

$$\frac{\partial}{\partial \theta_1^z} F_{a_1 \dots a_s}^1 + \frac{\partial U_{a_1 \dots a_s}^*}{\partial \theta_1^z} \quad (28)$$

$$+ \frac{1}{4\pi} \int \sum_{1 \leq a_{s+1} \leq N} \frac{\partial \psi_{a_1 a_{s+1}}}{\partial \theta_1^z} d\Omega_{s+1} = 0,$$

and since, for the dipole potential,

$$\int \psi_{a_1, a_{s+1}} d\Omega_{s+1} = 0, \quad \text{identically,}$$

we get

$$F_{a_1 \dots a_s}^1 = -U_{a_1 \dots a_s}^* \quad (29)$$

in agreement with the normalization condition. The second approximation is found in a similar way and has the form

$$F_{a_1 \dots a_s}^2 = \frac{1}{2} \left(U_{a_1 \dots a_s}^{*2} \right. \quad (30)$$

$$\left. + \frac{1}{4\pi} \sum_{a_{s+1}}^N \int \left(\sum_{a_i}^s \psi_{a_i a_{s+1}} \right)^2 d\Omega_{s+1} \right) + K_{a_1 \dots a_s}^{(2)},$$

where, in accord with the normalization condition, the constant K must be given by

$$K_{a_1 \dots a_s}^{(2)} = -\frac{1}{2} \frac{1}{(4\pi)^s} \left\{ \sum_{a_i a_h}^s \psi_{a_i a_h}^2 d\Omega_1 \dots d\Omega_s \right. \quad (31)$$

$$\left. + \frac{1}{4\pi} \int \sum_{a_{s+1}}^N \sum_{a_i}^s \psi_{a_i a_{s+1}}^2 d\Omega_1 \dots d\Omega_{s+1} \right\}.$$

Here we have studied the properties of the dipole potential ψ . In similar fashion the constants for

further approximations can be determined, but their calculation is not so simple.

In the presence of an external homogeneous electric field E and upon fulfillment of the condition

$$\gamma = p^2/d^3\mu \ll 1 \quad (32)$$

the computation scheme does not undergo any change.

These expressions for the distribution function permit us to construct a theory of the equilibrium properties of the dipole crystals under consideration. Numerical calculations and comparison with experiment, and also comparison with other theories which apply to the two very simple cases considered by us will be given in a subsequent paper. Separate consideration is necessary for the construction of a theory in the case of violation of the relation (32), while this case has the greatest interest. We hope in the near future to publish results which apply to this variant.

For the second of the problems considered by us the considerations developed above can without essential change be applied for the purpose of constructing a semiclassical theory of ferromagnetic bodies without use of the "sphericalizing" approximation.

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Relaxation between Electrons and the Crystalline Lattice*

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The relaxation between the electrons of a metal and the crystalline lattice (phonons) is considered. The state of the electrons and the lattice is described by equilibrium Fermi and Bose functions with different temperatures. The heat transfer coefficient connected with the "Cerenkov" radiation of sound waves by the electrons has been determined.

I THERE have appeared recently several experimental^{1,2} and theoretical^{3,5} papers illuminating the investigation of deviations from Ohm's law in metals. The departures from a linear relation between the current J and the field E , noted by

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Borovik,¹ are observed in the presence of a large current when the mean energy of the electrons differs appreciably from that which corresponds to the lattice temperature T . Since the time required to establish equilibrium in the electron gas is much less than the time for achieving equilibrium between the electrons and the lattice,⁴ we can consider that the electron gas is in a state of equilibrium, i.e., its state is described by the ordinary Fermi distribution function

$$n = [1 + \exp(\varepsilon - \varepsilon_0) / k\Theta]^{-1}, \quad (1)$$

$$\varepsilon_0 = (3n_0 / 8\pi)^{2/3} (2\pi\hbar)^2 / 2m,$$

n_0 is the conduction electron density, m is the effective electronic mass. The electron temperature Θ here differs from the lattice temperature ($\Theta > T$).

A similar state of partial equilibrium can arise not only in the passage of electric current through the metal. For example, in the passage of fast charged particles through material, the energy of the particle is spent in direct interaction with the ions of the lattice (nuclear collisions) and by interaction with electrons (ionization, polarization). Here a major part of the energy of the particles is spent in interaction with the electrons ($\sim 90\%$ for fission fragments, see Ref. 6). As a result, large local temperature differences arise between the electrons and the lattice. Subsequently, the transfer of energy from the electrons to the lattice takes place by means of a relaxation mechanism, described below.*

The temperature regimes of the electron gas (the maximum temperature difference between the electrons and the lattice, the relaxation time) are determined by the source of the heat and by the coefficient of heat transfer from the electrons to the lattice. At high temperatures ($T \gg T_0$, T_0 the

Debye temperature) the problem was solved by Ginzburg and Shabanskii.⁴ In the present work a method is used which permits us to find the heat transfer coefficient for arbitrary temperatures.

2. Let us compute the amount of energy transferred by the electrons (per unit volume) to the lattice in unit time (\bar{U}). Evidently,**/

$$\bar{U} = \int \dot{N}_f \hbar \omega_f (2\pi)^{-3} d\tau_f V. \quad (2)$$

Here N_f = change (per unit time) in the number of phonons with wave vector f and energy $\hbar \omega_f = \hbar s f$ (per unit volume), s = sound velocity, V = volume of the crystal. We designate the probability (per second) of transition of the electron from a state with wave vector k into a state with wave vector k' by $W_{k,k'} \delta(\varepsilon_k + \hbar \omega_f - \varepsilon_{k'})$; the δ -function obeys the law of energy conservation*

$$W_{k,k'} = (\pi U^2 / \rho V s^2) \omega_f; \quad f = k - k'. \quad (3)$$

Then⁸

$$\begin{aligned} \dot{N}_f = \int W_{k,k'} \{ & (N_f + 1) n_{k'} (1 - n_k) \\ & - N_f n_k (1 - n_{k'}) \} \\ & \times \delta(\varepsilon_k + \hbar \omega_f - \varepsilon_{k'}) (2\pi)^{-3} 2d\tau_{k'}. \end{aligned} \quad (4)$$

As is seen from (3) and (4), the change in the phononfunction takes place because of the "creation" and "annihilation" of the phonons. Similar processes can exist, since the velocities of the electrons are greater than the velocity of sound, which guarantees simultaneous fulfillment of the laws of conservation of momentum and energy. This shows that the creation of the phonon in the quantum-mechanical description corresponds to Cerenkov radiation of sound waves by electrons in the classical view (see below).

For identical temperatures of electrons and phonons (lattice) the expression under the integral vanishes identically for the equilibrium distribution function. In our case, when $T \neq \Theta$ and

$$N_f = [e^{\hbar \omega_f / kT} - 1]^{-1}, \quad (5)$$

we have

$$\dot{N}_f = \frac{m^2 U^2 \hbar \omega_f}{2\pi \hbar^4 \rho V s} \frac{e^{\hbar \omega_f / kT} - e^{\hbar \omega_f / k\Theta}}{(e^{\hbar \omega_f / kT} - 1)(e^{\hbar \omega_f / k\Theta} - 1)}. \quad (6)$$

For computation, we have considered that $\varepsilon_k = \hbar^2 k^2 / 2m$ and that $\varepsilon_0 \gg m s^2 / 2$ and $k\Theta$. The latter conditions are always satisfied beforehand. For Bi, for which $\varepsilon_0 \sim kT_0$, this limits the region of application of Eq. (6) to low temperatures.

*A separate publication will be devoted to consideration of the interaction of nuclear radiation with matter.

** We use the method developed by Akhiezer and Pomeranchuk⁷ for the calculation of the spin-lattice exchange energy.

*The following notation is used: ρ = density of the material ($\rho = M/d^3$, M is the mass of the lattice atom, V = lattice volume); U = constant interaction of the electron with the lattice that appears in the expression for the time to travel the mean free path (see, for example, Refs. 8,9 where it is denoted by C).

Substituting (6) in (2) and integrating, we get

$$\bar{U} = \frac{2}{(2\pi)^3} \frac{m^2 U^2 (kT_0)^5}{\hbar^7 \rho S^4} \quad (7)$$

$$\times \left\{ \left(\frac{\Theta}{T_0} \right)^5 \int_0^{\Theta} \frac{x^4 dx}{e^x - 1} - \left(\frac{T}{T_0} \right)^5 \int_0^T \frac{x^4 dx}{e^x - 1} \right\}.$$

In the limiting case of low temperatures ($T, \Theta \ll T_0$), we have

$$\bar{U} = \frac{2D_5}{(2\pi)^3} \frac{m^2 U^2 (kT_0)^5}{\hbar^7 \rho S^4} \frac{\Theta^5 - T^5}{T_0^5}; \quad (8)$$

$$D_5 = \int_0^\infty \frac{x^4 dx}{e^x - 1};$$

in the opposite case, at high temperatures ($T, \Theta \gg T_0$),

$$\bar{U} = \frac{m^2 U^2 (kT_0)^5}{2\hbar^7 \rho S^4 (2\pi)^3} \frac{\Theta - T}{T_0}. \quad (9)$$

Comparing Eqs. (8) and (9) with the expressions for the time of flight (Ref. 9, p. 184), we get for low temperatures,

$$\bar{U} = (2\pi^2 / 15) m S^2 n_0 \{1 / \tau(\Theta) - 1 / \tau(T)\}, \quad (10)$$

and for high,

$$\bar{U} = (\pi^2 / 6) m S^2 n_0 \{1 / \tau(\Theta) - 1 / \tau(T)\}. \quad (11)$$

Here, we understand by $\tau(\Theta)$ and $\tau(T)$ the time of free flight of electrons under the condition that the lattice temperature coincides with the electron temperature and is equal respectively to Θ or T ; n_0 is the number of electrons per unit volume.

If the temperature difference $\Theta - T$ is much less than the lattice temperature T , then

$$\bar{U} = \frac{2\pi^2}{3} \frac{m S^2 n_0}{\tau(T)} \frac{\Theta - T}{T} \quad (12)$$

$$(T \ll T_0; \Theta - T \ll T);$$

$$\bar{U} = \frac{\pi^2}{6} \frac{m S^2 n_0}{\tau(T)} \frac{\Theta - T}{T} \quad (13)$$

$$(T \gg T_0; \Theta - T \ll T);$$

The latter expression differs from Eq. (4) of Ref. 4 by the appearance of the factor $\pi^2 / 6$ in place of the 2 in the earlier expression.

If the lattice temperature is much less than the temperature of the electrons (hardly a practical case), then

$$\bar{U} = \begin{cases} \frac{2\pi^2}{15} \frac{m S^2 n_0}{\tau(\Theta)} & (\Theta \ll T_0; T \ll \Theta); \\ \frac{\pi^2}{6} \frac{m S^2 n_0}{\tau(\Theta)} & (\Theta \gg T_0; T \ll \Theta). \end{cases} \quad (14)$$

3. Equation (13) can be obtained by purely classical means. Let us consider for this purpose the radiation of sound waves by an electron moving through the lattice with constant velocity v . Similar consideration was given by Buckingham¹⁰ but the method used was excessively complicated.

The equations for excited vibrations of the elastic continuum have the form

$$\ddot{\mathbf{u}}_l - s^2 \Delta \mathbf{u}_l = - (U / \rho) \nabla \delta(\mathbf{r} - \mathbf{v}t), \quad (15)$$

\mathbf{u}_l is the longitudinal component of the displacement vector; the transverse component is equal to zero identically, since the actual force on the part of the electron is "longitudinal" [$\text{curl } \mathbf{F} = 0$; $\mathbf{F} = -(U / \rho) \nabla \delta(\mathbf{r} - \mathbf{v}t)$], the δ -character of the force shows that the electron, as a consequence of the screening action of the atom, lies within the limits of single cell].

The energy losses of the electron can be computed as the work done by the force exerted by the electron on the medium (cf. with Ref. 11),

$$\frac{d\varepsilon}{dt} = U \int \dot{\mathbf{u}}_l \nabla \delta(\mathbf{r} - \mathbf{v}t) dV. \quad (16)$$

Expanding the δ -function in a Fourier integral, we determine the Fourier components of the displacement vector from (15):

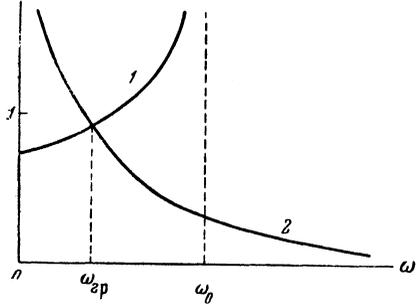
$$\mathbf{u}_k = \frac{iU}{\rho (2\pi)^3} \mathbf{k} \frac{e^{-i(\omega t - \mathbf{k}\mathbf{r})}}{\omega^2 - s^2 k^2}; \quad \omega = vk. \quad (17)$$

Hence

$$\mathbf{u} = \frac{iU}{\rho (2\pi)^3} \int \frac{\mathbf{k} e^{-i(\omega t - \mathbf{k}\mathbf{r})}}{\omega^2 - s^2 k^2} d\tau_{\mathbf{k}}. \quad (18)$$

Differentiating (18) with respect to the time and substituting in Eq. (16), making use of the properties

electric field E . Step-by-step solution of the problem reduces to the solution of a system of kinetic equations for the electronic and phonon distribution functions, in analogy with the known work of Davydov on semiconductors.¹⁴ A similar calculation for metals has been given recently.⁵ However the method, applied by the author, meets with several objections although the results obtained in Ref. 5 are evidently correct (see below) for not too high fields, i.e., for small differences of the electron and phonon temperatures.



Curve 1 — $\frac{1}{1 - \omega/\omega_0} - \frac{s_0^2}{v^2}$ as a function of ω , curve 2 — $\frac{1}{\omega^2} - \frac{s_0^2}{m^2 \omega^2}$ as a function of ω

If the difference of temperatures $\Theta - T$ is small in comparison with the lattice temperature T , then for the determination of the temperature of the electrons it is necessary to make use of the heat balance equation

$$c\dot{\Theta} = -\bar{U} + q \quad (26)$$

[$c = (\pi/2)^2 k n_0 kT/\epsilon_0$ is the electronic specific heat, q is the strength of the heat source]. In this case one can regard the phonon temperature as given, since the specific heat of the lattice down to very low temperatures ($\lesssim 1^\circ \text{K}$) is appreciably larger than the specific heat of the electrons. In our case, $q = \sigma E^2 = j^2/\sigma$, where $\sigma = ne^2\tau/m$ is the real conductivity of the metal. Within the limits of applicability of Matthiessen's rule⁹

$$1/\tau = 1/\tau(T) + 1/\tau_{\text{imp}} \quad (27)$$

where τ_{imp} is time of free flight relative to the impurity. From Eq. (26), making use of Eqs. (12) and (13), we find the equilibrium temperature of the electrons (for which $\dot{\Theta} = 0$):

$$\frac{k(\Theta - T)}{\epsilon_0} = \frac{3}{4\pi^2} \left(\frac{eLE}{\epsilon_0} \right)^2 \frac{l}{L} \frac{kT}{ms^2} \quad (T \ll T_0); \quad (28)$$

$$\frac{k(\Theta - T)}{\epsilon_0} = \frac{6}{\pi^2} \left(\frac{eLE}{\epsilon_0} \right)^2 \frac{kT}{ms^2} \quad (T \gg T_0). \quad (29)$$

Here $l = \tau(T)v_0$, $L = \tau v_0$. The expressions that have been obtained coincide, with accuracy to within a numerical factor, with the expressions found in Ref. 5 [Eq. (12)] and in Ref. 4.

Equation (28) cannot be regarded as correct down to absolute zero. It is evident from (14) that the heat emission will take place even for $T = 0$. This leads to the following value of the equilibrium electron temperature:

$$\Theta_p = T_0 \left\{ \frac{15}{2\pi^2} \frac{(eLE)^2}{\epsilon_0^2} \frac{\epsilon_0}{ms^2} \frac{l_0}{L} \right\}^{1/5}. \quad (30)$$

By l_0 is meant the value of the length of mean free path at the Debye temperature.*

Making use of Eq. (26), it is easy to find the relaxation time t_0 (it is identical with the time required to establish temperature equilibrium)

$$t_0 = \alpha\tau(T) (kT)^2 / ms^2 \epsilon_0. \quad (31)$$

Here $\alpha = 3/4$ for $T \ll T_0$ and $\alpha = 3$ for $T \gg T_0$. Since $\epsilon_0/k \sim 10^4$, $v_0 \sim 10^3$ s, then t_0 becomes of the same order as $\tau(T)$ for $T \sim 10^\circ \text{K}$. It is evident that our consideration would be invalid below this temperature if the metal did not contain impurities. However, for low temperatures, $\tau(T) \gg \tau$ practically always, so that even in this case the relaxation time (time for the transfer of energy from the electrons to the lattice) is significantly greater than the time for establishing the temperature inside the electron gas (for details, see Ref. 4).

If we consider that the lattice is maintained all the time at very low temperature ($T = 0$), then the established temperature is determined by Eq. (30) while the process of equalizing the temperature takes place according to the following law:

$$\Theta = \Theta_0 (1 + t/t_0)^{-1/5}, \\ t_0 \sim (kT_0)^2 (ms^2 \epsilon_0)^{-2/5} \tau_0 (L/l_0)^{3/5}.$$

The effect of the departure of the electron temperature from the phonon temperature on the electrical

*This is a purely formal definition: $\tau(T_0) = (T/T_0)^5 \times \tau(T)$. By $\tau(T)$ is understood the expression for the time of free flight, which is valid for temperatures considerably less than the Debye temperature.

of the δ -function, we find

$$\frac{d\varepsilon}{dt} = -\frac{iU^2}{(2\pi)^3 \rho} \int \frac{k^2 \omega d\tau_k}{\omega^2 - s^2 k^2}. \quad (19)$$

The integration is carried out over all k -space. In order that the integral have meaning, it is necessary to consider that s has a small imaginary addition (which later is made to approach zero) which corresponds to sound absorption in the medium. Here $s(-\omega) = s^*(\omega)$. With the help of the latter equality, Eq. (19) can be written as

$$\frac{d\varepsilon}{dt} = -\frac{U^2}{(2\pi)^2 \rho v} \times 2\text{Re} i \int_0^{\infty} \int_0^{\kappa_m} \frac{(\kappa^2 + \omega^2/v^2) \kappa d\kappa \omega d\omega}{\omega^2(1 - s^2/v^2) - s^2 \kappa^2}. \quad (20)$$

We transformed to a cylindrical coordinate system with its axis along the vector \mathbf{v} , κ_m is the maximum value of the component κ of the wave vector \mathbf{k} . We integrated over κ up to the upper limit κ_m keeping in mind the fact that for very short waves (over very short distances), the crystal cannot be regarded as continuous. Evidently κ_m must be of the order of π/d (d = lattice constant). For a more accurate choice of the value of κ_m see below.

Integrating first over κ , and noting that the component which does not contain the logarithm is known to be equal to zero, we get

$$\frac{d\varepsilon}{dt} = \frac{U^2}{(2\pi)^2 \rho v s^4} \times \text{Re} i \int_0^{\infty} \omega^3 d\omega \ln \left[1 + \frac{s^2 \kappa_m^2}{\omega^2 [(s^2/v^2) - 1]} \right]. \quad (21)$$

It is evident from Eq. (21) that the real contribution to the loss is made by those frequencies for which

$$1 + \frac{s^2 \kappa_m^2}{\omega^2 [(s^2/v^2) - 1]} < 0. \quad (22)$$

Then

$$\frac{d\varepsilon}{dt} = \begin{cases} 0 & (v < s), \\ -\frac{1}{16\pi} \frac{U^2 \kappa_m^4}{\rho v (1 - s^2/v^2)^2} & (v > s). \end{cases} \quad (23)$$

The latter expression coincides with the expression obtained in Ref. 10.

The divergence which takes place for $v=s$ (and, it would seem, which has no analog in Cerenkov radiation of electromagnetic waves), is

evidently connected with the fact that we have not taken dispersion into account (the dependence of the sound velocity on the frequency). Actually, it is evident from Eq. (22) that the divergence at $v=s$ is brought about by the fact that the limiting frequency becomes infinite (i.e., the frequency below which Eq. (22) is satisfied). Account of dispersion leads to finiteness of the limiting frequency for all values of the particle velocity.* For example, if we choose the dependence of the sound velocity on the frequency in the form

$$s = s_0 (1 - \omega/2\omega_0), \quad (24)$$

which corresponds to account of quadratic terms in the dependence of the frequency ω on the wave vector \mathbf{f} , then the limiting frequency ω_{lim} is determined by the equation and is finite for all values of

$$\frac{1}{1 - \omega_{\text{rp}}/\omega_0} - \frac{s_0^2}{v^2} = \frac{\kappa_m^2 s_0^2}{\omega_{\text{rp}}^2}$$

the electron's velocity (see Figure). We note that, with account of dispersion, there is always a region in which condition (22) is satisfied, i.e., there is a region of Cerenkov radiation of sound for any velocity. This is connected with the fact that, according to the assumption of Eq. (24), the velocity of sound for high frequencies ($\omega \sim \omega_0$) falls to zero (cf. Ref. 13).

It is evident that in the transfer of energy from the electron to the lattice, only those electrons take part which are located close to the Fermi surface,⁴ i.e.,

$$\bar{U} = \frac{1}{16\pi} \frac{U^2 \kappa_m^4 n_0}{\rho v_0} \frac{(\theta - T) k}{\varepsilon_0}. \quad (25)$$

We omit the factor $(1 - s^2/v_0^2)^2$, since $s \ll v_0$ (v_0 is the limiting velocity of the electrons). We choose κ_m so that Eq. (25) coincides with Eq. (9). Making use of the expression for ε_0 and T_0

= $\hbar \pi s / dk$ we obtain

$$\kappa_m = (3\pi^2)^{1/4} \pi / d,$$

which agrees in order of magnitude with the limiting wave vector π/d .

4. Up to now we have considered the temperature of the electrons as given. For its computation it is necessary to give a mechanism of heating the electrons. Let us consider the effect of a strong

*We recall that without account of dispersion, the energy losses of the electron in Cerenkov radiation also become infinite,¹² and for the same reason.

conductivity and the thermoelectric emission is considered in further detail in Ref. 4.

In conclusion the authors take this occasion to express their gratitude to E. S. Borovik for discussions on the problem.

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