, \partial v_k / \partial x_i = \partial J / \partial x_i. \quad (9)$$

Subtracting Eq. (9) from Eq. (7), we find

$$u^k \left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) = -T\frac{\partial S}{\partial x_i}. \quad (10)$$

These are the well-known vorticity equations of classical hydrodynamics, with the curl of the velocity replaced by the curl of the pseudovelocity. Multiplying Eq. (10) by u^i and contracting, we obtain immediately the equation of conservation of particle entropy

$$dS / ds = u^i\, \partial S / \partial x_i = 0. \quad (11)$$

We proceed to transform the equation of continuity (5). Putting $u^k = v^k/J$, we find

$$\frac{\rho}{J}\frac{\partial v^k}{\partial x_k} + J\frac{d(\rho / J)}{ds} = 0.$$

From Eq. (11) and (2) we deduce

$$\frac{d(\rho / J)}{ds} = \frac{\rho}{J^2}(\bar{a}^{-2} - 1)\frac{dJ}{ds}, \quad (12)$$

and so the continuity equation takes the form

$$(dv^k / \partial x_k) + (\bar{a}^{-2} - 1)\, dJ / ds = 0. \quad (13)$$

From Eq. (8) we have

$$J\, dJ / ds = v^i\, dv_i / ds$$

or

$$dJ / ds = u^i\, dv_i / ds = u^i u^k\, dv_i / dx_k. \quad (14)$$

Therefore, Eq. (13) becomes

$$[g^{ik} + (\bar{a}^{-2} - 1)\, u^i u^k]\, dv_i / \partial x_k = 0. \quad (15)$$

This equation, when both flow and sound velocities are small, reduces to the classical equation of propagation of sound, including the effect of wind-velocity. In terms of general coordinates, and in general relativity, the corresponding equation is

$$[g^{ik} + (\bar{a}^{-2} - 1)\, u^i u^k] \times [(\partial v_i / \partial x_k) - \Gamma^l_{ik} v_l] = 0. \quad (16)$$

The whole system of hydrodynamical equations is contained in Eqs. (10) and (15).

The theorem governing the circulation around a line moving with the fluid in a barotropic flow is obtained as follows. In a barotropic flow Eq. (7) gives

$$u^k\, \partial v_i / \partial x_k = \partial J / \partial x_i \quad (17)$$

$$\text{or} \quad \partial v_i / ds = \partial J / \partial x_i,$$

where d means differentiation along the path of a fluid-element. Let δ denote differentiation along the line around which we are considering the circulation of pseudovelocity

$$\Gamma = \oint v_i\, \delta x^i. \quad (18)$$

Then

$$d\Gamma / ds = \oint [(dv_i/ds)\, \delta x^i + v_i \delta\, dx^i / ds]$$
$$= \oint [(\partial J / \partial x_i)\, \delta x^i + v_i\, \delta u^i)$$
$$= \oint [\delta J + {}^1/_2 J \delta (u_i u^i)] = \oint \delta J = 0,$$

showing that Γ remains constant in time. Strictly speaking, we must say that the circulation of pseudovelocity around a fluid line is equal in two successive positions, if each fluid element along the line has lived through the same interval of proper-time in moving from the earlier to the later position. From this theorem follows the impossibility of creating or destroying vortices of pseudovelocity in a barotropic flow.

If in a barotropic flow the pseudovelocity is derived from a potential ($v_i = d\varphi/dx_i$), then the relativistic Euler equations follow automatically. In this case

$$g^{ik}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_k} = J^2,$$

and so

$$g^{ik}\frac{\partial\varphi}{\partial x_i}\frac{\partial^2\varphi}{\partial x_k\, \partial x_l} = J\frac{\partial J}{\partial x_l} = \frac{J}{\rho}\frac{\partial p}{\partial x_l}$$

which gives

$$u^k \, \partial v_k / \partial x_l = \rho^{-1} \, \partial p / \partial x_l,$$

i.e., Eq. (6). In a potential flow, the equation of sound-propagation (15) becomes

$$[g^{ik} + (\bar{a}^{-2} - 1) \, u^i u^k] \, \partial^2 \varphi / \partial x_i \, \partial x_k = 0. \qquad (19)$$

In a barotropic flow, $\bar{a}$ depends only on J, and hence, by Eq. (8) on the pseudovelocity; therefore, Eq. (19) contains only the potential φ and its derivatives, and the entire problem in this case reduces to the solution of the single equation (19).

[1] L. D. Landau and E. M. Lifshitz, *Mechanics of Continuous Media*, 2nd Edition, Moscow, 1954.

[2] I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 529 (1954).

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