$$\Phi(y) = (2\pi)^{-1/2} e^{-y^2/2}\left[1 + \frac{1}{10}\left(\frac{3mL_iE_0}{2\mu^2}\right)^{1/2}(y^3 - 3y)\right.$$
$$\left. + \frac{1}{28}\frac{mL_iE_0}{\mu^2}(y^4 - 6y^2 + 3) + \frac{3}{400}\frac{mL_iE_0}{\mu^2}(y^6 - 15y^4 + 45y^2 - 15) + \ldots\right]; \quad (16)$$

$$y^2 \approx (E - \overline{E})^2 (3L_i\mu^2/2mE_0^3)(1 + 3mE_0/2L_i\mu^2)^{-1}; \quad (17)$$

$$B_n \approx (2m/\mu^2)^{n-1}E^{2n-1}/(n-1)(2n-1)L_i(E_0), \; \overline{E} \ll E_0. \quad (18)$$

From Eq. (16) it is evident that for sufficiently large ratio $L_i m E_0 / \mu^2$, the distribution function differs substantially from a Gaussian one even at the end of the passage. The curve has a characteristic "tail" at the low energy and a sharp cutoff on the high energy side. Its maximum is displaced in the direction of greater energies relative to its center of gravity ($y = 0$).

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[2] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) 18, 759 (1948).

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The Propagation of Sound in Moving Helium II and the Effect of a Thermal Current upon the Propagation of Second Sound

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ROM ordinary hydrodynamics it is well known that an "entrainment" of sound occurs in a moving fluid. A similar phenomenon must take place in the hydrodynamics of a superfluid. Inasmuch as in the case of a superfluid two types of motion (normal, with a velocity $\mathbf{v}_n$, and superfluid, with a velocity $\mathbf{v}_s$) as well as two types of sound vibrations, propagated with different velocities, are possible, it is natural that the picture of sound propagation in a moving superfluid liquid should differ from the corresponding phenomenon in classical hydrodynamics.

Let sound oscillations of frequency ω be propagated in a direction characterized by the unit vector n (along the x-axis) through helium II in which normal and superfluid motions are taking place with the constant velocities $\mathbf{v}_n$ and $\mathbf{v}_s$. The wave vector k is equal to $\mathbf{n}\omega/u$, where u is the velocity of sound. We shall determine here the velocities of first and second sound in the moving Helium II under the assumption that the motion proceeds at velocities which are small by comparison with the velocity of sound. Let us write the complete set of hydrodynamic equations for Helium II:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0,$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_k}(p\delta_{ik} + \rho_n v_{ni}v_{nk} + \rho_s v_{si}v_{sk}) = 0,$$

$$\frac{\partial\rho s}{\partial t} + \operatorname{div}\rho s v_n = 0, \quad \frac{\partial\mathbf{v}_s}{\partial t} + \nabla\left(\Phi + \frac{v_s^2}{2}\right) = 0.$$

Here ρ is the density, s is the entropy per unit mass, p is the pressure, $\mathbf{j} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$ is the mass current density, ρ_n and ρ_s are the densities of the normal and superfluid components, and $\Phi = \Phi_0(p,T) - \rho_n(\mathbf{v}_n - \mathbf{v}_s)^2/2\rho$ is the thermodynamic potential, depending upon the relative velocity $\mathbf{v}_n - \mathbf{v}_s$. If we take the pressure and temperature as independent variables, then from the thermodynamic identity for the potential

$$\rho\, d\Phi = -\rho s\, dT + dp + (\mathbf{j} - \rho\mathbf{v}_s)\, d(\mathbf{v}_n - \mathbf{v}_s)$$

it follows that the density ρ and the entropy s are functions of the relative velocity $\mathbf{v}_n - \mathbf{v}_s$:

$$s = s_0 + \frac{\partial}{\partial T}\left(\frac{\rho_n}{2\rho}\right)(\mathbf{v}_n - \mathbf{v}_s)^2, \quad s_0 = -\left(\frac{\partial\Phi_0}{\partial T}\right),$$

$$\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{\partial}{\partial p}\left(\frac{\rho_n}{2\rho}\right)(\mathbf{v}_n - \mathbf{v}_s)^2,$$

$$\frac{1}{\rho_0} = \left(\frac{\partial\Phi_0}{\partial T}\right)_p$$

We shall look for increments linear in $\mathbf{v}_n$ and $\mathbf{v}_s$ to the velocity of sound for the stationary fluid. For this purpose it is necessary to rewrite the hydrodynamic equations to include terms quadratic in the velocity. We shall introduce the notation $\mathbf{v} = \mathbf{j}/\rho$ and $w = \mathbf{v}_n - \mathbf{v}_s$. The components of these vectors in the direction of the wave vector k we shall identify by the index k, and those in a plane perpendicular to k by the index $\perp$. In a traveling sound wave, all quantities depend upon the argument $(x - ut)$, derivatives with respect to which we shall indicate by a prime. To the accuracy we require, the hydrodynamic equations assume the form (with $U = u - v_k$)

$$-U\frac{\partial\rho}{\partial T}T' - U\frac{\partial\rho}{\partial p}p' - U\rho^2\frac{\partial}{\partial\rho}\left(\frac{\rho_n}{2\rho}\right)(w_k w_k' + w_\perp w_\perp') + \rho v_k' = 0, \quad (1)$$

$$p' + (2\rho_s\rho_n/\rho)\, w_k w_k' - \rho U v_k' = 0, \quad (2)$$

$$(\rho_s\rho_n/\rho)(w_k w_\perp' + w_k' w_\perp) - \rho U v_\perp' = 0, \quad (3)$$

$$\left[-\rho U\frac{\partial s}{\partial T} + w_k\frac{\partial}{\partial T}(\rho_s s)\right]T' + \left[-\rho U\frac{\partial s}{\partial p} + w_k\frac{\partial}{\partial p}(\rho_s s)\right]p' + \rho_s s w_k' = 0, \quad (4)$$

$$\left[-s - Uw_k\frac{\partial}{\partial T}\left(\frac{\rho_n}{\rho}\right)\right]T' - U\frac{\partial}{\partial p}\left(\frac{\rho_n}{p}\right)w_k p' + \left(\frac{\rho_n}{\rho}U - \frac{3\rho_n\rho_s}{\rho^2}w_k\right)w_k'$$
$$-\left(\frac{\rho_n}{\rho}v_\perp + \frac{\rho_s\rho_n}{\rho^2}w_\perp\right)w_\perp' - \frac{\rho_n}{\rho}w_k v_k' - \frac{\rho_n}{\rho}w_\perp v_\perp' = 0, \quad (5)$$

$$[v_\perp' - (w_\perp\rho_n/\rho)'] = 0. \quad (6)$$

Simultaneous solution of this system of equations will yield equations determining the values of the velocity of sound u. In this process the components of the equations of motion perpendicular to the wave vector k [Eqs. (3) and (6)] can be separated from the remaining equations. This is due to the fact that the components $v_\perp$ and $w_\perp$ enter into Eqs. (1) and (5) only in terms of the second order. In this way we find:

$$\left(U^2\frac{\partial\rho}{\partial p} - 1\right)\left\{\left(\rho_n\rho U^2\frac{\partial s}{\partial T} - s^2\rho\rho_s\right) - Uw_k\left(4\rho_s\rho_n\frac{\partial s}{\partial T} - 2\rho s\frac{\partial\rho_n}{\partial T}\right)\right\} = 0; \quad (7)$$

$$U - w_k\rho_s/\rho = 0. \quad (8)$$

The roots of Eq. (7) determine the velocities of first and second sound in moving Helium II. For first sound we find a solution which is identical with that known from ordinary hydrodynamics: $u_1 = c_1 + v_k$, $c_1^2 = \partial p/\partial\rho$. For the velocity of second sound we have:

$$u_2 = c_2 + v_k + w_k\left(\frac{2\rho_s}{\rho} - \frac{s}{\rho_n}\frac{\partial\rho_n}{\partial T}\frac{\partial T}{\partial s}\right), \quad (9)$$

$$c_2^2 = \frac{\rho_s}{\rho_n}s^2\frac{\partial T}{\partial s}.$$

From (8) we find that the oscillations of the transverse components of velocity $v_\perp$ and $w_\perp$ in a plane perpendicular to the direction of the wave vector are propagated with the velocity

$$u = v_k + w_k\rho_s/\rho = \mathrm{v}_n\mathrm{k}/k. \quad (10)$$

According to (6) no superfluid motion takes place in a direction perpendicular to the wave vector k. The direct implication of the solution (10) is that the transverse oscillations of the normal velocity are propagated with a velocity equal to v_n. These oscillations of the normal velocity are analogous to the viscous waves of an ordinary fluid.

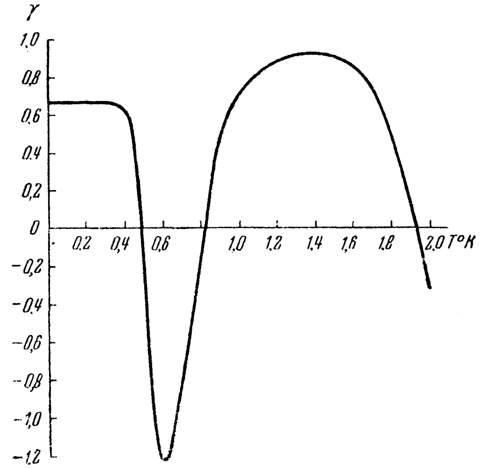
The velocities c_1 and c_2 are the velocities of sound for stationary helium. If the magnitudes of the velocities in the sound wave are small by comparison with the constant velocity components, it will be possible to neglect the nonlinear effect arising from the variation of c_1 and c_2 along the profile of the sound wave (see Ref. 1 for further details).

From Eqs. (9) it follows that if a constant thermal current is present in Helium II it will carry the second sound along. In the thermal current j = 0 and, consequently, v= 0 as well. Further, we have $\mathrm{w} = \mathrm{v}_n - \mathrm{v}_s = \mathrm{v}_n\rho/\rho_s$. In this case Eq. (9) assumes the form:

$$u_2 = c_2 + \frac{(\mathrm{v}_n\mathrm{k})}{k}\frac{\rho}{\rho_s}\left(\frac{2\rho_s}{\rho} - \frac{s}{\rho_n}\frac{\partial\rho_n}{dT}\frac{\partial T}{\partial s}\right) \quad (11)$$
$$= c_2 + \gamma v_{nk}.$$

The quantity v_n is related to the thermal current density q by means of the equation $\mathrm{q} = \rho s T \mathrm{v}_n$. The coefficient γ preceding v_{nk} in (11) is a function of temperature of order unity. The dependence of γ on temperature is shown in the figure. In the temperature region above 1 ° K, where only the roton components of thermodynamic quantities are of importance, we have, approximately, $\rho_n \sim s$ and $\gamma = (\rho_s - \rho_n)/\rho_s$. From the fact that γ changes sign at several temperatures it follows that the "entrainment" of the second sound can take place either in the direction of the thermal current or opposite to it.

Finally, we comment on the effect of a thermal current upon the velocity of second sound in standing waves. From the symmetry of the standing waves, it follows that this effect will be quadratic with respect to the ratio of the "entrainment" velocity to the velocity of second sound. Even for velocities $\mathrm{v}_k \sim 1$ m/sec, the



effect upon the velocity of second sound in a standing wave is of the order of a fraction of a percent:

In conclusion, I thank L. D. Landau for his consideration of these results.

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Remarks on One Variant of the Equations of a Nonlocal Field

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IN a nonlocal field theory, where the wave-function of a particle $U(x_\mu, \xi_\mu)$ depends on "internal" variables ξ_μ as well as the ordinary space-time variables x_μ $(\mu = 1, 2, 3, 4)$ it is natural to interpret the orbital angular momentum of "internal" motion as the intrinsic angular momentum (spin) of the particle. We will assume the function U to be scalar; then the equation of Markov[1] for a free particle in the momentum representation has the form:

$$\left\{\omega^2 k_\mu^2 - \frac{\partial^2}{\partial r_\mu^2} + r_\mu^2 + \frac{2}{k_\mu^2}\left[\left(k_\mu \frac{\partial}{\partial r_\mu}\right)^2 - (k_\mu r_\mu)^2\right]\right\}\chi(k_\mu, r_\mu) = 0, \quad (1)$$

where

$$\chi(k_\mu, r_\mu) = \int \exp\{-ik_\mu x_\mu\}\, U(x_\mu, r_\mu)\, d^4x,$$

and λ and ω are constants with the dimensions of lengths. We introduce the notation $f = -\omega^2 k^2_\mu$; this is the square of the mass measured in units of $\hbar/\omega$. In a rest system Eq. (1) assumes the form

$$\left(-\frac{\partial^2}{\partial r_i^2} + r_i^2 - \frac{\partial^2}{\partial r_0^2} + r_0^2\right)\chi = f\chi, \quad (2)$$

where $i = 1, 2, 3$ and r_0 is the real variable $r_0 = -ir_4$. The solutions of (2) will be sought in the form $X(r_\mu) = g(r_i)\,\Phi(r_0)$, separating the dependence on space and time variables. On the function X we impose the requirement of boundedness in all of the 4-space of internal coordinates.

For the functions $g(r_i)$ and $\Phi(r_0)$ we get the equations

$$(-(\partial^2/\partial r_i^2) + r_i^2)\,g = kg, \quad (3)$$

$$(-(\partial^2/\partial r_0^2) + r_0^2)\,\Phi = (f-k)\,\Phi, \quad (4)$$

where k is a constant of separation of variables. The solution of Eq. (3) in spherical coordinates $r = |\mathbf{r}|, \theta, \varphi$, as is well known, has the form

$$g_{klm}(r, \theta, \varphi) = Y_{lm}(\theta, \varphi)\, r^l e^{-r^2/2} L^{l+1/2}_{(k-2l-3)/4}(r^2),$$

where Y_{lm} is the spherical function and L is the associated Laguerre polynomial. Here the quantity k assumes the values

$$k = 4n + 2l + 3, \quad (5)$$

where $l = 0, 1, 2, \ldots$; $n = 0, 1, 2 \ldots$

Thus for given k the internal angular momentum l can assume the values $0, 2, \ldots, (k-3)/2$ or $k, 3, \ldots, (k-3)/2$ depending on whether k is odd or even. The projection of the internal angular momentum m assumes the values $|m| \leqslant l$.

Equation (4) has bounded solutions only for

$$f - k = 2n_0 + 1, \quad n_0 = 0, 1, 2, \ldots \quad (6)$$

Its solution then has the form

$$\Phi_{n_0}(r_0) = H_{n_0}(r_0)\, e^{-r_0^2/2},$$

where H_n is the Hermite polynomial. From condition (6) we obtain that for given f the quantity k can assume the values $3, 5, \ldots, f-1$. However, from conditions (5) and (6) it is evident that f can assume the values $2\beta + 4$