On Polarization Effects in the Radiation of an Accelerated Electron

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The polarization effects in the radiation of an accelerated electron are investigated by quantum methods, but in classical approximation. The formulas obtained describe both the linear and circular polarization.

1. AMPLITUDES OF THE LINEAR AND CIRCULAR POLARIZATIONS OF THE PHOTON FIELD

In considering the spontaneous emission of photons, we must choose the commutation relations for the amplitudes of the vector potential in the form

$$a_{s'}^{\prime +} a_s = 0, \quad a_s a_{s'}^{\prime +} = (\delta_{ss'} - \varkappa_s \varkappa_{s'} / \varkappa^2)\, \delta_{\varkappa, \varkappa'}. \tag{1}$$

For the determination of the polarization properties of the radiation it is necessary, moreover, to resolve the vector potential amplitude **a** into components such that each will characterize a definite state of polarization. In the study of the linear polarization one must resolve the amplitude of the vector potential **a** into two mutually perpendicular components (cf, e.g., Ref. 1):

$$\mathbf{a} = \beta_2 q_2 + \beta_3 q_3 = \mathbf{a}_2 + \mathbf{a}_3, \tag{2}$$

$$\beta_2 = [\mathbf{nj}] / \sqrt{1 - (\mathbf{nj})^2}, \tag{3}$$

$$\beta_3 = (\mathbf{n}(\mathbf{nj}) - \mathbf{j}) / \sqrt{1 - (\mathbf{nj})^2},$$

$$q_s q_{s'}^+ = \delta_{ss'}, \quad s, s' = 2, 3. \tag{4}$$

The vector **j** must be fixed in some specified direction, for which one can take, for example, that of the magnetic field vector **H**.

In the study of the circular polarization one must resolve the vector potential into two components in a different way:

$$\mathbf{a} = \gamma_1 q_1 + \gamma_{-1} q_{-1} = \mathbf{a}_1 + \mathbf{a}_{-1}, \tag{5}$$

where the vectors $\vec{\gamma}$ are related to the vectors $\vec{\beta}$ by

$$\gamma_\lambda = 2^{-1/2} (\beta_2 + i\lambda\beta_3), \tag{6}$$

and $\lambda, \lambda' = +1, -1$, $q_\lambda q_{\lambda'}^+ = \delta_{\lambda\lambda'}$. For $\lambda = 1$ we find the intensity of the radiation having left-circular polarization, and for $\lambda = -1$ that with right-circular polarization.

We write the expression for the intensity of the radiation in the form:

$$W_i = \sum_\nu \int_0^\pi d\theta \sin\theta\, W_i(\nu, \theta), \tag{7}$$

with the spectral and angular distribution of radiative intensity

$$W_i(\nu, \theta) = ce^2 S_i. \tag{8}$$

The symbol i will denote the state of polarization ($i = 2, 3, 1, -1$), and θ is the angle that the wave vector $\varkappa$ makes with the z axis. The quantity

$$S = (\bar{\alpha}\mathbf{a})(\bar{\alpha}^+\mathbf{a}^+) \tag{9}$$

is related to the amplitudes of the photon field and the matrix element

$$\bar{\alpha} = \int \psi_{n'}^+ e^{-i\varkappa\mathbf{r}} \alpha \psi_n d^3x,$$

describing the transition of the electron from the quantum state n to the state $n' = n - \nu$. In the study of the linear polarization we shall have to determine two expressions for S:

$$S_2 = (\bar{\alpha}\mathbf{a}_2)(\bar{\alpha}^+\mathbf{a}_2^+) = \bar{\alpha}_x \bar{\alpha}_x^+ (1 - \varkappa_x^2 / (\varkappa_x^2 + \varkappa_y^2)) \tag{10}$$

$$+ \bar{\alpha}_y \bar{\alpha}_y^+ (1 - \varkappa_y^2 / (\varkappa_x^2 + \varkappa_y^2))$$

$$- (\bar{\alpha}_x \bar{\alpha}_y^+ + \bar{\alpha}_y \bar{\alpha}_x^+) \varkappa_x \varkappa_y / (\varkappa_x^2 + \varkappa_y^2),$$

$$S_3 = (\bar{\alpha}\mathbf{a}_3)(\bar{\alpha}^+\mathbf{a}_3^+) = \bar{\alpha}_z\bar{\alpha}_z^+\left(1 - \frac{\varkappa_z^2}{\varkappa_x^2+\varkappa_y^2}\right) - (\bar{\alpha}\boldsymbol{\varkappa})(\bar{\alpha}^+\boldsymbol{\varkappa})\left(\frac{1}{\varkappa^2} - \frac{1}{\varkappa_x^2+\varkappa_y^2}\right) - (\bar{\alpha}_x\bar{\alpha}_z^+ + \bar{\alpha}_z\bar{\alpha}_x^+)\frac{\varkappa_x\varkappa_z}{\varkappa_x^2+\varkappa_y^2} - (\bar{\alpha}_y\bar{\alpha}_z^+ + \bar{\alpha}_z\bar{\alpha}_y^+)\frac{\varkappa_z\varkappa_y}{\varkappa_x^2+\varkappa_y^2}. \quad (11)$$

Equation (10) describes the photons with polarization vector (i.e., the electric field vector) in the plane of the orbit (σ component), and Eq. (11) is for those with polarization vector perpendicular to the plane of the orbit (π component). Both formulas are written with the convention that the vector **j** is directed along the z axis, i.e., along **H**.

In the study of the circular polarization we have for the quantity S the two values

$$S_{\pm 1} = (\bar{\alpha}\mathbf{a}_{\pm 1})(\bar{\alpha}^+\mathbf{a}_{\pm 1}^+) = {}^1/_2(\bar{\alpha}\,\bar{\alpha}^+) - {}^1/_2(\bar{\alpha}\mathbf{n})(\bar{\alpha}^+\mathbf{n}) \pm (i/2)(\mathbf{n}\,[\bar{\alpha}\,\bar{\alpha}^+]), \quad (12)$$

with the quantities $S_{\pm 1}$ not depending on the direction of the vector **j**. As is well known, in the classical case, the polarization is characterized by the ratio between the squares of the amplitudes of two mutually perpendicular oscillations, and also the phase difference δ between these oscillations, which for an unpolarized beam takes all possible values. In our case we shall set the squares of the amplitudes of the oscillations proportional to W_2 and W_3, and the phase will be given by

$$\sin\delta = (W_{-1} - W_1)/2\sqrt{W_2 W_3}.$$

It is obvious that the total intensity of the radiation will be equal to the sum of the appropriate components

$$W = W_2 + W_3 = W_{-1} + W_1. \quad (13)$$

Thus, in order to give a complete specification of the polarization properties of the radiation, not knowing the phases, we must calculate not only the intensities of the linear polarizations (W_2 and W_3), but also those of the circular polarizations (W_{-1} and W_1).

For unpolarized radiation we have

$$W_2 = W_3 = W_{-1} = W_1 = \frac{1}{2}W. \quad (14)$$

In the case of linear polarization

$$W_2 = W,\ W_3 = 0. \quad (15)$$

If the radiation has circular polarization, then

$$W_{-1} = W,\ W_1 = 0. \quad (16)$$

The simultaneous presence of partial linear and partial circular polarization

$$W_2 \neq W_3,\ W_{-1} \neq W_1 \quad (17)$$

corresponds to elliptically polarized radiation, since $\delta \neq 0$.

2. RADIATION OF THE ACCELERATED ELECTRON WITH INCLUSION OF POLARIZATION EFFECTS

As has been shown (cf. Refs. 2 and 3), when an electron moves in a homogeneous magnetic field the frequency of the radiation is given in extreme relativistic approximation by

$$\omega = c\varkappa = (E_n - E_{n'})/\hbar = \nu c/R, \quad (18)$$

where R is the radius of the orbit and

$$E_n = \sqrt{2eHc\hbar + m^2c^4} \quad (19)$$

is the energy of the electron. When one goes over to the classical approximation ($\hbar \to 0$), which involves the use of the asymptotic formula connecting generalized Laguerre polynomials with Bessel functions (cf. Ref. 4)

$$\lim_{n\to\infty,\ \nu,\ z\neq\infty} I_{n,\,n-\nu}(z^2/4n) = J_\nu(z), \quad (20)$$

the matrix elements take the form:

$$|\bar{\alpha}_x|^2 = \beta^2 J_\nu'^2(\nu\beta\sin\theta), \quad (21)$$

$$|\bar{\alpha}_y|^2 = \sin^{-2}\theta J_\nu^2(\nu\beta\sin\theta);$$

$$\bar{\alpha}_x\bar{\alpha}_y^+ = -\bar{\alpha}_y\bar{\alpha}_x^+ = (i\beta/\sin\theta)J_\nu(\nu\beta\sin\theta)J_\nu'(\nu\beta\sin\theta);$$

$$|\bar{\alpha}_z|^2 = \bar{\alpha}_x\bar{\alpha}_z^+ = \bar{\alpha}_z\bar{\alpha}_x^+ = \bar{\alpha}_y\bar{\alpha}_z^+ = \bar{\alpha}_z\bar{\alpha}_y^+ = 0.$$

Then we get a formula for the radiation density that indicates not only the spectral and angular distribution but also the polarization:

$$W_i(\nu, \theta) = (e^2\beta^2 c / 2\pi R^2)\, \nu^2 \{\beta\lambda_2 J'_\nu(\nu\beta\sin\theta) - \lambda_3 \operatorname{ctg}\theta J_\nu(\nu\beta\sin\theta)\}^2. \quad (22)$$

In calculating the intensities of polarized radiation we must set the constants λ_2 and λ_3 equal to the following values: a) for the linearly polarized beam with polarization vector lying in the plane of the orbit ($i = 2$, σ component)

$$\lambda_2 = 1, \ \lambda_3 = 0, \quad (23)$$

b) for the linearly polarized beam with polarization vector perpendicular to the plane of the orbit ($i = 3$, π component)

$$\lambda_2 = 0, \ \lambda_3 = 1. \quad (24)$$

As is well known, in the extreme relativistic case the direction of the vector $\varkappa$ almost coincides with the direction of motion of the electron (i.e., the vector $\varkappa$ is directed along the tangent to the circle). In this case the polarization vector for $\lambda_2 = 1$, $\lambda_3 = 0$ is directed along the radius toward the center, and for $\lambda_2 = 0$, $\lambda_3 = 1$ it is directed almost along the z axis (Fig. 1).

In calculating the intensity of circularly polarized radiation we must set

$$\lambda_2 = \pm\lambda_3 = 2^{-1/2}. \quad (25)$$

From Eq. (22) we see that for the νth harmonic the light propagated along some arbitrary direction θ will as a rule be elliptically polarized.

The ratio of the squared amplitudes and the value of the phase that characterize this elliptical polarization are

$$\frac{W_3(\nu, \theta)}{W_2(\nu, \theta)} = \frac{\operatorname{ctg}^2\theta J^2_\nu(\nu\beta\sin\theta)}{\beta^2 J'^2_\nu(\nu\beta\sin\theta)}; \quad (26)$$

$$\delta = \arcsin\frac{\cos\theta}{|\cos\theta|} = \pm\frac{\pi}{2}.$$

From the latter equation it is seen that for $\theta = \pi/2$ (i.e., in the plane of the motion of the

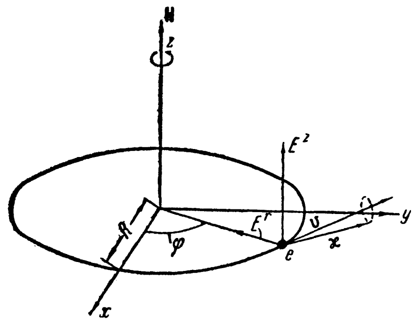

FIG. 1

electron) the radiation becomes linearly polarized, since $W_3 = 0$. For $0 < \theta < \pi/2$, i.e., in a direction making an acute angle with the magnetic field, the light emitted is elliptically polarized with right-handed rotation, and for $\pi/2 < \theta < \pi$, with left-handed rotation. With increase of $|\cos\theta|$ the elliptical polarization begins to go over into circular. In particular, for the nonrelativistic case ($\beta \ll 1$), when almost the whole intensity is in the fundamental mode ($\nu = 1$), we have

$$J_1(\beta\sin\theta) \approx {}^1/_2\,\beta\sin\theta, \qquad J'_1(\beta\sin\theta) \approx {}^1/_2. \quad (27)$$

Then the ratio in Eq. (26) becomes

$$W_3(\theta) / W_2(\theta) = \cos^2\theta. \quad (28)$$

From this one sees that the beam in the direction of the magnetic field ($\theta = 0$) must be right-circularly polarized, and that directed against the field ($\theta = \pi$) is left-circular. This result for the nonrelativistic case is well known and is used in the analysis of the Zeeman effect.

To study the extreme relativistic case $1 - \beta^2 \ll 1$ it is necessary to approximate the Bessel functions by the formulas (cf. Ref. 2):

$$J'_\nu(\nu\beta\sin\theta) = \frac{\varepsilon}{\pi\sqrt{3}} K_{2/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right), \quad (29)$$

$$J_\nu(\nu\beta\sin\theta) = \frac{\sqrt{\varepsilon}}{\pi\sqrt{3}} K_{1/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right),$$

where $\epsilon = 1 - \beta^2\sin^2\theta$. Then the intensity of the polarized radiation [cf. Eq. (22)] will depend on the harmonic number ν and on the angle θ in the following way

$$W_i(\nu, \theta) = (e^2\beta c / 2\pi R^2)(\nu^2 / 3\pi^2) \quad (30)$$

$$\times \left[\lambda_2 \varepsilon K_{2/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right) - \cos\theta \lambda_3 \sqrt{\varepsilon} K_{1/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right)\right]^2.$$

To find the dependence of the degree of polarization on the angle θ alone, we sum the expression (30) over the harmonics ν. Since in the extreme relativistic case the maximum of the intensity lies in the high frequency region, we can replace the summation over ν by an integration. Then, using the integrals

$$\int_0^\infty \nu^2 K_{2/3}^2\left(\frac{\nu}{3}\varepsilon^{3/2}\right) d\nu = \frac{21}{16}\pi^2\varepsilon^{-9/2}; \quad (31)$$

$$\int_0^\infty \nu^2 K_{1/3}^2\left(\frac{\nu}{3}\varepsilon^{3/2}\right) d\nu = \frac{15}{16}\pi^2\varepsilon^{-9/2};$$

$$\int_0^\infty \nu^2 K_{1/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right) K_{2/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right) d\nu = \frac{6\pi}{\sqrt{3}}\varepsilon^{-4/2},$$

we find that

$$W_i(\nu, \theta)\sin\theta\, d\theta \quad (32)$$

$$= \sin\theta\, d\theta \int_0^\infty W_i(\nu, \theta)\, d\nu = W f_i(\xi)\, d\xi,$$

where $\xi = \cos\theta / \sqrt{1-\beta^2}$ and W is the total energy radiated per unit time,

$$W = {}^2/_3 (e^2 c / R^2)(E / mc^2)^4. \quad (33)$$

For the function $f_i(\xi)$ we have the following expression:

$$f_i(\xi) = \frac{3}{2}\left\{\frac{7}{16}\lambda_2^2(1+\xi^2)^{-5/2}\right. \quad (34)$$

$$\left. + \frac{5}{16}\lambda_3^2\xi^2(1+\xi^2)^{-7/2} - \frac{4}{\pi\sqrt{3}}\lambda_2\lambda_3\xi(1+\xi^2)^{-3}\right\}.$$

Curves of the dependence of f_i on ξ (i.e., on the angle θ) for the linearly polarized components are given in Fig. 2. The curves f_2 and f_3 characterize the radiated intensity of the linearly polarized light. The curves f_1 and f_{-1} of Fig. 3 characterize the intensity of the circularly polarized radiation, namely, the left-circular ($\lambda_2 = \lambda_3 = 1/\sqrt{2}$) and the right-circular ($\lambda_2 = -\lambda_3 = 1/\sqrt{2}$), respectively.

Finally, the curves f_0 (cf. Figs. 2 and 3) correspond to the total intensity of the radiation, equal to the sum of the linearly polarized intensities (Fig. 2) or of the circularly polarized intensities (Fig. 3). These curves are identical in the two diagrams.

From the curves given it is seen that in the plane of the electron's motion ($\theta = \pi/2$) the light emitted must be linearly polarized, with its polarization vector directed along the axis of r. For $\theta < \pi/2$ the radiation has predominantly right-circular polarization, and for $\theta > \pi/2$, left-circular.

Integrating Eq. (30) with respect to the angle θ, we find the polarization properties of the radiation as they depend on the harmonic order ν. Making use of the integrals

$$\int_0^\pi \varepsilon^2 K_{2/3}^2\left(\frac{\nu}{3}\varepsilon^{3/2}\right)\sin\theta\, d\theta \quad (35)$$

$$\approx \frac{\pi\varepsilon_0}{\sqrt{3}\nu}\left\{\int_{2/3\nu\varepsilon_0^{3/2}}^\infty \frac{K_{2/3}(x)}{x}dx + 3K_{2/3}\left(\frac{2}{3}\nu\varepsilon_0^{3/2}\right)\right\};$$

$$\int_0^\pi \varepsilon\cos^2\theta K_{1/3}^2\left(\frac{\nu}{3}\varepsilon^{3/2}\right)\sin\theta\, d\theta$$

$$\approx \frac{\pi\varepsilon_0}{\sqrt{3}\nu}\int_{2/3\nu\varepsilon_0^{3/2}}^\infty \frac{K_{2/3}(x)}{x}dx;$$

$$\varepsilon_0 = 1 - \beta^2 = (mc^2/E)^2$$

and introducing the variable $y = (2\nu/3)(mc^2/E)^3$, we obtain

$$W_i(\nu)\, d\nu = \int_0^\pi W_i(\nu, \theta)\sin\theta\, d\theta = W\varphi_i(y)\, dy, \quad (36)$$

where

$$\varphi_i(y) = \frac{9\sqrt{3}}{16\pi}y\left\{(\lambda_2^2 + \lambda_3^2)\right. \quad (37)$$

$$\left.\times \int_y^\infty K_{5/3}(x)\, dx + (\lambda_2^2 - \lambda_3^2) K_{2/3}(y)\right\}.$$

Summing Eq. (37) over the states of polarization, we obtain for the spectral intensity of the radiation the formula found in Ref. 5 (cf. also Ref. 6). From this it is seen that on integration over the angles the circular polarization disappears completely, since the intensities for the two cases, right-circular polarization ($\lambda_2 = \lambda_3 = 1/\sqrt{2}$) and left-circular polarization ($\lambda_2 = -\lambda_3 = 1/\sqrt{2}$), become equal.

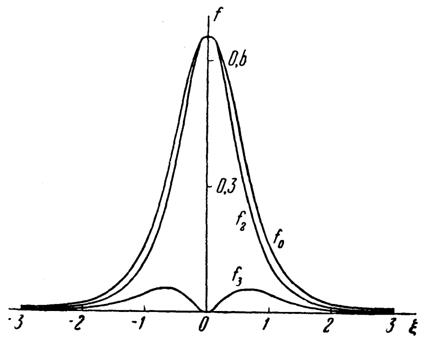

FIG. 2

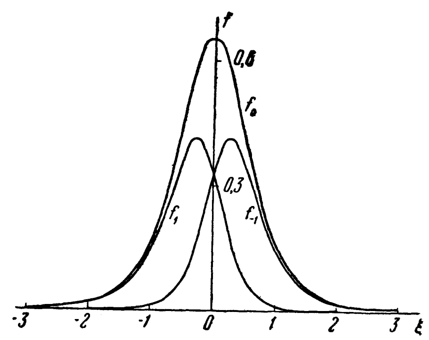

FIG. 3

The radiation intensities of the linear polarizations of the light will be different. Curves of the dependence of φ on y (i.e., on the harmonic order ν) are shown in Fig. 4. It is seen from the curves that the bulk of the intensity is in the radiation corresponding to the σ component (cf. the curve of φ_2 for $\lambda_2 = 1$, $\lambda_3 = 0$). The total intensity of the radiation is represented by the curve φ_0.

Integrating the expression (37) over all frequencies, we find the total intensities of the polarized radiations (cf. also Ref. 4):

$$W_2 = W \int_0^\infty \varphi_2(y)\,dy = {}^7/_8 W, \qquad (38)$$

$$W_3 = W \int_0^\infty \varphi_3(y)\,dy = {}^1/_8 W. \qquad (39)$$

These same values can be obtained also from Eq. (32) by integrating over the variable ξ.

The theory of the polarization of the radiation of an accelerated electron can also find applications in the study of the radio radiation of the sun and the galaxy, in which polarization properties have already been found experimentally. But the field of radio astronomy in question requires special consideration and is beyond the scope of our present problem.

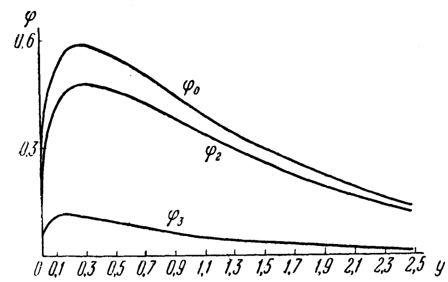

FIG. 4

Note added in proof: Recently, the polarization effects from the accelerated electron, as a function of the direction of emission, have been studied experimentally[7] for particular monochromatic parts of the spectrum ($\lambda \sim 5500$ A, etc.) by the group headed by F. A. Korolev at the 260 mev synchrotron. Theoretical curves drawn by Eq. (30) for the σ and π components (these curves had approximately the shape of the curves for the whole spectrum shown in Fig. 2) agreed fairly well with the corresponding experimental data.

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