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On the Possibility of Introducing an Effective Dielectric Constant at High Frequencies

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It is shown that it is possible to introduce an effective dielectric constant $\epsilon_{\text{eff}} = (4\pi/cZ)^2$ in the anomalous skin-effect range.

1. AS is well known[1,2], the introduction of an effective dielectric constant $\epsilon_{\text{eff}} = (4\pi/cZ)^2$ (where $Z = R + iX$ is the surface impedance of the metal) is based on the fact that Z is practically independent of the polarization of the incident wave and of the angle of incidence φ. According to Ginzburg, this is so if

$$|\varepsilon'| \gg 1, \quad \varepsilon' = \varepsilon(\omega) - 4\pi i\sigma/\omega \;; \tag{1}$$

a rigorous proof has been given for frequencies and temperatures at which the normal skin effect occurs*.

Condition (1) is equivalent to the requirement that

$$|\delta| \ll \lambda \; (\delta = c/\omega\sqrt{\varepsilon'}, \quad \lambda = c/\omega),$$

i.e., that the depth of penetration of the electromagnetic wave into the metal be much smaller than the wavelength in vacuum. The latter condition is satisfied even in the anomalous skin-effect range; this fact suggests that in this range, as well as under normal skin effect, the surface impedance may be independent of the polarization and of the angle of incidence. The present communication is devoted to a rigorous proof of this proposition* .

2. By definition,

$$Z_s = (4\pi/c)\, E_x(0)/H_y(0);$$

$$Z_p = -(4\pi/c)\, E_y(0)\; H_x(0).$$

The index s corresponds to polarization of the electric field in the plane of the metal surface, the index p to polarization in the plane of incidence. We consider oblique incidence (at angle φ) of an electromagnetic wave of frequency ω upon a half-space ($z > 0$) occupied by a metal. On the usual assumptions of the theory of the anomalous skin effect[3], we find the following expressions for the surface impedance:

1) In the case of specular reflection of electrons from the metal-vacuum boundary:

$$Z_s = i\frac{8\delta}{c\lambda}\int_0^\infty \frac{dt}{t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi}; \tag{2}$$

$$Z_p = i\frac{8\delta}{c\lambda}\int_0^\infty \frac{1 - \sin^2\varphi\,\dfrac{1 - i\gamma^2(\lambda/\delta)^2 t^2 K_2(tl/\delta)}{1 - i(\lambda/\delta)^2 K_2(tl/\delta)(1+\gamma^2 t^2)}}{t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi + i\sin^2\varphi\,\dfrac{K_3(tl/\delta)}{1 - i(\lambda/\delta)^2 K_2(tl/\delta)(1+\gamma^2 t^2)}}\, dt.$$

Here

$$\delta = \frac{c}{\sqrt{2\pi\sigma\omega}}; \quad \lambda = \frac{c}{\omega}; \quad \sigma = \frac{\sigma_0}{1 + i\omega\tau};$$

$$l = \frac{l_0}{1 + i\omega\tau}; \quad \gamma = \frac{l}{\lambda\sqrt{6\omega\tau}},$$

σ_0 is the static conductivity of the metal, $l_0 = v_0\tau$ is the length of the free path, v_0 is the limiting Fermi velocity and τ is the relaxation time of the conduction electrons.

$$K_1(w) = 3w^{-3}[(1 + w^2)\,\mathrm{arctg}\, w - w], \quad K_1(0) = 3;$$

$$K_2(w) = 6w^{-3}[w - \mathrm{arctg}\, w], \quad K_2(0) = 2;$$

$$K_3(w) = K_2(w) - K_1(w);$$

for $w \gg 1$, $K_1(w) \approx \pi/6w$ and $K_2(w) \approx 6/w^2$.

2) With diffuse scattering of the electrons from the metal surface, we have:

* $\epsilon(\omega)$ is the dielectric constant of the metal, σ is its conductivity, and ω is the angular frequency of the electromagnetic wave.

* The Proof given in Ref. 1 (Sec. 2) seems to us not quite convincing, since it assumes in advance the existence of ϵ'.

$$Z_s = i\frac{4\pi}{c}\frac{\delta}{\lambda}\left(\alpha + 1 + \sum_{j=1}^{n} s_j - n\right)^{-1}, \qquad (3)$$

$$Z_p = i\frac{4\pi}{c}\frac{\delta}{\lambda}\left(\beta + 1 + \sum_{j=1}^{m} p_j - m\right)^{-1};$$

$$\alpha = \frac{1}{\pi} \times \int_0^\infty \ln\left\{\frac{(t^2+1)^{n-1}\ (t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi)}{(t^2+s_1^2)(t^2+s_2^2)\ldots(t^2+s_n^2)}\right\}dt; \qquad (4)$$

$$\beta = \frac{1}{\pi}\int_0^\infty dt \times \ln\left\{\frac{(t^2+1)^{m-1}\left[t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi + i\sin^2\varphi\dfrac{K_3(tl/\delta)}{1 - i(\lambda/\delta)^2K_2(tl/\delta)(1+\gamma^2t^2)}\right]}{(t^2+p_1^2)(t^2+p_2^2)\ldots(t^2+p_m^2)}\right\};$$

s_j ($j = 1, 2, \ldots, n$) are the roots of the characteristic equation

$$t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi = 0,$$

in the plane of the complex variable t. p_j ($j=1, 2, \ldots, n$) are the roots of the equation

$$t^2 + iK_1(tl/\delta) - (\delta/\lambda)^2\cos^2\varphi + i\sin^2\varphi\frac{K_3(tl/\delta)}{1 - i(\lambda/\delta)^2K_2(tl/\delta)\ (1+\gamma^2t^2)} = 0$$

in the same plane. The other symbols are as before.

In the derivation of Eqs. (2)-(4), the collision integral was written in the form

$$(\partial f/\partial t)_{\rm col} = (f - \bar{f})/\tau.$$

The bar over f means an average over angles in momentum space. With oblique incidence, $\overline{f_1} \neq 0$ (f_1 is the nonequilibrium contribution to the Fermi distribution function), because of the presence of a z component of the electromagnetic field. Calculation of the term $\overline{f_1}/\tau$ leads to the appearance, in the right member of the kinetic equation, of a term $(v_0^2/12\pi\sigma_0)$ div $\mathbf{E}$ (because $\overline{f_1}$ is proportional to the charge density $\rho = (1/4\pi)$ div $\mathbf{E}$). For good metals, this term is negligible in comparison with $\mathbf{E}\cdot\mathbf{v}_0$, up to frequencies in the infrared region of the spectrum.

It is evident from Eqs. (2)-(4) that in the general case ($\varphi \neq 0$), Z_s differs from Z_p and depends on three parameters of the dimensions of a length: δ, the skin-depth of penetration of the wave into the metal; l, the length of the free path of the electrons; and λ, the length of the wave in vacuum. We will determine how important this dependence is in various cases.

At all frequencies up to $\omega \lesssim 10^{16}$ sec^{-1}, and at any temperatures, the wavelength λ is large in comparison with the skin depth δ and the path length l. In fact,

$$\left|\frac{l}{\lambda}\right| = \frac{v_0}{c}\frac{\omega\tau}{(1+\omega^2\tau^2)^{1/2}} \ll 1;$$

$$\left|\frac{\delta}{\lambda}\right| = \left(\frac{\omega}{2\pi\sigma_0}\right)^{1/2}(1+\omega^2\tau^2)^{1/4} \ll 1,$$

since even for $\omega\tau \gg 1$

$$|\,\delta/\lambda\,| \approx \omega/\sqrt{2\pi ne^2/m} \sim \omega\cdot 10^{-16}.$$

It is therefore permissible, in the integrands in (2) and (4), to go to the limit $/\lambda \to \infty$; then we get directly $Z_p = Z_s$. Under these conditions, the surface impedance is independent of the angle of incidence φ and essentially agrees with the expression obtained in Ref. 3 [Eq. (21)]. This is correct to the first nonvanishing term in $1/\lambda$, for arbitrary ratio between l and δ.

The same argument (cf. Sec. 1) proves the possibility of introducing an effective dielectric constant in all ranges of frequency and temperature.

3. We will calculate a correction to the surface impedance, depending on the angle of incidence. In the limiting case of normal skin effect, when

$$|\,l/\delta\,| = (v_0/c)(2\pi\sigma_0/\omega)^{1/2}\omega\tau(1+\omega^2\tau^2)^{-3/4} \ll 1$$

(low frequencies, high temperatures), the largest small parameter --- in terms of which it is convenient to expand --- is δ/λ. Then l/δ may be set equal to zero everywhere. The result, as was to be expected, is that we arrive at the known expression for the surface impedance under normal skin effect [cf., for example, Eq. (2.2) in Ref. 1].

In the opposite limiting case $|l/\delta| \gg 1$, which is realized at high frequencies ($10^6 \leq \omega \leq 10^{11}$) and low temperatures (anomalous skin effect), the expansion parameter is l/λ. We approximate $K_j(tl/\delta)$ asymptotically for large values of the argument and expand expressions (2) and (4) in powers of l/λ. After some calculations we find:

Specular reflection of electrons from the surface:

$$Z_p(\varphi) = Z_p(0)\left\{1 - \sin^2\varphi\,\frac{9\pi^{1/3}}{16}\left(1+\frac{i}{\sqrt{3}}\right)\beta p^{-1/3}\right\}; \quad (5)$$

$$Z_p(0) = Z_s(0) = {}^8/_9\,(\sqrt{3}\,\pi\omega^2 l_0/c^4\sigma_0)^{1/3}\,(1+i\sqrt{3});$$

Diffuse reflection:

$$Z_p(\varphi) = Z_p(0)\left\{1 - \sin^2\varphi\,\frac{\pi^{2/3}}{32}\,\frac{\beta^2 p^{-2/3}}{1+p^2q^2}\right. \quad (6)$$

$$\times\left[5pq + \frac{11}{\sqrt{3}} + \frac{2}{\pi}\left(\frac{pq}{\sqrt{3}}+1\right)\ln a\right.$$

$$\left.\left.+\, i\left(\frac{11pq}{\sqrt{3}} - 5 + \frac{2}{\pi}\left(\frac{pq}{\sqrt{3}}-1\right)\ln a\right)\right]\right\};$$

$$Z_p(0) = Z_s(0) = (\sqrt{3}\,\pi\omega^2 l_0/c^4\sigma_0)^{1/3}(1+i\sqrt{3}),$$
$$a = 8p/\pi\sqrt{6}\beta^3.$$

Here we have used the notation of Ref. 1:

$$\beta = v_0/c;\; p = \sqrt{3/2}\,(l_0/\delta_0)\,(\omega\tau)^{-3/2}$$

$$= (3\pi/\omega)\,(v_0/c)\,(ne^2/3\pi m)^{1/2};$$

$$q = \sqrt{2/3}\,(\delta_0/l_0)\,(\omega\tau)^{1/2}$$

$$= (2c/\sqrt{3}\,l_0)\,(m/4\pi ne^2)^{1/2};\; pq = (\omega\tau)^{-1}.$$

The coefficients of $\sin^2\varphi$ in Eqs. (5) and (6) are of order 10^{-3} to 10^{-5} for representative values of the quantities occurring in them: $\beta \sim 10^{-2}$, $\omega \sim 10^{11}$ sec^{-1}, $l_0 \sim 10^{-3}$ cm., $\delta_0 \sim 10^{-5}$ cm.

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[1] V. L. Ginzburg and G. P. Motulevich, Usp. fiz. nauk 55, 469 (1955).

[2] V. L. Ginzburg, Dokl. Akad. Nauk SSSR 97, 999 (1954).

[3] G.E.H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) A195, 336 (1948).

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