The Uncertainty Relation and Fluctuation Theory

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As is well known, one of the fundamental bases of quantum mechanics is the uncertainty principle, first obtained by Heisenberg. In operator notation, it has the form

$$p_x x - x p_x = \hbar / i. \qquad (1)$$

The attempts of the representatives of the Copenhagen school of physicists (Bohr, Heisenberg and others) to consider the uncertainty principle as the result of uncontrollable interaction between the object and the measuring apparatus did not develop the physical significance of this important relation, not to mention the fact that such an interpretation led to a subjective, idealistic understanding of the phenomenon of the microworld (for details, see Ref. 1, p. 183).

As was pointed out by Blokhintsev (see Ref. 2, p. 154), the connection between corpuscular and wave properties, for the presence of many particles, can be correctly established with the aid of the introduction of statistical ensembles. However, statistical ensembles are not suitable for the interpretation of the uncertainty principle in its application to an isolated electron.

We attempt to construct a theory of electron motion (in isolated examples, for the time being), in which the quantum effects would be considered as the result of the singular effect of a collection of virtual particles which form the vacuum. An analogy can be established between the quantum effects and the theory of fluctuations, for example, in the investigation of radial vibrations in the theory of the "radiating" electron.

Making use of quantum theory, one of us, together with Ternov[3], succeeded in showing that the square of the amplitude of excitations (due to radiation in the direction of an incident photon with energy $\Delta E = \hbar\omega$) of macroscopic radial vibrations (which make up the "macro-atom") will increase according to the law

$$(\Delta R)^2 = R^2 (1 - q)^2 (\Delta E / E)^2. \qquad (2)$$

On the other hand, this same quantum formula can be obtained in semi-classical fashion. For this purpose we must consider, in the classical equations of the vibration of the electron (radial component), the fluctuating forces which are statistically independent:

$$F_{\text{fluct}} = [R \Delta E / (1 - q) c^2] \, \delta' (t - t_i), \qquad (3)$$

where q is the power of decay of the magnetic field in the neighborhood of the equilibrium orbit ($H \sim R^{-q}$, R = radius). In the identity of the two methods, we are inclined to see the connection (including even the quantitative side of the question) between the quantum method and the theory of fluctuations, where the so-called Markov chains occur, i.e., the statistical independence of consecutive processes.

We shall attempt to connect the quantum character of the motion of an electron in the microworld with the fluctuations of virtual photons. Similar attempts have already been made by a series of authors (see, for example, Welton, Ref. 4 and Kalitsin, Ref. 5).

Let us consider the equation of motion of a harmonic oscillator in the field of virtual photons;

$$m\ddot{x} = - m\omega_0^2 x - e (E_x + E_x^i), \qquad (4)$$

where

$$E_x^i = -\frac{1}{c}\frac{\partial A_x^i}{\partial t} = -\frac{2}{3}\frac{e}{c^3}\dddot{x}, \quad E = -\frac{1}{c}\frac{\partial A_x}{\partial t} \qquad (5)$$

is the self-acting electric field, i.e., unradiated longitudinal photons.

Then, with the help of division of the operator, we find the following expression for x

$$x = - L^{-3/2} \sum_{\vec{\varkappa}} \frac{e\omega}{mc} \sqrt{\frac{2\pi c\hbar}{\varkappa}} \qquad (6)$$

$$\times \left(\frac{i a_x \exp\{-i\omega t + i\vec{\varkappa}\mathbf{r}\}}{\omega_0^2 - \omega^2 - i\gamma\omega^3} + \text{complex conjugate} \right),$$

where $\gamma = 2e^2 / 3mc^3$, $\omega = c\varkappa = \omega_0 y$, and a_x is the quantum amplitude of the vector-potential.

In the given problem, the momentum of the particles must be equated to

$$p_x = m\dot{x} - \frac{e}{c}(A_x + A_x^i) \qquad (7)$$

$$= - L^{-3/2} \sum_{\vec{\varkappa}} \frac{e\omega_0^2}{c} \sqrt{\frac{2\pi c\hbar}{\varkappa}}$$

$$\times \left(\frac{a_x \exp\{-i\omega t + i\vec{\varkappa}\mathbf{r}\}}{\omega_0^2 - \omega^2 - i\gamma\omega^3} + \text{complex conjugate} \right).$$

It is evident from (6) and (7) that, thanks to a consideration of the field of virtual photons, the classical quantities x and p_x are operators which do not commute with each other:

$$p_x x - x p_x = \frac{\hbar}{i} \frac{2}{\pi} \gamma\omega_0 J, \qquad (8)$$

$$J = \int_0^\infty \frac{y^2 dy}{(y^2-1)^2 + \gamma^2\omega_0^2 y^6}.$$

In deriving the last formula, we took into consideration the permutation relations

$$[a_s a_{s'}^+]_- = \delta_{ss'} - \varkappa_s^0 \varkappa_{s'}^0.$$

The integral J does not diverge and has the value

$$J = \pi/2\gamma\omega_0 + O(\gamma\omega_0). \qquad (9)$$

Therefore, in first approximation, we obtain permutation relations which coincide with the permutation relations (1) of wave theory.

Making use of the operator expressions (6) and (7) for x and p_x, we can obtain the energy levels for the harmonic oscillator. If the momentum is given, not in the form (8), but in the form

$$p_x = m\dot{x}, \qquad (10)$$

as was done, for example, in Ref. 8, then an additional factor of 2 appears on the right side of Eq. 1. It is of interest to note that when photons are absent ($a_s^+ a_{s'} = 0$) it is better to use Eq. (7) for the momentum in the expression of zero point energy

$$E_0 = (p_x^2/2m) + {}^1/_2 m\omega_0^2 x^2, \qquad (11)$$

rather than Eq. (10) as would have been more natural in the given case.

Giving the momentum by Eq. (7), we find that the zero-point energy will automatically contain the necessary subtraction terms, leaving the finite quantity

$$E_0 = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0^2 e^2}{3\pi c^3 m}\left(\ln\frac{3c^3 m}{2e^2\omega_0} - 1\right). \qquad (12)$$

The first term is the well-known expression for the zero-point energy without vacuum terms, and the second is an additional energy caused by the vacuum action.

A strict quantum electrodynamical derivation of the corresponding formula gives an expression very close to Eq. (12):

$$E_0^{\text{qm}} = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_0^2 e^2}{3\pi c^3 m}\left(\ln\frac{mc^2}{\hbar\omega_0} - 0{,}2\right). \qquad (13)$$

It is of interest to observe that if we limit the integral of type (9) (which is also encountered in the calculation of the zero-point energy), to relativistic frequencies of vibration $\omega_{max} = 2mc^2/\hbar$, as was done by Weisskopf[9], then we obtain Eq. (13) for the additional energy, with the help of the fluctuation method.

Thus a classical system which describes the motion of an electron in its interaction with the second quantized field of photons (actually radiated, or only virtual) is the same as in quantum mechanics.

[1] A. Sokolov and D. Ivanenko, *Quantum Theory of Fields*, Moscow-Leningrad, 1952.

[2] D. Blokhintsev, *Fundamentals of Quantum Mechanics*, Moscow-Leningrad, 1949.

[3] A. Sokolov and I. Ternov, Dokl. Akad. Nauk SSSR **97**, 823 (1954); J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 431 (1955); Soviet Phys. JETP **1**, 227 (1955).

[4] T. Welton, Phys. Rev. **74**, 1157 (1948).

[5] N. Kalitsin, J. Exptl. Theoret. Phys. (U.S.S.R.) **25**, 407 (1953).

[6] A. Sokolov, Vestnik, Moscow State Univ. **2**, (1947); J. Exptl. Theoret. Phys. (U.S.S.R.) **18**, 280 (1948).

[7] D. Ivanenko and A. Sokolov, *Classical Theory of Fields*, Moscow-Leningrad, 1951.

[8] E. Adirovich and M. Podgoretskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **26**, 150 (1954).

[9] V. Weisskopf, Rev. Mod. Phys. **21**, 305 (1949).

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Radiation Resonance in Synchrotrons

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The present research deals with a resonance which can take place under certain conditions in synchrotrons, due to the presence of radiation, and which can bring about amplification of the amplitude of betatron oscillations. The main points are the following:

1. In the absence of radiation, the betatron oscillations in synchrotrons are described by an equation of the form