Nuclear Density and the Distribution of Orbital Angular Momentum in Nuclei

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A connection is found between the state of a nucleus, which is characterized by the minimum energy obtained in Ref. 1 and the distribution of the mean square orbital angular momentum of the nucleons in the nucleus. The number of particles in the nucleus for which nucleons in the l-state first appear has been determined.

THE statistical model of a nucleus with non-uniform density distribution of nucleons was investigated in Ref. 1. The parameters of the density function were obtained from the condition of saturation of the binding energy and the density. In the present work, starting out from the particle density obtained in Ref. 1, the distribution of the mean square orbital momentum of the nucleons is calculated and its relation to the state of the nucleus which is characterized by the minimum energy is found. In the statistical Thomas-Fermi model, the mean square of the orbital momentum of the nucleon can be expressed by the following equation (the treatment is carried out only for neutrons, since the same formulas are also valid for protons):

$$\langle L_N^2\rangle_{\text{av}} = \frac{1}{N}\int_0^\infty L^2 n_N(L)\,dL. \tag{1}$$

Here $n_N(L)dL$ is the number of neutrons with orbital momentum between L and $L+dL$ which, according to the Thomas-Fermi model, is equal to

$$n_N(L) = \frac{4L}{\pi\hbar^3}\int_{r_1}^{r_2}\sqrt{r^2P_n^2(r)-L^2}\,\frac{dr}{r}. \tag{2}$$

N is the total number of neutrons in the nucleus. The limits of integration should be so chosen that the expression under the integral would be real for $r_1 \le r \le r_2$.

With the help of (2), Eq. (1) yields[2]

$$\langle L_N^2\rangle_{\text{av}} = \frac{8}{15\pi N\hbar^3}\int_0^\infty \frac{dr}{r}[rP_n(r)]^5. \tag{3}$$

The maximum momentum of the neutron $P_n(r)$, according to Eq. (12) of Ref. 1, is connected with the density of neutrons $\rho_n(r)$ by the relation

$$P_n(r) = (3\pi^2)^{1/3}\hbar\rho_n^{1/3}(r). \tag{4}$$

Taking (4) into consideration, we express the mean value of (3) by the neutron density distribution function:

$$\langle L_N^2\rangle_{\text{av}} = \frac{8(3\pi^2)^{5/3}\hbar^2}{15\pi N}\int_0^\infty r^4\rho_n^{5/3}(r)\,dr. \tag{5}$$

Equation (5), together with the renormalization condition

$$\int \rho_n(r)\,d\tau = N \tag{6}$$

for a known density of particles ρ gives the distribution of the mean square angular momentum of the neutrons (or protons) in the nucleus. On the other hand, we have for the mean value (from the shell model of the nucleus),

$$\langle L_N^2\rangle_{\text{av}} = \frac{1}{N}\sum_{N_i=1}^{N} l_{N_i}(l_{N_i}+1), \tag{7}$$

where L_{N_i} is the orbital quantum number of the N_i the neutron. It is clear from this that a definite requirement follows from the shell model for the probable form of the density distribution of particles in nuclei, viz: only for a choice of a convergent form of the density function of the particles can we obtain results from Eq. (5) which agree with experimental values of (7).

For our case of the density of distribution of particles, determined by Eq. (42) of Ref. 1,

$$\rho_n(r) = {}^1/_2\rho(r) = \rho_{0n}e^{-(r-R_0)/a}, \tag{8}$$

we, making use of Eqs. (6) and (5), obtain after integration

$$\langle L_N^2\rangle_{\text{av}} = \frac{3}{25}2^{-1/3}(9\pi)^{2/3}\hbar^2N^{2/3} \tag{9}$$
$$\times\frac{1+3\varepsilon_0+\frac{36}{5}\varepsilon_0^2+\frac{12\cdot 27}{25}\varepsilon_0^3+\frac{24\cdot 81}{125}\varepsilon_0^4+\frac{243\cdot 24}{625}\varepsilon_0^5}{(1+3\varepsilon_0+6\varepsilon_0^2+6\varepsilon_0^3)^{5/3}}$$

We get precisely the same equation for protons if we take Eq. (8) for $\rho_p(r)$ and replace N by Z.

Choice of the same function for neutrons and for protons in the problem under consideration is justified by the fact that the experimental data on shells indicate an identical distribution of angular momenta for neutrons and protons. Thus in the case of the density distribution (8), the mean square angular momentum of neutrons and protons in the nucleus is a function of only the single parameter $\epsilon_0 = a/R_0$. One can choose for this parameter only such values which, upon substitution in (9) give results which agree with (7).

If we choose

$$\varepsilon_0 \approx 0.2, \tag{10}$$

we obtain, in the mean, a satisfactory agreement with the shell model. The graph of the function $\langle L^2 \rangle_{av}$, obtained from Eq. (9), and also from the shell model (7), is plotted in Fig. 1. The solid line in Fig, 1 corresponds to the value of Eq. (10).

It follows from Eq. (10) that the thickness of the surface layer a increases in proportion to the radius R_0 of the rest of the nucleus (with constant density). Inasmuch as $R_0 \sim A^{1/3}$ [see Ref. 1, Eq. (47) and Tables 1 and 2], then $a \sim A^{1/3}$ also.

Consequently, the agreement of the data on the mean square angular momentum of the neutrons and protons in the nucleus (according to the Thomas-Fermi model), with the shell model has as a consequence that, for $A \gtrsim 100$, the thickness of the surface layer ought to be proportional to $A^{1/3}$. Therefore, there is no necessity (as was done in Ref. 2 for the Born-Yang density) to choose the surface layer independent of A.

We establish the connection between the minimum energy, obtained in Ref. 1, and the distribution of the angular momentum in the nucleus. As can be seen from the data of Tables 1 and 2[1], the value of the parameter q, which corresponds to the value of ϵ_0 of Eq. (10), is equal to

$q \approx 3.6$ for the case of Eq. (61) of Ref. 1 (11)

$q \approx 3$ for the case of Eq. (62) of Ref. 1 (12)

Substituting Eqs. (10) and (11) into Eq. (63) of Ref. 1, we obtain for the parameters R_0, a, $\overline{r}_0$ and ρ_0 the values:

$$R_0 = 0.944 \cdot 10^{-13} A^{1/3}; \tag{13}$$

$$a = 0.189 \cdot 10^{-13} A^{1/3};$$

$$\overline{r}_0 = r_0^* A^{1/3} = 1.133 \cdot 10^{-13} A^{1/3};$$

$$\rho_0 = 0.394\, k_0^3 = 0.915 / (4\pi/3)\, r_0^{*3}.$$

For the pair of values (10 and (12) we get

$$R_0 = 1.133 \cdot 10^{-13} A^{1/3}; \tag{14}$$

$$a = 0.227 \cdot 10^{-13} A^{1/3};$$

$$\overline{r}_0 = 1.36 \cdot 10^{-13} A^{1/3};$$

$$\rho_0 = 0.228\, k_0^3 = 0.915 / (4\pi/3)\, r_0^{*3}.$$

The nuclear density (8), with the values of (13) and (14) for the parameters R_0, a and ρ_0 gives the correct value for the binding energy E/A and the distribution of the mean square orbital momentum, which agrees with the shell model. The values of the parameters (13) which characterize the model of the nucleus (8) are closer to the values found from the experimental data relating to the nuclear scattering of electrons and nucleons of high energy[3] than those of (14). It was shown in Ref. 3 that the theoretically obtained angular distribution of the scattering of electrons (with energy of 15.7 mev) by nuclei of silver and gold with nucleon density distribution of the form (8) agrees with the results of experiments if we assume that

$$R_0 = 0.792\, r_0 A^{1/3};\ a = 0.208\, r_0 A^{1/3}; \quad r_0 \approx 1 \cdot 10^{-13}\ \text{cm};\ \rho_0 = 0.87 / (4\pi/3)\, r_0^3. \tag{15}$$

For the density distribution of nucleons (51) in Ref. 1, we obtain the following expression from Eq. (5):

$$\langle L_N^2 \rangle_{av} = 0.417\, \hbar^2 N^{2/3}. \tag{16}$$

For constant particle density of Eq. (58), Ref. 1, Eq. (5) gives

$$\langle L_N^2 \rangle_{av} = 0.885\, \hbar^2 N^{2/3}. \tag{17}$$

The graphs of the functions (16) and (17) are shown in Fig. 1. It is evident from the figure that constant nuclear density gives higher values for the mean square angular momentum (see also Ref. 2), but that the nuclear density of the form of Eq. (51) in Ref. 1 is too low by comparison with experimental data.

2. Let us find the relation between the density distribution (8) and the first appearance of the nucleon with high angular momentum in the nucleus. In the limits of applicability to the

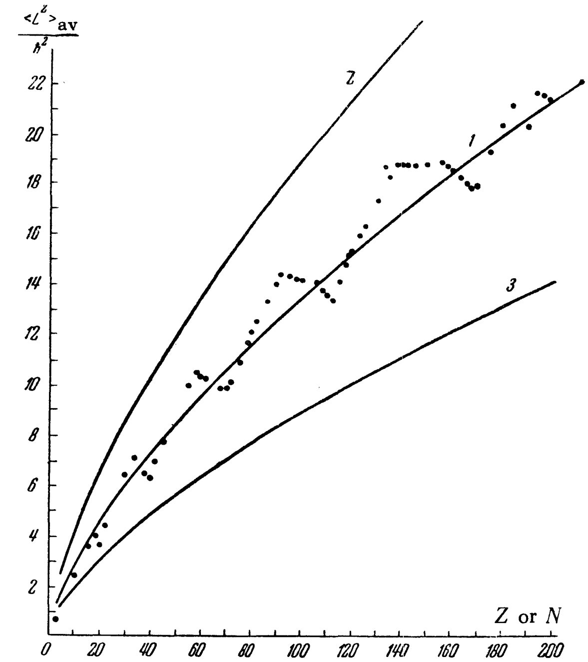

FIG. 1. Comparison of the mean square orbital momentum of nucleons, computed according to the model of Thomas-Fermi, with values computed from the shell model. Points are the data of the shell model: *1* -- for nonuniform density (8); *2* -- for constant density; *3* -- for density of particles of the form of Eq. (51) of Ref. 1.

nucleus of the statistical model of Thomas-Fermi, the following correlation was established between the structure of the shells and the density of particles in the nucleus[4]

$$N_l = \gamma\,(2l+1)^3, \qquad (18)$$

where

$$\gamma = \frac{1}{24\pi^2}\left[\frac{N}{r^3\rho\,(r)}\right]_{r=r_m}; \qquad (19)$$

$$\frac{1}{r} = -\frac{1}{3}\frac{d}{dr}\,[\rho\,(r)\,/\,N]. \qquad (20)$$

Here N_l is the number of particles which correspond to nuclei which have no nucleons in the l-state; $\rho(r)$ is the density distribution of nucleons in the nucleus; r_m is the root of Eq. (20). Substituting the value of (8) in Eq. (20) in place of $\rho(r)$, we find

$$r_m = 3a. \qquad (21)$$

With the help of Eqs. (8), (6) and (21), we obtain N_l and the coefficient γ from Eqs. (18) and (19) as functions of the parameter ϵ_0 in the form

$$N_l = \gamma\,(2l+1)^3, \qquad (22)$$

where

$$\gamma = (1\,/\,18\pi\cdot 27)$$
$$\times(1 + 3\varepsilon_0 + 6\varepsilon_0^2 + 6\varepsilon_0^3)\,\varepsilon_0^{-3}e^{3-1/\varepsilon_0}. \qquad (23)$$

The dependence of N_l and γ on ϵ_0, computed by Eqs. (22) and (23) are represented in Fig. 2 (for each l). It is evident from these graphs that for $\epsilon_0 \gg 1$, the values of N_l for all l are almost independent of ϵ_0.

For the value

$$\varepsilon_0 = a\,/\,R_0 = 0.375 \qquad (24)$$

the neutron (or proton) critical numbers computed from Eq. (22) are equal to

$l = 0$	1	2	3	4	5	6
$N_l =$	1.54	7.12	19.54	41.53	75.83	125.16
Next Integer	2	8	20	42	76	126

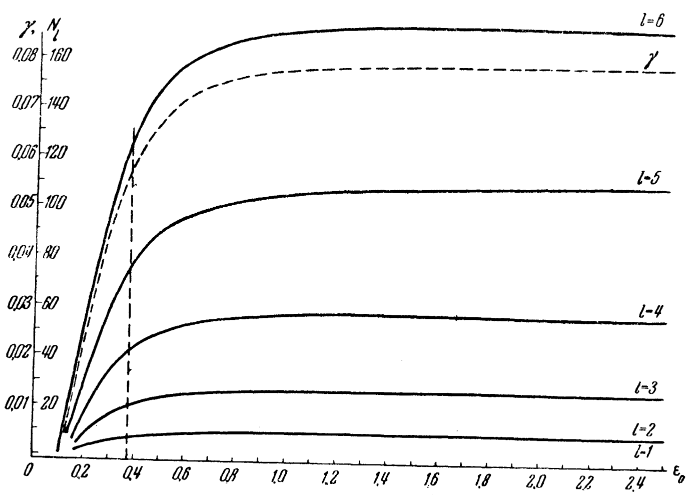

FIG. 2. N_l and γ as functions of $\epsilon_0 = a/R_0$.

These numbers agree well with the values of N_l determined in the quantum-mechanical shell model; these latter are, respectively, equal (square potential well) to 2, 8, 20, 40, 70, 112.

We find from the dependence $q = q(\epsilon_0)$ in Ref. 1 (see Fig. 2) that the value of the parameter q which corresponds to $\epsilon_0 = 0.375$ is equal to

$$q = 3.935 \text{ for case (61) of Ref. 1} \tag{25}$$

$$q = 3.115 \text{ for case (62) of Ref. 1.} \tag{26}$$

Substituting (25) and (24) in Eq. (63) of Ref. 1, we find the following values for the parameters:

$$R_0 = 0{,}718 \cdot 10^{-13} A^{1/3}; \tag{27}$$

$$a = 0{,}269 \cdot 10^{-13} A^{1/3};$$

$$\bar{r}_0 = r_0^* A^{1/3} = 0{,}987 \cdot 10^{-13} A^{1/3}; \rho_0$$

$$= 0{,}514\, k_0^3 = 0{,}792 / (4\pi/3)\, r_0^{*3}.$$

For the values of Eq. (24) and (26), we have

$$R_0 = 0.907 \cdot 10^{-13} A^{1/3}; \quad a = 0.340 \cdot 10^{-13} A^{1/3}; \tag{28}$$

$$\bar{r}_0 = r_0^* A^{1/3} = 1.247 \cdot 10^{-13} A^{1/3};$$

$$\rho_0 = 0.255\, k_0^3 = 0.792 / (4\pi/3)\, r_0^{*3}.$$

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